Control in multidimensional chaotic systems by small perturbations

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An algorithm is given by which the number of visits to the ε neighborhood of an arbitrary point of a chaotic attractor is increased. When the perturbation is sufficiently large, the trajectory becomes periodic.

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In recent years we have witnessed the results of work by many scientists directed toward the use of chaotic phenomena by way of control by some form of small external perturbations. The dynamical system under consideration is described by

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x},\mathbf{p}) , \tag{1}$$

where $\mathbf{x}=(x_1,x_2,\ldots,x_n)$ defines the state of the system and $\mathbf{p}$ is the vector of the control parameters. With slight changes of $\mathbf{p}$ the chaotic system can be controlled, obtaining periodic motion [1], targeting trajectories [2], or achieving synchronization [3]. Although these methods are applicable in principle to systems with arbitrary dimension, due to technicalities their use is practical only for systems with low fractal dimensions.

In another approach [4], instead of changes of the control parameter, small perturbations $\mathbf{g}(\mathbf{x})$ are added to the system, with the aim of achieving control and some desirable effects. The dynamical system to be considered is then given by

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x},\mathbf{p})+\mathbf{g}(\mathbf{x}) . \tag{2}$$

The main difficulty is how to make the appropriate choice of perturbation with the restriction that it is sufficiently small. Most papers dealing with the subject of control of the behavior of nonlinear systems are devoted to methods of stabilizing some periodic state. However, one can think of situations where a periodic outcome is neither desirable nor, strictly speaking, possible, as is the case in weather comportment, population dynamics, and functioning of markets. On the other hand, some particular states in the evolution of a system may have a property which is considered beneficial and therefore a valuable aim in the control of the system. With this motivation in mind, in this paper we address the following question: How to control the trajectory in order to increase the number of visits in the neighborhood of a specified point on the chaotic attractor? We provide an algorithm which gives a desired control. Although the algorithm was developed in order to obtain an answer to the above question, it can be applied to causing the motion to become periodic when large enough perturbations are used, but still much smaller than the size of the attractor. Moreover, the dimension of the chaotic dynamical system is not essential for the application of the algorithm, i.e., it can be applied to such systems of arbitrary dimension.

First we define the natural ε frequency at a point $\bar{\mathbf{x}}$ which belongs to the attractor $\mathcal{A}$ of the dynamical system (1). Let $T(\mathbf{f},\mathbf{x}_0)$ denote the trajectory obtained from (1), with an initial point $\mathbf{x}_0$ at $t=0$ [here $\mathbf{f}$ is a solution of (1), with an initial point $\mathbf{x}_0$]. Consequently $\bar{\mathbf{x}}\in T(\mathbf{f},\mathbf{x}_0)$ means that $\exists\tilde{t}\in(0,\infty)$ such that $\bar{\mathbf{x}}=\mathbf{f}(\mathbf{x}_0,\tilde{t})$. For a given $\bar{\mathbf{x}}\in\mathcal{A}$ and $\Theta\in R$, $\Theta>0$ and $\varepsilon>0$, let $k=k(\Theta)$ be the number with the property that $\exists 0\le t_1'<t_1''<t_2'<t_2''<\cdots t_k'<t_k''\le\Theta$, such that, if $\mathbf{x}_0\notin O(\bar{\mathbf{x}},\varepsilon)$,

$$\forall t\in(t_i'<t_i''),\quad \mathbf{f}(\mathbf{x}_0,t)\in O(\bar{\mathbf{x}},\varepsilon)$$

and

$$\forall t\in[t_i'',t_{i+1}'],\quad \forall t\in[0,t_1'],\quad \forall t\in[t_k'',\Theta] ,$$
$$\mathbf{f}(\mathbf{x}_0,t)\notin O(\bar{\mathbf{x}},\varepsilon) ,$$

where $O(\bar{\mathbf{x}},\varepsilon)$ is the open ball with its center at $\bar{\mathbf{x}}$ and a radius ε.

We will call the natural ε frequency (or, briefly, the ε frequency) of the trajectory of the dynamical system at $\bar{\mathbf{x}}$ the limit

$$\mu(\bar{\mathbf{x}},\varepsilon)=\lim_{\Theta\to\infty}\frac{k(\Theta)}{\Theta} . \tag{3}$$

In fact the ε frequency measures how often the trajectory of the dynamical system visits the open ball $O(\bar{\mathbf{x}},\varepsilon)$. Note that when $\varepsilon\to 0$ the value of the ε frequency tends to the value of the natural measure of the attractor of the dynamical system at point $\bar{\mathbf{x}}$ [5,6].

Now we will measure the ε frequency, at some point $\bar{\mathbf{x}}$, and then try to increase it using small perturbations. We will test the algorithm on several chaotic dynamical systems: the three dimensional (3D) Lorenz system, the four dimensional hyperchaotic Rössler system [7], and the nonautonomous hyperchaotic system of two coupled Duffing oscillators [8] which are described, respectively, by the following equations:

$$\frac{dx}{dt}=\sigma(y-x)\ ,$$
$$\frac{dy}{dt}=rx-y-xz\ , \tag{4}$$
$$\frac{dz}{dt}=xy-bz\ ,$$

where the values of the parameters are $\sigma=16$, $b=4.0$, and $r=45.92$:

$$\frac{dx}{dt}=-(y+z)\ ,$$
$$\frac{dy}{dt}=x+ay+w\ , \tag{5}$$
$$\frac{dz}{dt}=b+xz\ ,$$
$$\frac{dw}{dt}=cw-ez\ ,$$

where $a=0.25$, $b=3.0$, $c=0.05$ and $e=0.5$:

$$\frac{d^2x_1}{dt^2}+a\frac{dx_1}{dt}+x_1^3=b\cos(ct)\ ,$$
$$\frac{d^2x_2}{dt^2}+e\frac{dx_2}{dt}+x_2^3=x_1\ , \tag{6}$$

where $a=0.1$, $b=10.0$, $c=1.0$, and $e=0.12$.

We can illustrate this by taking some examples. Different points of the chaotic attractors have different ε frequencies. The point $\bar{\mathbf{x}}_1=(-9.909\,750\,544\,297\,916,\ -15.079\,781\,802\,800\,559,\ 30.394\,713\,806\,945\,177)$ of the Lorenz system, with $\varepsilon=1.0$, has $\mu(\bar{\mathbf{x}}_1,1)=0.097$, and for the point $\bar{\mathbf{x}}_2=(9.425\,870\,275\,350\,913,\ 16.811\,983\,540\,820\,339,\ 22.095\,203\,328\,432\,007$, of the same system, with $\varepsilon=1.0$, the ε frequency is $\mu(\bar{\mathbf{x}}_2,1)=0.058$.

The value $\varepsilon=1.0$ is chosen to be of the order of 1% of the size of the attractor (the attractor of the Lorenz system is contained within a 3D box with approximate size $64\times87\times75$, and a diagonal approximately equal to 131).

Now we describe the algorithm which will increase μ. We replace system (1) with one given by Eq. (2), where $\mathbf{g}(\mathbf{x})$ is a real n-component function of $\mathbf{x}$, which takes small values, i.e., $|\mathbf{g}(\mathbf{x})|\leq\delta$, and δ is relatively small (less than 0.01% of the size of the attractor) fixed value.

The main question now is how to choose $\mathbf{g}$, in order to increase μ. We offer the following solution.

First we take a random point $\bar{\mathbf{x}}\in\mathcal{A}$, and we look for a point $\mathbf{y}\in O(\bar{\mathbf{x}},\varepsilon)\cap\mathcal{A}$, such that the trajectory which starts at $\mathbf{y}$ has a property that will return back in $O(\bar{\mathbf{x}},\varepsilon)$ for the shortest time. It is not possible to find such a point exactly in the general case, since we have to integrate the system for $t\in(0,\infty)$. However, in practice we will examine the trajectory of the system for a reasonably long time interval $t\in(0,\ 100\,000)$, looking for such a point. When we finally decide upon the choice of the point $\mathbf{y}$, we will determine a part of the trajectory $T(\mathbf{f},\mathbf{y})$, starting in $\mathbf{y}$ and returning back again in $O(\bar{\mathbf{x}},\varepsilon)$. We denote that part of the trajectory by $\hat{T}$ and the points on $\hat{T}$ by $\hat{\mathbf{y}}$.

The next step is to control the trajectory of the chaotic dynamical system, using the knowledge of $\hat{T}$. We adopt the following strategy. The system is left to evolve according to the unperturbed equations (1), and if the trajectory overlaps with $\hat{T}$ there is not need to intervene with the external perturbative term $\mathbf{g}(\mathbf{x})$. In the case when the trajectory differs from $\hat{T}$, a perturbation is applied by which the actual trajectory is pushed toward the nearest point on $\hat{T}$. This is made by a small amount $\mathbf{g}(\mathbf{x})$ with a modulus that does not exceed δ.

Figure 1 depicts schematically the idea of the algorithm. The algorithm can be precisely described as follows.

Preparatory phase

Step 1. Take a random point $\bar{\mathbf{x}}\in\mathcal{A}$, for which one wants to increase the number of visits to its ε neighborhood.

Step. 2. Calculate the ε frequency, and at the same time find the point $\mathbf{y}$.

Step. 3. Determine the trajectory $\hat{T}$.

Free uncontrolled phase

Step 4. Integrate the system without any perturbation.

Controlling phase

Step 5. Start with control at a random point $\mathbf{x}\in\mathbf{T}$.

Step 6. If the distance between $\mathbf{x}$ and $\hat{\mathbf{T}}$ is greater than δ (which implies than $\mathbf{x}\notin\hat{\mathbf{T}}$), then find a point $\hat{\mathbf{y}}\in\hat{\mathbf{T}}$ which is nearest to $\mathbf{x}$, and push $\mathbf{x}$ to a position in a direction toward $\hat{\mathbf{y}}$, with the restriction that the perturbation does not exceed δ.

If the distance between $\mathbf{x}$ and $\hat{T}$ is less than δ, select a perturbation $\mathbf{g}(\mathbf{x})$ which will bring the system onto $\hat{T}$.

Step. 7. Continue to integrate system (1) starting from its later position for a small time step $\Delta\tau$, reaching some point $\mathbf{x}$. Go to step 6.

A remark is necessary concerning step 6, where a

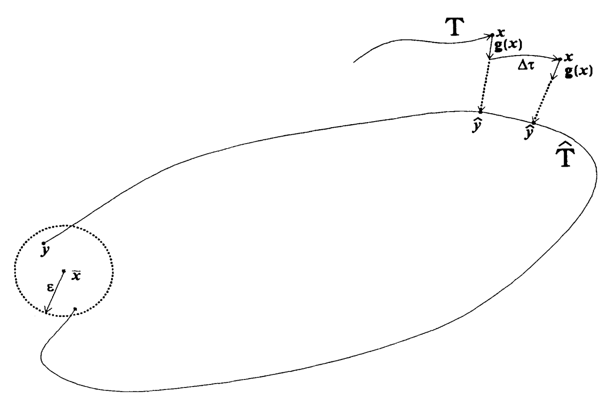

FIG. 1. Schematic presentation of the algorithm presented in this paper.

search of the closest point $\hat{\mathbf{y}} \in \hat{\mathbf{T}}$ for a given $\mathbf{x}$ is made. The number of calculated points of $\hat{\mathbf{T}}$ can be relatively large, and so the time consumption for searching the nearest point on $\hat{\mathbf{T}}$ is too large if in each step it is to be done by a linear search along $\hat{\mathbf{T}}$. Instead, in searching the nearest point on $\hat{\mathbf{T}}$ we proceed in the following manner. When we start the process of control, the initial $\hat{\mathbf{y}} \in \hat{\mathbf{T}}$ closest to $\mathbf{x}$ is found with a linear search through $\hat{\mathbf{T}}$. Then in subsequent searches the last closest point $\hat{\mathbf{y}}$ is used to look in both directions of $\hat{\mathbf{T}}$ for the point nearest to the later position of $\mathbf{x}$ (see Fig. 1).

Here we give some results obtained by the above algorithm. It is interesting to note that we found a threshold value $\hat{\delta}$ for the amount of the maximal perturbation δ, whereby a periodicity is reached, and the ε frequency μ no longer increases. In that case the maximal $\hat{\mu}$ for the threshold value is near the inverse of the time length of the shortest trajectory $\hat{T}$.

Figure 2(a) gives the calculated ε frequency as a function of the perturbation δ around the point $\bar{\mathbf{x}}_1$ of the Lorenz system. The threshold value, as we can see, is reached for $\hat{\delta}=0.0085$, which is 0.006% of the size of the attractor. The shortest trajectory takes 1.39 times units, so the maximal frequency is $\hat{\mu} \approx 1/1.39 \approx 0.719$. The integration of the Lorenz system was made by the Runge-Kutta method of order 4, with a time step $\Delta\tau=0.01$. The ε frequency for the same point calculated with a smaller time step $\Delta\tau=0.005$ is represented in Fig. 2(b). In this case the necessary perturbation required to make the trajectory periodic becomes respectively smaller, i.e., $\hat{\delta}=0.0054$.

Figure 3(a) shows the shortest trajectory running through $O(\bar{\mathbf{x}}_1, 1)$. The periodic trajectory resulting from the application of the algorithm, using perturbations with maximal amount of $\hat{\delta}=0.0085$, is shown in Fig. 3(b).

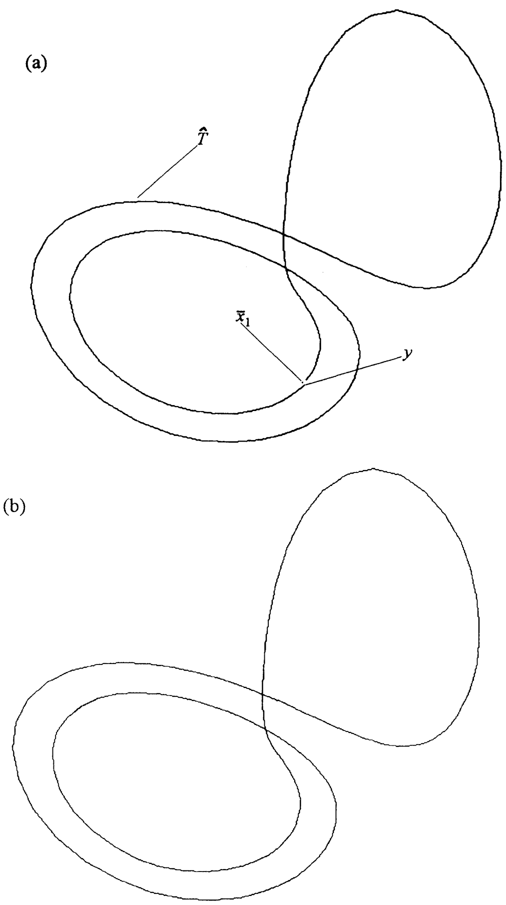

FIG. 3. (a) The points $\bar{\mathbf{x}}_1$ and $\mathbf{y}$, and the shortest trajectory $\hat{T}$. (b) Periodic trajectory obtained with $\hat{\delta}=0.0085$ for the point $\bar{\mathbf{x}}_1$.

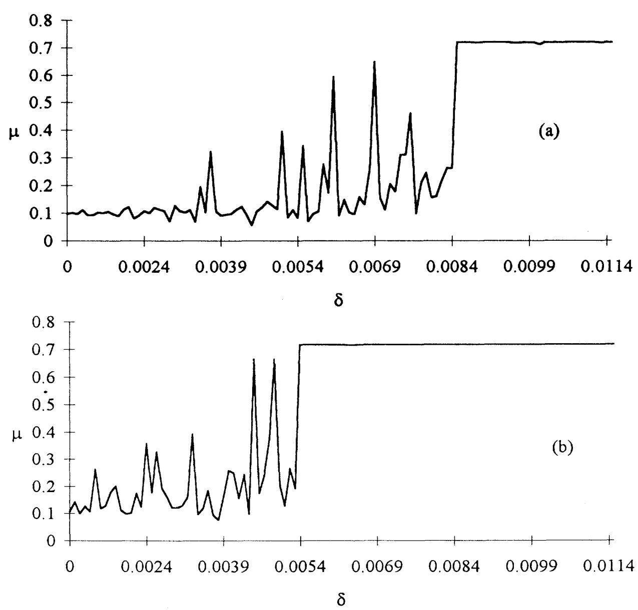

FIG. 2. (a) μ as a function of the perturbation δ for $\Delta\tau=0.01$ calculated in $\bar{\mathbf{x}}_1=(-9.909\,750\,544\,297\,916, -15.079\,781\,802\,800\,559, 30.394\,713\,806\,945\,177)$ of the Lorenz system. (b) μ as a function of the perturbation δ for $\Delta\tau=0.005$ calculated in $\bar{\mathbf{x}}_1$.

When the value of δ is less than the threshold value $\hat{\delta}$, the ε frequency μ is more frequently increased than not, although the system behaves similarly to the chaotic system. For example, for the value $\delta=0.0061$ we obtained that $\mu \approx 0.59$, which is more than six times greater than the value of μ when the system is not controlled. Figure 4 shows the resulting chaotic trajectory in the Lorenz system obtained for $\delta=0.0061$.

Figure 5 gives the calculated ε frequency as a function of the perturbation δ around the point $\bar{\mathbf{x}} = (-39.942\,503\,140\,840\,444, 12.569\,036\,805\,223\,906, 0.075\,739\,698\,850\,500, 19.504\,748\,651\,452\,275)$ of the Rössler system. Over that point we took $\varepsilon=1.0$, and the Runge-Kutta integration step $\Delta\tau=0.005$. When the system was not controlled, we found that $\mu(\bar{\mathbf{x}},1)=0.002\,05$. The shortest trajectory we found for this point was 20.39 time units long, and so the maximal $\hat{\mu}$ we expected and numerically confirmed was 0.049. The threshold value, as we can see, was reached for $\hat{\delta}=0.001\,07$.

In Fig. 6 we give the calculated ε frequency as a function of the perturbation δ around the point

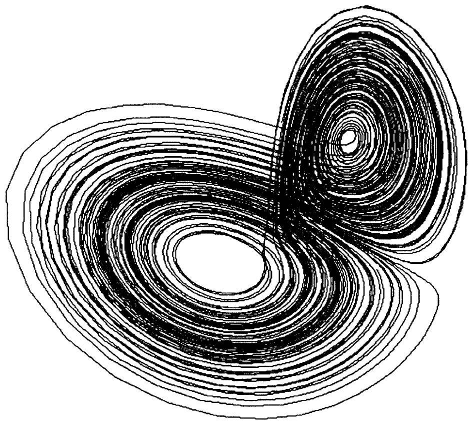

FIG. 4. Chaotic trajectory of the Lorenz system obtained with $\delta = 0.0061$, with increased ε frequency at the point $\bar{\mathbf{x}}_1$.

$\bar{\mathbf{x}} = (-1.835\,005\,385\,613\,494,\ 0.665\,126\,023\,934\,870,\ 0.452\,066\,898\,799\,946,\ 0.915\,365\,307\,670\,325)$ of the system of two coupled Duffing oscillators. Over that point we took $\varepsilon = 0.2$, and the Runge-Kutta integration step $\Delta\tau = 0.01$. When the system was not controlled, we found that $\mu(\bar{\mathbf{x}}, 0.2) = 0.000\,53$. The shortest trajectory we found for this point was 81.74 time units long, and so the maximal $\hat{\mu}$ we expected and experimentally confirmed was 0.0122. The threshold value, as we can see, is hard to determine more precisely, and lies between $\hat{\delta} = 0.000\,69$ and 0.000 71.

We have made more than 1000 numerical experiments with the Lorenz system, and over 300 numerical experiments with the other two dynamical system, randomly choosing various starting points from the attractor, and then applying the algorithm. We did not find a point for which the ε frequency cannot be increased. However there were many cases when the maximal $\hat{\mu}$ for the threshold value $\hat{\delta}$ was not near the inverse of the time length needed for the shortest trajectory $\hat{T}$. Also we found many cases when the resulting controlled trajectory gave an ε frequency which was sometimes much larger than expected. The following examples will illustrate these phenomena.

For point $\bar{\mathbf{x}}_2$ in the Lorenz system, as we mentioned for

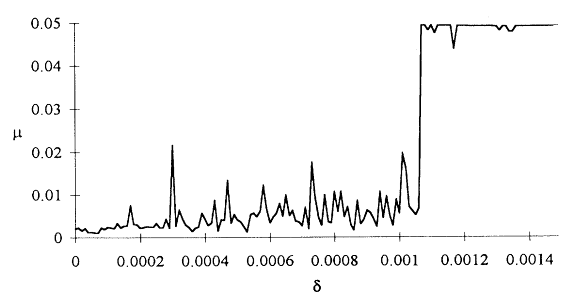

FIG. 5. μ as a function of the perturbation δ for $\Delta\tau = 0.005$ calculated at the point $\bar{\mathbf{x}} = (-39.942\,503\,140\,840\,444,\ 12.569\,036\,805\,223\,906,\ 0.075\,739\,698\,850\,500,\ 19.504\,748\,651\,452\,275)$ of the Rössler system.

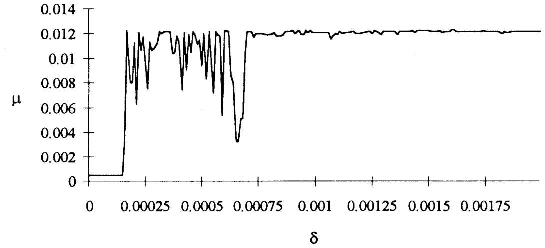

FIG. 6. The ε frequency as a function of the perturbation δ for the time step $\Delta\tau = 0.01$ calculated at the point $\bar{\mathbf{x}} = (-1.835\,005\,385\,613\,494,\ 0.665\,126\,023\,934\,870,\ 0.452\,066\,898\,799\,946,\ 0.915\,365\,307\,670\,325)$ of two coupled Duffing oscillators.

$\varepsilon = 1.0$, the ε frequency is $\mu = 0.058$. The threshold value for this point is reached for $\hat{\delta} = 0.0097$, which is 0.007% of the size of the attractor. The shortest trajectory takes 1.85 units of time, so the maximal frequency we could reach was $\hat{\mu} \approx 1/1.85 \approx 0.54$. However, this is not the case here. The maximal frequency we reached was 0.437, which is more than seven times greater than the value of μ when the system was not controlled, but still lower than 0.54. The reason behind this difference is that $\hat{T}$ and the imposed periodic trajectory differ considerably, and the motion along the latter takes more time.

More interesting situations were found in the hyperchaotic dynamic system of two coupled Duffing oscillators. The behavioral pattern for reaching the periodicity while increasing the maximal amount of δ was also found here, but we investigated what happened when applying greater δ in controlling (but still much smaller than 0.1% of the size of the hyperchaotic attractor). In Fig. 7(a) we give the calculated ε frequency

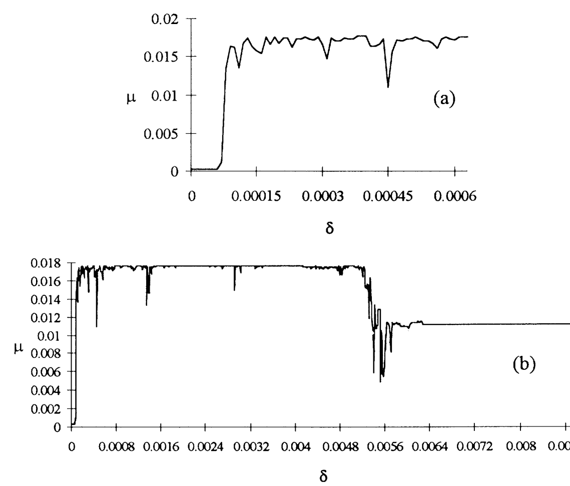

FIG. 7. (a) μ as a function of the perturbation δ for $\Delta\tau = 0.005$ calculated at the point $\bar{\mathbf{x}} = (-0.984\,624\,500\,154\,097,\ -1.190\,400\,032\,890\,617,\ 0.020\,945\,314\,102\,096,\ -3.697\,367\,069\,430\,180)$ of two coupled Duffing oscillators. (b) μ as a function of the perturbation δ for $\Delta\tau = 0.005$ calculated at the same point but for a larger interval of δ.

as a function of δ, for the point $\bar{\mathbf{x}}=(-0.984\,624\,500\,154\,097, -1.190\,400\,032\,890\,617, 0.020\,945\,314\,102\,096, -3.697\,367\,069\,430\,180)$, $\varepsilon = 0.2$, and $\Delta\tau=0.005$. The shortest trajectory we found was 56.49 time units long, so the expected ε frequency was 0.0177. Indeed, for some value of δ between 0.0001 and 0.0002 that ε frequency was reached. But when we applied control with larger δ, we obtained a very interesting graphic, shown in Fig. 7(b).

Figure 8 shows the calculated ε frequency as a function of the perturbation δ around the point $\bar{\mathbf{x}}=(3.296\,867\,550\,146\,056, 4.448\,041\,459\,252\,043, -1.129\,934\,661\,608\,241, 1.909\,228\,284\,447\,795)$ of the system of two coupled Duffing oscillators. Over that point we took $\varepsilon=0.2$, and the Runge-Kutta integration step $\Delta\tau=0.01$. When the system was not controlled, we found that $\mu(\bar{\mathbf{x}},0.2)=0.000\,15$. The shortest trajectory we found for this point was 113.21 time units long, and so the maximal $\hat{\mu}$ we expected and numerically confirmed was 0.0088. The threshold value, as we can see (the arrow shows the area where we found $\hat{\delta}$), is $\hat{\delta}=0.0077$. However, when we continued to increase δ, the ε frequency continued to grow, and for $\delta=0.0175$ μ increased to the value of 0.157, which is more than 1000 times greater than the ε frequency of the noncontrolled system.

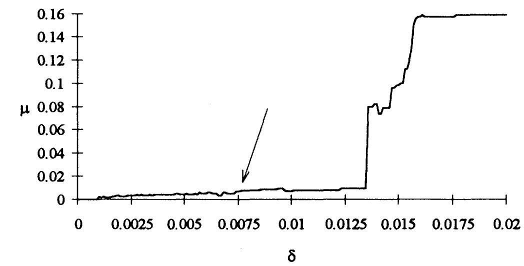

FIG. 8. μ as a function of the perturbation δ for $\Delta\tau=0.01$ calculated at the point $\bar{\mathbf{x}}=(3.296\,867\,550\,146\,056, 4.448\,041\,459\,252\,043, -1.129\,934\,661\,608\,241, 1.909\,228\,284\,447\,795)$ of two coupled Duffing oscillators.

In conclusion, we have implemented an algorithm for control of chaotic dynamical systems using small external perturbations. The numerical evidence demonstrates that by applying the algorithm the frequency of visits to the neighborhood of a randomly selected point on the strange attractor, starting from an arbitrary point on the attractor, is increased in most cases. When the perturbations are sufficiently strong, although still much smaller than the size of the attractor, the motion can be caused to become periodic. How large the perturbation necessary to make the system periodic should be depends on the time step $\Delta\tau$ of the integration process. For smaller values of $\Delta\tau$ the perturbations are more frequent and therefore can be more gentle. By providing several examples, we have explicitly shown that our algorithm is applicable to controlling the comportment of multidimensional chaotic systems.

Interested readers can contact the first author to obtain the program code written in C.

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