Signature of chaos in nonlinear drift-wave-induced transport

Th. Eickermann and K.H. Spatschek
Institut für Theoretische Physik, Heinrich-Heine-Universität Düsseldorf, D-40225 Düsseldorf, Germany
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For a drift-type instability in low-temperature magnetized plasmas, the transition from collisional to convective transport is investigated. First, regular coherent modes dominate, but with increasing control parameter, spatially coherent but time-chaotic states appear. The signature of nonlinear dynamics with subsequent chaos is a crossover in the magnetic-field dependencies of the particle diffusion coefficients.

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Anomalous transport is an outstanding problem of statistical physics with important consequences for many areas of physics. In plasma physics, many attempts, starting from quasilinear and weak-turbulence theories, have been undertaken to model this important phenomenon. Although some features could be explained through effective collision frequencies in a more or less *ad hoc* manner, others are still not understood at all. The reason may be that anomalous transport does not have one single reason—there are many aspects of this challenging phenomenon [1–6]. To contribute to the theory of the multiple-faced anomalous transport with applications to plasmas is the aim of the present paper.

The linear (classical and neoclassical) transport theory is very well developed [7], although due to geometrical complications quite sophisticated theories are needed. It is well known that some of the linear transport coefficients do not reveal the experimental results. The most prominent examples are the electron heat conductivity and the perpendicular (to an external magnetic field B) particle diffusion coefficient. Even in simple collisional plasmas, many transport results are anomalous, and on the basis of the very detailed linear predictions we know that nonlinearities must be responsible for the anomalies. However, inclusion of nonlinearities is not straightforward. When, e.g., using the weak-turbulence description for an unstable drift-wave situation, a dual cascade process appears. We get a condensation at small wave numbers k, and in general a more detailed description, taking into account higher nonlinearities, may become necessary. The appearance of large scale spatially coherent structures hints at another process which might be dominant in the nonlinear state: convective transport caused by nonlinear structures. And, indeed, during recent years it became evident that new entities like vortices, cavitons, and solitons can take part in the nonlinear dynamics with severe consequences for transport beyond the linear limit.

Convective cells can appear in the transcritical region of drift-wave instabilities due to bifurcations in space, and it is expected that the (e.g., magnetic-field) parameter dependence of the transport coefficients changes during these bifurcations. Such a scenario seems to be consistent with, e.g., the measured L-H transitions in tokamaks. But from the basic physics point of view at least two problems remain open. (i) How can one explain the observed frequency spectra? Definitely, the expected spatially coherent structures should not correspond to a quiescent plasma state. (ii) Wave-number spectra show a broad range of variations which, at first glance, seems to be inconsistent with the appearance of a spatially coherent convective state. While the second problem may find its explanation in the dual cascade behavior in driftlike situations, the first question is closely related to low-dimensional chaos. The implications for transport have not been considered in detail so far, and it is the purpose of this paper to investigate them by examining a simple model. The main idea is, and it will be proved in this paper, that spatially coherent convective states (which correspond to the small-k region of the dual cascade wave-number spectrum) can become chaotic in time without losing their spatial coherence. Another finding of this paper is that such a behavior exhibits a pronounced signature in the nonlinear drift-wave-induced transport. The paper presents a clear example of a relation for plasmas that has been hypothesized about in various forms for a long time, but rarely demonstrated concretely — a link between plasma diffusion coefficients and a transition to, or between, different regimes of chaotic behavior.

We start from a reduced model for nonlinear drift waves in weakly ionized plasmas. It originates, in a standard derivation, from the transport equations for densities, momenta, and temperatures under the additional assumptions of (i) quasineutrality, (ii) drift approximation in the plane perpendicular to the external magnetic field, and (iii) isothermal electrons and cold ions:

$$\partial_t p - \kappa\partial_y\varphi + \delta\nabla_\perp^2(\varphi - p) + \delta^{-1}\nabla_\parallel^2(\varphi - p) + \eta\partial_y(\varphi - p) + \{\varphi, p\} = 0\ , \quad (1)$$

$$(1+\partial_t)\,\nabla_\perp^2\varphi + \delta\nabla_\perp^2\,(\varphi - p) - \nabla_\parallel u + \delta^{-1}\nabla_\parallel^2\,(\varphi - p) - \eta\partial_y p + \{\varphi, \nabla_\perp^2\varphi\} = 0\ , \quad (2)$$

$$(1+\partial_t)\,u + \{\varphi, u\} = -\nabla_\parallel\varphi\ . \quad (3)$$

Here, $p := \delta n_-/n_0$, where δn_- is the electron density fluctuation and n_0 is the background electron density. The electrostatic potential φ is normalized by k_BT_-/e, time by τ_+, the mean ion collision time with neutral species, and the perpendicular lengths by $\rho_s = v_{\text{th}-}/|\Omega_i|$. The density gradient is in the x direction (x, y are perpendicular to the local $\vec{B}$ direction), $\kappa = d\ln n_0/dx$, $\delta = m_-\tau_+/m_+\tau_-$ (ratio of ion to electron mobility), and $\eta = 2/R$ (curvature of the external magnetic field B). The parallel (to $\vec{B}$) ion velocity is normalized by the ion sound speed ($u = u_{+\|}/c_s$), and the parallel length by $c_s\tau_+$. Note that $\{\ ,\ \}$ is the Poisson bracket, and B enters the equations only via the boundary conditions since ρ_s has been used as the unit length.

A further simple reduction, being used here, leads to a classical model [8]. It consists of two equations for the normalized density fluctuations (p) and the normalized electrostatic field (φ) and follows from Eqs. (1)–(3) in the limit (a) $\delta\nabla_\perp^2(\varphi-p) \gg \delta^{-1}\nabla_\|^2(\varphi-p)$ and (b) $u\equiv 0$. In the following we call that simplification of Eqs. (1)–(3) the Simon model, since Simon and co-workers [8] have used such equations extensively:

$$\partial_t p - \kappa\partial_y\varphi + \delta\nabla_\perp^2(\varphi-p) + \eta\partial_y(\varphi-p) + \{\varphi,p\} = 0\ , \tag{4}$$

$$(1+\partial_t)\nabla_\perp^2\varphi + \delta\nabla_\perp^2(\varphi-p) - \eta\partial_y p + \{\varphi,\nabla_\perp^2\varphi\} = 0\ . \tag{5}$$

Equations (4) and (5) are valid for weakly ionized plasmas. Equations similar to (4) and (5) occur in ionospheric turbulence; for the latter case Huba *et al.* [9] investigated the transition to chaos by performing a three-mode truncation in analogy to the Lorenz model for the Rayleigh-Bénard instability. Furthermore, it should be noted that models with similar mathematical structures were derived for fully ionized plasmas by several authors, e.g., by Hasegawa *et al.* [10], Hamaguchi [11], Carreras *et al.* [12], and Diamond *et al.* [13].

The model (4) and (5) contains the instability mechanism (dissipative drift instability of an inhomogeneous plasma due to collisions with neutral species [14]) self-consistently. The onset of instability depends on the magnetic field as a control parameter, and the model is therefore appropriate to follow the unstable modes from their onset up to saturation. The main dissipation mechanisms, in a weakly ionized plasma, are collisions with neutral species. The distribution of the latter is considered as fixed. From the usual quasilinear ansatz for the electron current Γ_-,

$$\Gamma_- \approx \langle\delta n_- v_E\rangle \approx \sum_k ik_y\varphi_k^*\delta n_{-k}\frac{n_0k_BT_-}{eB\rho_s} + \text{c.c.}\ , \tag{6}$$

it is obvious that a phase shift between density and potential fluctuations is necessary for a net particle flux. The average is over the y direction, and v_E is the $\vec{E}\times\vec{B}$ velocity in the fluctuating electrostatic field. If the phase shift is known, e.g., from gyrokinetic or collisional models, it remains to determine the spectrum $|\varphi_k|^2$ which is a very difficult task. Even when one succeeds in calculating the spectrum from a wave-kinetic equation, the whole procedure suffers from the various approximations inherent in weak-turbulence theories.

The procedure used here is to determine the first bifurcations in the transcritical region either numerically or analytically. Then no further basic assumptions need to be made to calculate the fluxes

$$\Gamma_+ = -\langle p\partial_y\varphi + \partial_y\varphi\partial_{xx}\varphi + \partial_x\varphi\rangle\ , \tag{7}$$

$$\Gamma_- = -\langle p\partial_y\varphi - \delta\partial_x(\varphi-p)\rangle\ , \tag{8}$$

besides the natural assumptions that in the ion flux (7) we take into account the $\vec{E}\times\vec{B}$ velocity and the nonlinear ion polarization drift, in addition to the usual collisional transport, whereas for the electrons in Eq. (8) only the $\vec{E}\times\vec{B}$ velocity dominates the nonlinear transport. The collisional contributions [the last terms in (7) and (8), respectively] lead in the usual way to the ambipolar diffusion coefficient (in nondimensional units)

$$\tilde{D}_a = (\delta/1+\delta) \sim B^{-2}T_-^{3/2}\ . \tag{9}$$

We use the latter as the reference value for the anomalous transport coefficients.

Close to the onset of instability ($B > B_c$, where B_c is the critical value of B) the nonlinear behavior of Eqs. (1)–(3) can be investigated by perturbation theory, leading to a complex Ginzburg-Landau equation [15,16] for the amplitude A of the most unstable mode as the order parameter,

$$\partial_T A - \alpha(B-B_c)A - \gamma_y\partial_Y^2 A - \gamma_z\partial_Z^2 A + \beta|A|^2A = 0. \tag{10}$$

Here T, Y, and Z are scaled time and space coordinates, respectively; $\alpha, \gamma_y, \gamma_z$, and β are coefficients which have been calculated explicitly. For not too small δ values (which typically occur in weakly ionized plasmas), the coefficients are such that the solutions of Eq. (10) correspond to convection rolls with no structure in the z direction and stable against modulations in the z direction. The latter behavior supports the previous reduction to the simplified model (4) and (5). In Eq. (10), the factor $B-B_c$ indicates that the mode under consideration is marginal at $B = B_c$. As we shall show below, the analytical prediction of stable saturated convection cells agrees very well with the direct simulation of the coupled partial differential equations (4) and (5). Typical simulation results are shown in Fig. 1 for two values of B, (a) $B = 1.5B_c$ and (b) $B = 2.4B_c$, i.e., already well above the critical value of the control parameter.

To interpret these drawings we have to mention first that we use Dirichlet boundary conditions in the x direction and periodic boundary conditions in the y direction. Furthermore, a snapshot at a certain time t would reveal $2d$ mode structures which show large convection cells "short-circuiting" the density gradient in the x direction. These convection cells are periodic in y, but have alternatively humps (solid lines) and dips (broken lines). From Fig. 1, which shows contour lines for φ and p, we can recognize that the $\varphi \approx p$ relation is broken (i.e., we have no Boltzmann distribution), and the phase shift is responsible for anomalous transport. The dynamics of the convection cells can be seen by the oscillations, which, for the control parameter chosen in the frequency spectrum of Fig. 2, are already quasiperiodic. We have to empha-

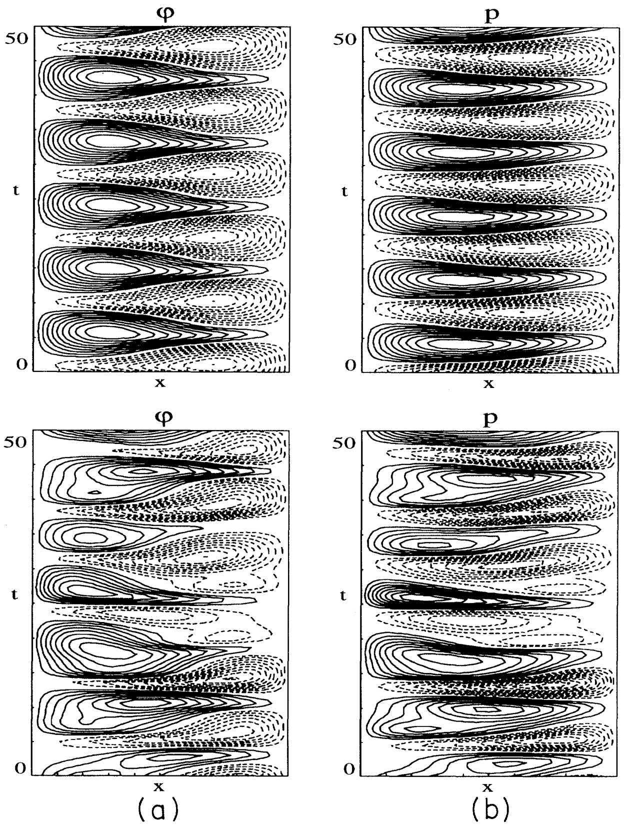

FIG. 1. Contour lines for potential φ (left) and density p (right) at fixed position y for different times t. (a) The upper examples are for $B = 1.5B_c$, whereas (b) the lower graphs are for $B = 2.4B_c$.

size that a very interesting scenario of nonlinear dynamics takes place which is accompanied by a crossover in the B dependence of the flux.

With increasing B, the convective rolls start oscillating with an additional frequency, and quite detailed di-

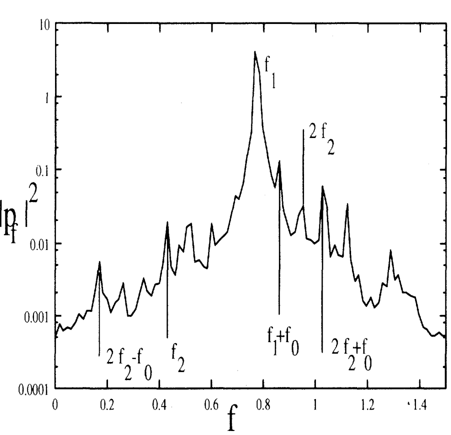

FIG. 2. Power spectrum $|P_f|^2$ of the (Fourier transformed) density fluctuation p versus frequency $f = \omega/2\pi$ in a quasiperiodic state with three frequencies f_0, f_1, and f_2 for $B = 2.4B_c$.

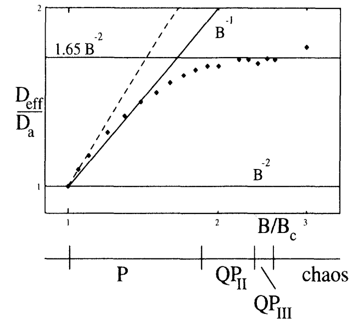

FIG. 3. Effective diffusion coefficient (normalized by the ambipolar diffusion coefficient D_a) as a function of the magnetic-field strength B. Depending on B, the system is single periodic (P), quasiperiodic with two frequencies ($QP_{\rm II}$), quasiperiodic with three frequencies ($QP_{\rm III}$), and then chaotic, respectively. The dots mark results from numerical experiments and the dashed line is the analytical prediction derived from the Ginzburg-Landau theory. The solid lines are inserted to characterize the various B dependencies.

agnostic measurements show that a transition to chaos in time occurs similar to the Ruelle-Takens-Newhouse quasiperiodic route [17]. A typical frequency spectrum for quasiperiodic motion is shown in Fig. 2. Note that it corresponds to the space-time plot of Fig. 1(a). Most interesting now is the signature of this behavior in the particle transport. In Fig. 3 we have plotted the effective diffusion coefficient as a function of B. The effective diffusion coefficient is defined via the relation $\Gamma_- \approx \Gamma_+ \approx -D_{\rm eff}\nabla n$.

The dots mark results from numerical experiments, and the dashed line is the analytical prediction derived from Ginzburg-Landau theory; see Eq. (10). In the slightly unstable region $B \geq B_c$ we obtain (in contrast to the ambipolar diffusion $D_a \sim B^{-2}$ for $B < B_c$) an anomalous dependence $D_{\rm eff} \sim B^{-1}$, where the effective diffusion coefficient $D_{\rm eff} \approx |\Gamma/\kappa|$ for $\Gamma_- \approx \Gamma_+$. Thus, at $B \geq B_c$ a crossover from B^{-2} dependence to B^{-1} dependence occurs. This result is understandable because of the onset of convective motion in addition to the still present collisional transport. The signature of chaos at larger B values is a change back to $D_{\rm eff} \sim B^{-2}$. Because of the chaotic jittering in time of the convective rolls, parts of the convective contributions are averaged out, and a collision-dominated regime appears again. However, the jittering convective motion still forms a plateau on which the classical part is superimposed.

Next we should mention that the behavior reported here can be modeled [13,16] by some low-dimensional system of ordinary differential equations (ODE's). The reason for this, at first glance, surprising fact is that the chaotic attractor of the partial differential equations (4) and (5) has a low dimension (correlation dimension $D_2 \approx 5.5 \pm 0.5$ for $B = 2.6B_c$), and thus a reduced ODE model works [18]. Similarly to hydrodynamic situations, truncated Fourier modes are not appropriate, whereas a Karhunen-Loève approximation leads to an extremely good conformity with the numerical results. More details will be published elsewhere.

Let us now comment on another interesting aspect.

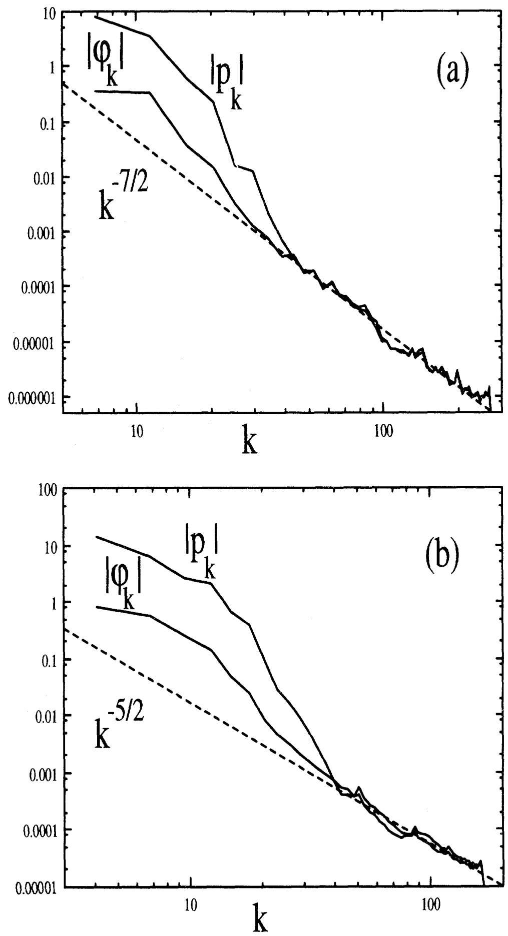

FIG. 4. Wave-number spectra of potential φ and density fluctuation p for (a) $B = 1.5B_c$ (time-periodic case), and (b) $B = 2.6B_c$ (time-chaotic state). Note that k is the averaged transverse wave number.

We have used a two-field model [the Simon model (4) and (5) for the two scalar fields p and φ]. Very often, such models are further reduced to one-field models for the simple reasons of (i) easier handling and, more important, (ii) universal behavior. However, the second aspect is usually valid only in a very small parameter regime. Leaving the latter, the more complicated multiple-field models are obligatory. But from the one-field description we can already learn a lot which, although not quantitatively, qualitatively can be utilized in the algebraically more complicated models. In our case the dual cascade process (energy towards small k and entropy towards large k) shown for a Hasegawa-Mima-type equation [10] is helpful. Although convective rolls are generated, part of the spectral transfer is also to large k, and thus when monitoring the wave-number spectra we find characteristic occupations there. In Fig. 4 we show two typical results for (a) $B = 1.5B_c$ and (b) $B = 2.6B_c$.

The first case corresponds to a periodic oscillation whereas the second one is the result for a time-chaotic situation. Very typical are the algebraic dependences close to the dissipative region. This fact can be understood from Pao's theory [19] and will be discussed at another location. More important in this context is that again a signature of chaos appears, here in form of different algebraic k dependences for large k in the wave-number spectra.

In summary, in this paper we have chosen a simple but classical model for a drift-type instability in weakly ionized plasmas in order to study the problem of transport at the onset of turbulence. Spatially coherent structures do play a significant role for convective transport. At the same time, chaos in time may set in; the significance of the latter can be seen in the frequency and wave-number spectra. However, and maybe most important, the signature of chaos also appears in the magnetic-field dependence of the particle transport. More detailed diagnostics and analytical models are in progress. Finally, when using similar models for fully ionized plasmas the qualitative conclusions are similar, showing that the phenomena investigated here are universal.

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