Resonances of nonlinear oscillators

Christoph Wargitsch* and Alfred Hübler
Center for Complex Systems Research, The Beckman Institute and the Department of Physics, University of Illinois at Urbana-Champaign, 405 North Mathews Avenue, Urbana, Illinois 61801
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We show that nonlinear oscillators have a large response to special aperiodic driving forces. If these forces are selected to minimize the driving effort for a given terminal energy, these forces are given by the time-reflected transient of the unperturbed dynamics (the "principle of the dynamical key"). We provide a proof of this principle. We find that these optimal forcing functions have very similar dynamics for several different norms. We present a quantitative comparison of the energy transfer for sinusoidal and optimal driving forces. We find that aperiodic driving forces are most effective for large nonlinearity and small friction. We show that this optimal control is stable for several important systems.

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I. INTRODUCTION

Galileo Galilei was probably the first to discuss resonances in "Discorsi a Dimostrazioni Matemache" (1638) [1]. He used the term resonance ("risonanza") to describe the response of music instruments to sound waves (lat. "resonare" = to echo). In the 18th century Huygens [2] discovered that two mechanical clocks with a slightly different speed tend to synchronize if they are mounted on the same wall. In the synchronized state the energy exchange is extremal [3] and the oscillators are said to be at resonance [4–7]. Later it has been shown that for a large class of weakly coupled self-sustained oscillators, the period of the coupled system matches closely with the period of the fastest oscillator [8]. This phenomena seems to be the key paradigm for control in neural regulatory systems, such as the coupling of the sine node and the atrio-ventricular node in the human heart [9].

Resonance is also a foundational paradigm in physical sciences and engineering. In quantum physics, Bohr [10] introduced the term resonance frequency for the radiation frequency of a decaying energy state. Breit and Wigner [11] used the concept of resonance to describe a sharp peak in cross sections for certain scattering events of particles in nuclear physics. In engineering and applied physics, most textbooks and review papers [12–15] define resonance by the largest response of a driven damped linear oscillator. At resonance, the driving frequency of an external sinusoidal force matches the natural frequency of the driven oscillator. This causes a large energy transfer to the oscillator. Technical applications of this concept are mostly in the area of electric engineering, e.g., *RLC* circuits in which the power consumption [15] of the circuit reaches its maximum at resonance. Spectroscopic instruments are based on resonant driving forces since this yields a large signal to noise ratio for harmonic and weakly nonlinear oscillators.

For highly nonlinear oscillators, the response to sinusoidal driving forces is typically small and complicated [3,16]. It can be determined with the Lindstedt-Poincaré method, Guckenheimer and Holmes's averaging methodologies [6], Lichtenberg and Lieberman's secular perturbation approach [4], or Nayfeh and Mooks's multiple scales methodologies [5]. The resulting signal-to-noise ratio for spectroscopic instruments is small for sinusoidal driving forces. Recently, it has been conjectured that a large energy transfer to nonlinear oscillators can be achieved by a special class of aperiodic driving forces [17]. This yields a high signal-to-noise ratio, which can be used for system identification with general resonance spectroscopy [18]. Other numerical studies of optimal controls of nonlinear quantum systems confirm this observation [19].

In the present paper, we show analytically that optimal driving forces have the same dynamics as the time-reflected transient dynamics of the unperturbed system. This paradigm was conjectured earlier by Hübler [17] and is called the "principle of the dynamical key" [16]. In this paper the smallest driving force, which makes it possible to achieve a certain energy transfer, is called resonant. In general, we use the $\mathcal{L}_2$ norm [20] to measure the size of the forcing function. In addition, we study the resonance conditions such as minimum reaction power and we compare the resulting forcing functions. In Sec. II, we calculate optimal driving forces for nonlinear oscillators. We introduce different types of resonances and show how to prove the principle of the dynamical key. In Sec. III, we compare the effectiveness of driving between sinusoidal and optimal driving for a variety of nonlinear oscillators. An analytical estimation and a numerical calculation of the energy gain, which a linear oscillator can achieve if it is optimally driven, is given in Sec. IV. In Sec. V, we show that the control with optimal forcing functions is stable for a big class of oscillators. In Sec.

*Permanent addresses: Physik Department, Technische Universität München, Theoretische Physik/8, James-Franck-Str.1, D-85356 Garching, Germany.

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VI we take a look at optimal driving forces, which satisfy different norms than the quadratic norm.

II. OPTIMAL DRIVING FORCES OF OSCILLATORS, PRINCIPLE OF THE DYNAMICAL KEY

We consider a one-dimensional damped oscillator which is driven by the time-dependent driving force $F(t)$,

$$\ddot{x} + \eta\dot{x} + \frac{\partial V}{\partial x} = F(t) \ , \tag{2.1}$$

where η is the friction coefficient and $V(x)$ is a time-independent potential. We assume that the potential V has a minimum at $x = x_{\rm min}$ and $V(x_{\rm min}) = 0$. The energy of the oscillator is $E(t) = \frac{1}{2}\dot{x}(t)^2 + V(x(t))$. In the following, we introduce three different types of optimal driving.

A. Strong resonance type A

First, we consider the situation where the terminal energy $E(T) = E$ and the initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ are fixed but T is arbitrary. We search for a forcing function which minimizes the "driving effort" $\bar{F}^2 = \int_0^T F^2 dt$. This problem can be solved by a variation of the functional $S = \int_0^T L_g dt$. The Lagrange function L_g is given by

$$L_g = L + \lambda K \delta_D(t-T), \tag{2.2}$$

where λ is a constant Lagrange multiplier and $\delta_D(t-T)$ is Dirac's δ function. L is given by

$$L(x,\dot{x},\ddot{x},F,t) = |F|^2 + \mu(t)\left(\ddot{x} + \eta\dot{x} + \frac{\partial V}{\partial x} - F\right), \tag{2.3}$$

where the Lagrange multiplier $\mu(t)$ is time dependent since the equation of motion [Eq. (2.1)] is a nonintegral constraint. K ensures that the terminal energy hits the required value E:

$$K(x(T),\dot{x}(T)) = \frac{1}{2}\dot{x}(T)^2 + V(x(T)) - E = 0, \tag{2.4}$$

where E is constant. The Lagrange problem then can be taken to be

$$\delta S = \delta \int_0^T |F|^2 + \mu(t)\left(\ddot{x} + \eta\dot{x} + \frac{\partial V}{\partial x} - F(t)\right) + \lambda\left(\frac{1}{2}\dot{x}^2 + V(x) - E\right)\,\delta_D(t-T)dt = 0, \tag{2.5}$$

where δS is the variation of S. The first term of the integral represents the minimization of the effort $\bar{F}^2$, the second term contains the equation of motion (2.1), and the third term is due to the boundary condition for the terminal energy $E(T)$. In the Appendix the corresponding Euler-Lagrange equations, Eqs. (A32)–(A35), for the variational scheme (2.5) with free terminal time T are calculated. This leads to the differential equation for the forcing function $F(t)$:

$$\ddot{F}(t) - \eta\dot{F}(t) + \frac{\partial^2 V}{\partial x^2}F(t) = 0, \tag{2.6}$$

and a set of boundary conditions,

$$F(T) = 2\eta\dot{x}(T) \ , \tag{2.7}$$

$$\dot{F}(T) = 2\eta\left(\eta\dot{x}(T) - \left.\frac{\partial V}{\partial x}\right|_T\right). \tag{2.8}$$

For Eq. (2.6), we use the trial solution

$$F(t) = \alpha\eta\dot{x}(t). \tag{2.9}$$

This trial solution satisfies the boundary conditions [Eqs. (2.7), (2.8)] for $\alpha = 2$. Another solution for Eqs. (A32)–(A35) is given by $\alpha = 0$ if $E(T) < E(0)$. Therefore, the forcing function to solve the Lagrange problem [Eq. (2.5)] is given by

$$F(t) = \begin{cases} 2\eta\dot{x}(t) & \text{for } E(T) > E(0) \\ 0 & \text{otherwise .} \end{cases} \tag{2.10}$$

If the desired terminal energy is smaller than the initial energy, the optimal driving force is zero. This means that the oscillator is cooling itself down through dissipation.

In the following, we assume that the terminal energy $E(T)$ is larger than the initial energy $E(0)$, i.e., $F(t) = 2\eta\dot{x}(t)$. If one substitutes this solution for the Lagrange problem in Eq. (2.1), the differential equation for the driven oscillator,

$$\ddot{x} - \eta\dot{x} + \frac{\partial V}{\partial x} = 0 \ , \tag{2.11}$$

is obtained. We introduce the auxiliary variable $y(t)$, which is defined by the time-reflected dynamics of the unperturbed system, i.e., $y(t) := x_u(T-t)$. $x_u(t)$ is a transient solution of the unperturbed dynamics

$$\ddot{x}_u(t) + \eta\dot{x}_u(t) + \frac{\partial V}{\partial x_u(t)} = 0, \tag{2.12}$$

for the boundary conditions $x_u(T) = x_0$, $\dot{x}_u(T) = \dot{x}_0$, $\frac{1}{2}\dot{x}_u(0)^2 + V(x_u(0)) = E(T)$. Therefore, the time derivatives of $y(t)$ are $\dot{y} = -\frac{dx_u}{dt}$, $\ddot{y} = \frac{d^2x_u}{dt^2}$. If one substitutes $x_u(t)$ in Eq. (2.12) through $y(t)$, the equation then reads

$$\ddot{y}(t) - \dot{y}(t) + \frac{\partial V}{\partial y(t)} = 0. \tag{2.13}$$

Equation (2.13) has the same structural form as Eq. (2.11). Therefore, a solution of the Lagrange problem [Eq. (2.5)] is given by

$$F(t) = 2\eta\dot{y}(t), \tag{2.14}$$

where $\dot{y}(0) = \dot{x}_0$ and $y(0) = x_0$.

In conclusion, we find that the optimal forcing function is proportional to the velocity of the time-reflected transient of the unperturbed dynamics [Eq. (2.14)]. This is the "principle of the dynamical key," i.e., the system reacts most sensitively to a special forcing function which is closely related to the dynamics of the unperturbed system. The transient of the nonlinear damped oscillator [Eq. (2.12)] is, in general, aperiodic. Therefore, optimal forcing functions are, in general, aperiodic too. The response of an optimally driven oscillator is given by Eq. (2.11), which is equal to the time-reflected dynamics of the unperturbed system [Eq. (2.13)].

With the optimal driving half of the energy, which is supplied when the driving force is dissipated, i.e., $\frac{E_D}{E_i} = 0.5$, the input energy supplied by the optimal driving force is given by $E_i = \int_0^T F(t)\dot{x}(t)dt = \int_0^T 2\eta\dot{x}^2(t)dt$ and the dissipated energy is given by $E_D = \int_0^T \eta\dot{x}^2(t)dt$. Therefore, the terminal energy $E(T)$ is 50% of the energy input E_i for minimum effort forcing.

B. Medium resonance

For medium resonance, we fix the dissipated energy E_D, in addition to the terminal energy $E(T)$ and the initial conditions $x(0) = x_0, \dot{x}(0) = \dot{x}_0$, and minimize the driving effort $\bar{F}^2$. The terminal time T is arbitrary. This situation can be described by the Lagrange problem,

$$\delta \int_0^T |F|^2 + \kappa\eta\dot{x}(t)^2 + \mu(t)\left(\ddot{x} + \eta\dot{x} + \frac{\partial V}{\partial x} - F(t)\right) + \lambda\left(\frac{1}{2}\dot{x}^2 + V(x) - E\right)\delta_D(t-T)dt = 0, \tag{2.15}$$

where κ is a constant Lagrange multiplier. The corresponding Euler-Lagrange equations [Eqs. (A24)–(A28), for detailed calculation see Appendix] read

$$\ddot{\mu} - \dot{\mu}\eta + \mu\frac{\partial^2 V}{\partial x^2} - 2\kappa\eta\ddot{x} = 0, \tag{2.16}$$

$$2F(t) - \mu(t) = 0, \tag{2.17}$$

for $0 < t < T$ with the boundary conditions,

$$\lambda\dot{x}(T) + \mu(T) = 0, \tag{2.18}$$

$$F(T)^2 - \kappa\eta\dot{x}(T)^2 - \mu(T)F(T) + \mu(T)\left.\frac{\partial V}{\partial x}\right|_T + \dot{x}(T)\dot{\mu}(T) = 0, \tag{2.19}$$

$$\lambda\left.\frac{\partial V}{\partial x}\right|_T + 2\kappa\eta\dot{x}(T) + \mu(T)\eta - \dot{\mu}(T) = 0. \tag{2.20}$$

A solution for these necessary conditions is given by

$$F(t) = \alpha\eta\dot{x}(t), \tag{2.21}$$

$$\mu(t) = 2\alpha\eta\dot{x}(t), \tag{2.22}$$

$$\lambda = -2\alpha\eta, \tag{2.23}$$

$$\kappa = \eta\alpha(\alpha - 1), \tag{2.24}$$

where α is a real constant. The constant α is fixed through the constraint for the energy dissipation E_D:

$$E_D = \int_0^T \eta\dot{x}(t)^2 dt = \frac{1}{\alpha - 1}\int_0^T \dot{x}\left(\ddot{x} + \frac{\partial V}{\partial x}\right)dt \tag{2.25}$$

$$= \frac{1}{\alpha - 1}[E(T) - E(0)]. \tag{2.26}$$

This yields to

$$\alpha = \frac{E(T) - E(0)}{E_D} + 1. \tag{2.27}$$

The driving effort $\bar{F}^2$ for a given energy dissipation then reads

$$\bar{F}^2 = \int_0^T \alpha^2\eta^2\dot{x}(t)^2 dt = \frac{\alpha^2\eta}{\alpha - 1}\int_0^T \dot{x}\left(\ddot{x} + \frac{\partial V}{\partial x}\right)dt = \frac{\alpha^2\eta}{\alpha - 1}[E(T) - E(0)]. \tag{2.28}$$

Strong resonance is a special case of medium resonance. For strong resonance, i.e., $\alpha = 2$ the effort $\bar{F}^2$ [Eq. (2.28)] has a minimum and the energy dissipation [Eq. (2.26)] is 50% of the energy input E_i. For a zero dissipation $E_D \longrightarrow 0$, the constant α becomes infinity, i.e., $\alpha \longrightarrow \infty$.

C. Weak resonance

For weak resonance, we minimize the reflected energy $E_R = \int_0^T F(t)\dot{x}\Theta(-F(t)\dot{x})dt$, where Θ is the Heavyside step function. This condition is equal to a minimal reaction power and a perfect impedance match. A solution of the Lagrange problem is given by

$$F(t) = \mathrm{sgn}(\dot{x}(t))\gamma(t), \tag{2.29}$$

where $\gamma(t)$ is an arbitrary function of time that has to be positive for all times, i.e., $\gamma(t) \geq 0$ for $0 \leq t \leq T$. The reflected energy E_R and the reaction power are equal to zero for all three types of resonance.

Medium resonance is a special case of weak resonance if $\gamma(t)$ has the form $\gamma(t) = \alpha\eta|\dot{x}(t)|$. Strong resonance is a special case of weak resonance for $\gamma(t) = 2\eta|\dot{x}(t)|$.

D. Strong resonance type B

We introduce a second type (B) of strong resonance and show that type A and type B are equivalent. For strong resonance type B the driving effort $\bar{F}^2$ and the initial conditions $x(0) = x_0, \dot{x}(0) = \dot{x}_0$ are fixed and the terminal energy $E(T)$ is to be maximized. The described situation sets up the Lagrange problem,

$$\delta S = \delta \int_0^T \bar{\lambda}|F|^2 + \bar{\mu}\left(\ddot{x} + \eta\dot{x} + \frac{\partial V}{\partial x} - F(t)\right) + \left(\frac{1}{2}\dot{x}^2 + V(x) - E\right)\delta_D(t-T)dt = 0, \quad (2.30)$$

where the driving effort $\bar{F}^2$ is constant and the final energy E_T is maximal. If one divides S through $\bar{\lambda}$ the Eqs. (2.5), (2.30) become equal and the optimal forcing is the same as for strong resonance type A, i.e., $F(t) = 2\eta\dot{x}(t)$ [Eq. (2.14)]. The Lagrange parameters now are given by $\bar{\mu} = \frac{\mu}{\lambda}, \bar{\lambda} = \frac{1}{\lambda}$.

III. NUMERICAL COMPARISONS FOR NONLINEAR OSCILLATORS

In this section, we study the effectiveness of sinusoidal forcing, i.e.,

$$F(t) = F_0 \cos(\Omega t - \Phi), \quad (3.1)$$

and optimal driving (strong resonance type B). Ω is the driving frequency and Φ is the initial phase. In order to determine the target energy for optimal forcing functions, we numerically integrate the Eqs. (2.1), (2.13), (2.14) for a fixed effort $\bar{F}^2$ and for the initial conditions $x(0) = 0$, $y(0) = 0$, $\dot{x} = 0$, and $\dot{y}(0) = 0.001$. In order to determine the maximum of the total energy $E_S(T_S)$ that a sinusoidal driven oscillator can achieve for fixed effort $\bar{F}^2$, we optimize numerically the parameters of the driving force Ω, T. The optimal terminal time is T_S and the optimal driving frequency is Ω_S. The ratio of the terminal energy for optimal driving $E(T)$ over $E_S(T_S)$ we call energy gain G,

$$G = \frac{E(T)}{E_S(T_S)}. \quad (3.2)$$

A. Varying order of nonlinearity

The goal of investigation in this subsection is to find a relation between strength of nonlinearity and the energy gain G. The nonlinearity of a potential is changed in two different ways. One approach is to change the weight coefficient W of the inharmonic term of the potential $V = \frac{x^2}{2} + W\frac{x^8}{8}$. Then the equation of motion (2.1) reads

$$\ddot{x} + \eta\dot{x} + x + Wx^7 = F(t), \quad (3.3)$$

where $F(t)$ is given either through the sinusoidal driving force [Eq. (3.1)] or through Eq. (2.14). Figure 1 shows the energy gain G for various weights W. The energy gain increases for large weight W. Figure 2 illustrates the gain G for different friction coefficients η for $W = 1.0$. For small friction coefficients, the energy of the optimal driven oscillator sharply increases. For strong damping ($\eta > 0.3$) the friction dominates over the nonlinearity. Thus the driven oscillator behaves like a linear oscillator and has a similar energy gain. For small friction

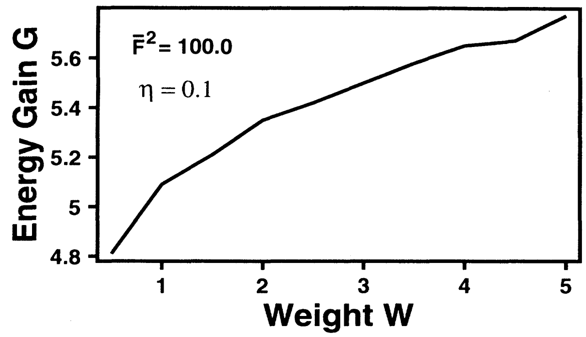

FIG. 1. Energy gain for increasing weight of the nonlinearity. The oscillator potential is $V = \frac{x^2}{2} + W\frac{x^8}{8}$. Here, the friction coefficient η is 0.1 and the driving effort $\bar{F}^2 = 100.0$.

coefficients ($\eta < 0.3$), the energy gain for the nonlinear oscillators is much larger than for the linear oscillator.

Another way to vary the nonlinearity is to change the exponent p of the nonlinear term in the oscillator potential $V = \frac{x^2}{2} + \frac{x^p}{p}$. The equation of motion (2.1),

$$\ddot{x} + \eta\dot{x} + x + x^{p-1} = F(t) \ , \quad (3.4)$$

is integrated for each p and the energies are compared. One finds again that the effectiveness of optimal driving becomes larger for strong nonlinearities (Fig. 3). For large driving efforts, the gain grows faster with increasing nonlinearity. This can be explained by the amplitude frequency shift of a nonlinear oscillator. A large driving effort $\bar{F}^2$ causes a large amplitude and, therefore, the frequencies shift rapidly. Since the frequency of sinusoidal driving is fixed, the driving force does not follow the natural frequency of the nonlinear oscillator and the gain for optimal driving is enhanced. Strong and weak nonlinear oscillators exhibit qualitatively the same relation between energy gain and the friction coefficient but strong nonlinear oscillators can achieve larger energy gains G. In summary, we find that the energy gain G is large for strong nonlinearities and weak friction.

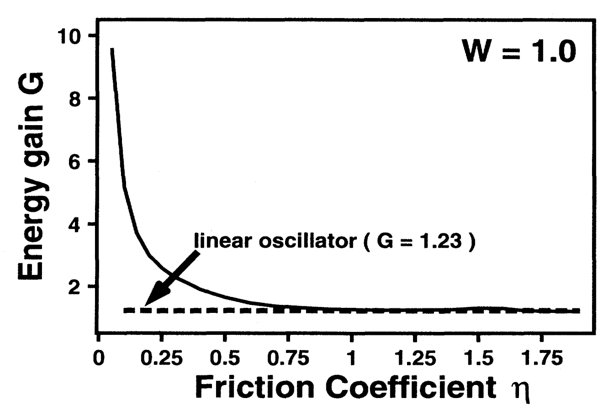

FIG. 2. Energy gain for nonlinear oscillator for weight $W = 1.0$ and driving effort $\bar{F}^2 = 100.0$.

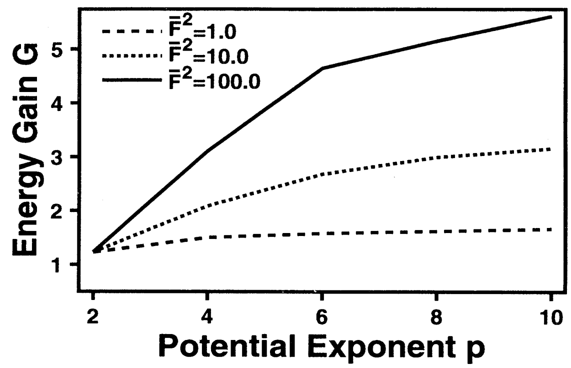

FIG. 3. Energy gain for different potential exponents of the nonlinear potential. The driving effort is $\bar{F}^2 = 100.0$

B. Duffing oscillator

As a further example for the optimal driving of a nonlinear oscillator, we consider the Duffing oscillator with a double-well potential $V = -\frac{x^2}{2} + \frac{x^4}{4} + \frac{1}{4}$. The energy gain for various friction coefficients for this oscillator is illustrated in Fig. 4. The initial values of the simulation are $\dot{x}(0) = \dot{y}(0) = 0$, $x(0) = 1.0$, $y(0) = 0.999$. The energy gain G is especially large if the terminal energy is close to the local maximum of the potential. This occurs for the given initial condition when the friction coefficient is $\eta \approx 0.25$.

IV. LINEAR OSCILLATOR

Finally, we consider the linear oscillator with the potential $V = \frac{x^2}{2}\omega^2$ and the equation of motion,

$$\ddot{x} + \eta\,\dot{x} + \omega^2 x = F(t). \tag{4.1}$$

A. Analytical estimation of the gain

In this section, we provide an analytical estimation for the gain G for the linear oscillator [Eq. (4.1)]. First we study the case where $F(t)$ is sinusoidal [Eq. (3.1)]. An exact solution of Eq. (4.1) is

$$x(t) = \frac{1}{B}\Bigg\{F_0[(\omega^2-\Omega^2)\cos(\Omega t-\Phi)+\eta\Omega\sin(\Omega t-\Phi)] + e^{-\frac{\eta t}{2}}\Bigg[F_0(\Omega^2-\omega^2+x_0B)\cos(\tilde{\omega}t) + -\eta F_0(\Omega^2+w^2)+x_0\eta B+2v_0B\Bigg]\frac{\sin(\tilde{\omega}t)}{2\tilde{\omega}}\Bigg\}, \tag{4.2}$$

where $\tilde{\omega} = \sqrt{\omega^2 - \frac{\eta^2}{4}}$ and $B = \eta^2\Omega^2 + (\Omega^2-\omega^2)^2$. In the following we assume the set of initial conditions $x(0) = x_0 = 0$, $\dot{x}(0) = v_0 = 0$, $\Phi = 0$, and $\omega = 1$. The effort to drive the linear oscillator is fixed and is approximately $\bar{F}^2 = \int_0^T F^2(t)dt \approx 0.5\,F_0^2\,T$, where $\bar{F}^2$ is an arbitrary constant. Therefore, the terminal time T can be substituted by

$$T = \frac{2\bar{F}^2}{F_0^2}. \tag{4.3}$$

We determine $\dot{x}(t)$ with Eq. (4.2) and calculate the terminal energy $E(T) = \frac{1}{2}\dot{x}(T)^2 + V(x(T))$ analytically. Then we write all products of trigonometric functions as sums of trigonometric functions with multiple angles [21] and make the following simplifications: $\sin(\tilde{\omega}t) \longrightarrow 0$, $\cos(\tilde{\omega}t) \longrightarrow 0$, $\sin[(\tilde{\omega}+\Omega)t-\Psi] \longrightarrow 0$, $\cos[(\tilde{\omega}+\Omega)t-\Psi] \longrightarrow 0$, $\sin[(\tilde{\omega}-\Omega)t+\Psi] \longrightarrow \sin(\Psi)$, $\cos[(\tilde{\omega}-\Omega)t+\Psi] \longrightarrow \cos(\Psi)$, $\sin[(\tilde{\omega}-\Omega)t-\Psi] \longrightarrow \sin(-\Psi)$, $\cos[(\tilde{\omega}-\Omega)t-\Psi] \longrightarrow \cos(-\Psi)$, $\cos[2(\Omega t-\Psi)] \longrightarrow 0$, where $\Psi = \arctan(\frac{\eta\Omega}{1-\Omega^2})$. This procedure corresponds to an averaging of one period of the forcing function. Then we neglect all sum terms with friction coefficients η with higher than linear order, i.e., $\eta^4 \longrightarrow 0$, $\pm(4-\eta^2) \longrightarrow \pm 4$ since the friction is assumed to be small, i.e., $\eta < 0.3$. A numerical study shows that we can assume $\Omega \approx 1$ for small friction. Then the terminal energy reads

$$E_S(T) = -\frac{F_0^2(-1+2e^{\frac{\eta T}{2}}-e^{\eta T})}{2\eta^2 e^{\eta T}}. \tag{4.4}$$

Then we eliminate T with Eq. (4.3). To compute the maximal terminal energy $E_S(T_S)$ as a function of F_0, we expand $\frac{\partial E_S(T)}{\partial F_0}$ in a Taylor series up to second order and set it to zero,

$$\frac{\partial E_S(T)}{\partial F_0} \approx \bar{F}^2\eta - \frac{(\bar{F}^2)^2\eta^2}{2F_0^2} - \frac{(\bar{F}^2)^3\eta^3}{6F_0^4} = 0. \tag{4.5}$$

From Eq. (4.5), we obtain an approximative value for the optimal F_0,

$$F_0 = \frac{1}{2}\sqrt{\left(1+\sqrt{\frac{11}{3}}\right)\bar{F}^2\eta} \approx 0.85\sqrt{\eta\bar{F}^2}. \tag{4.6}$$

With this value for F_0, T_S then reads

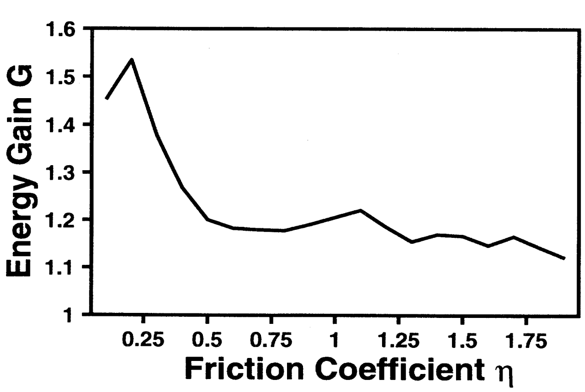

FIG. 4. Energy gain for the Duffing oscillator with double-well potential.

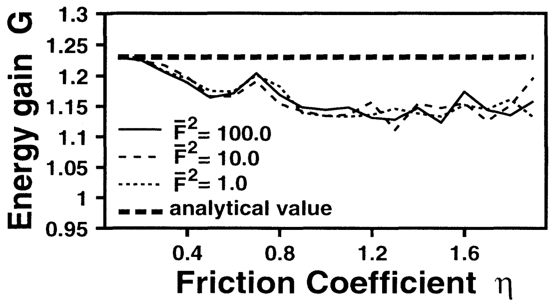

FIG. 5. The energy gain G for the linear oscillator is close to the analytical estimation for small friction coefficients η.

$$T_S \approx \frac{\bar{F}^2}{0.425\sqrt{\eta \bar{F}^2}}. \tag{4.7}$$

For Eq. (4.7), the maximal terminal energy for sinusoidal driving is approximately $E_S(T_S) \approx 0.203\bar{F}^2/\eta$. In the case of strong resonance, $E(T)-E(0) = E_D$ in Eq. (2.27), then Eq. (2.28) gives $\bar{F}^2 = 4\eta[E(T) - E(0)]$. The maximal terminal energy of the optimally driven oscillator is equal to $E(T) \approx \frac{\bar{F}^2}{4\eta}$, if we assume that the initial energy $E(0)$ is very small and can be neglected. The gain G for linear oscillators is then

$$G = \frac{E(T)}{E_S(T)} \approx 1.23 + O(\eta^2), \tag{4.8}$$

for small η, i.e., $\eta < 0.3$. From Eq. (4.8), we conclude that even for linear oscillators there is a 23% gain for aperiodic driving forces given by Eq. (2.14) for small η.

B. Numerical results

In the numerical simulation, we found that the energy gain G is approximately constant over the investigated range of friction coefficients $\eta = 0.1, ..., 1.9$. The peak at $\eta \approx 0.7$ coincides with a jump of the optimal driving frequency Ω_S of the sinusoidal driving force from $\Omega_S \approx 1.0$ for weak friction to $\Omega_S \approx 0$ for strong friction. That means sinusoidal driving forces with frequency zero (constant driving) become more efficient than oscillating forcing functions for large friction coefficients. When the friction coefficient is close to the transition between constant force and sinusoidal force, sinusoidal forcing is comparatively inefficient. Therefore, the gain has a local maximum. The numerical value for G is close to the analytical estimation (Fig. 5). The initial values of the simulation are $x(0) = \dot{x}(0) = \dot{y}(0) = 0$, $y(0) = 0.001$, and the driving effort is $\bar{F}^2 = 1.0$.

V. STABILITY OF OPTIMAL CONTROL

Resonant forcing of nonlinear oscillators is an optimal control problem where the goal dynamics [Eq. (2.13)] is obtained by a variation principle [Eq. (2.5)]. The control is considered to be stable if the difference between the goal dynamics $y(t)$ and the dynamics of the driven oscillator $x(t)$ does not increase. We measure this difference with the quantity S:

$$S = \ln\left(\frac{\dot{\epsilon}_i}{\dot{\epsilon}_f}\right), \tag{5.1}$$

where

$$\dot{\epsilon}_i = \max_{0<t<\Delta t_0}\left\{\left(\frac{\dot{\epsilon}(t)}{\dot{y}(t)}\right)^2\right\}, \tag{5.2}$$

$$\dot{\epsilon}_f = \max_{T-\Delta t_T<t<T}\left\{\left(\frac{\dot{\epsilon}(t)}{\dot{y}(t)}\right)^2\right\}, \tag{5.3}$$

where $\epsilon = x - y$. $\Delta t_0(\Delta t_T)$ is the period of time between the first (last) two maxima of $\dot{x}(t)$. The control is stable for $S \geq 0$. The stability parameter for some examples of the investigated oscillators is shown in Fig. 6. The nu-

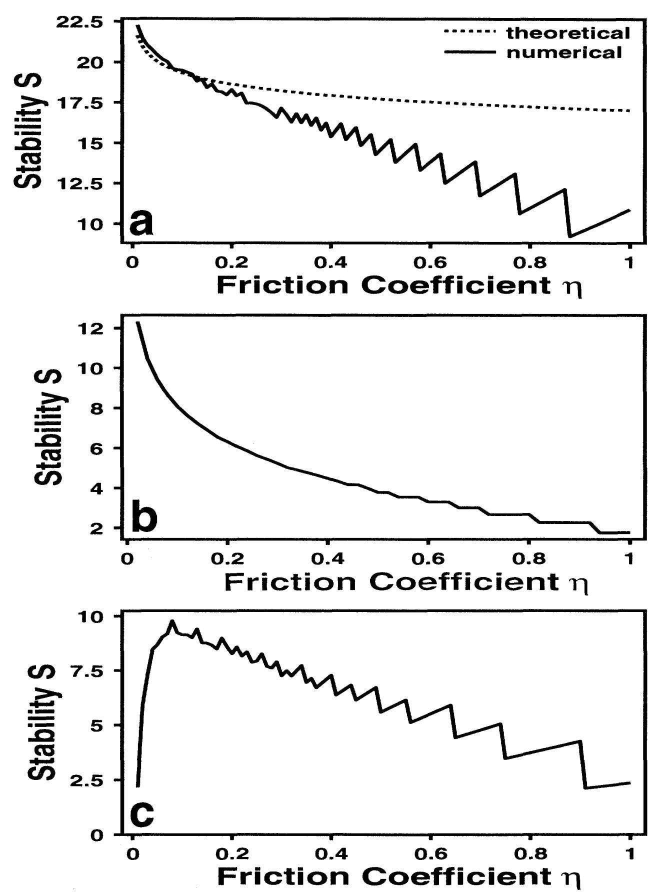

FIG. 6. Stability of control of the optimal driven oscillators: (a) linear oscillator $V = \frac{x^2}{2}$, (b) nonlinear oscillator $V = \frac{x^4}{4}$, and (c) the Duffing oscillator $V = -\frac{x^2}{2} + \frac{x^4}{4} + \frac{1}{4}$. The driving effort is $\bar{F}^2 = 100.0$. The initial values of the simulation are $x(0) = \dot{x}(0) = \dot{y}(0) = 0$, $y(0) = 0.001$ for (a) and (b) and $x(0) = 1.0$, $\dot{x}(0) = 0.0$, $y(0) = 0.999$, $\dot{y}(0) = 0.0$ for (c).

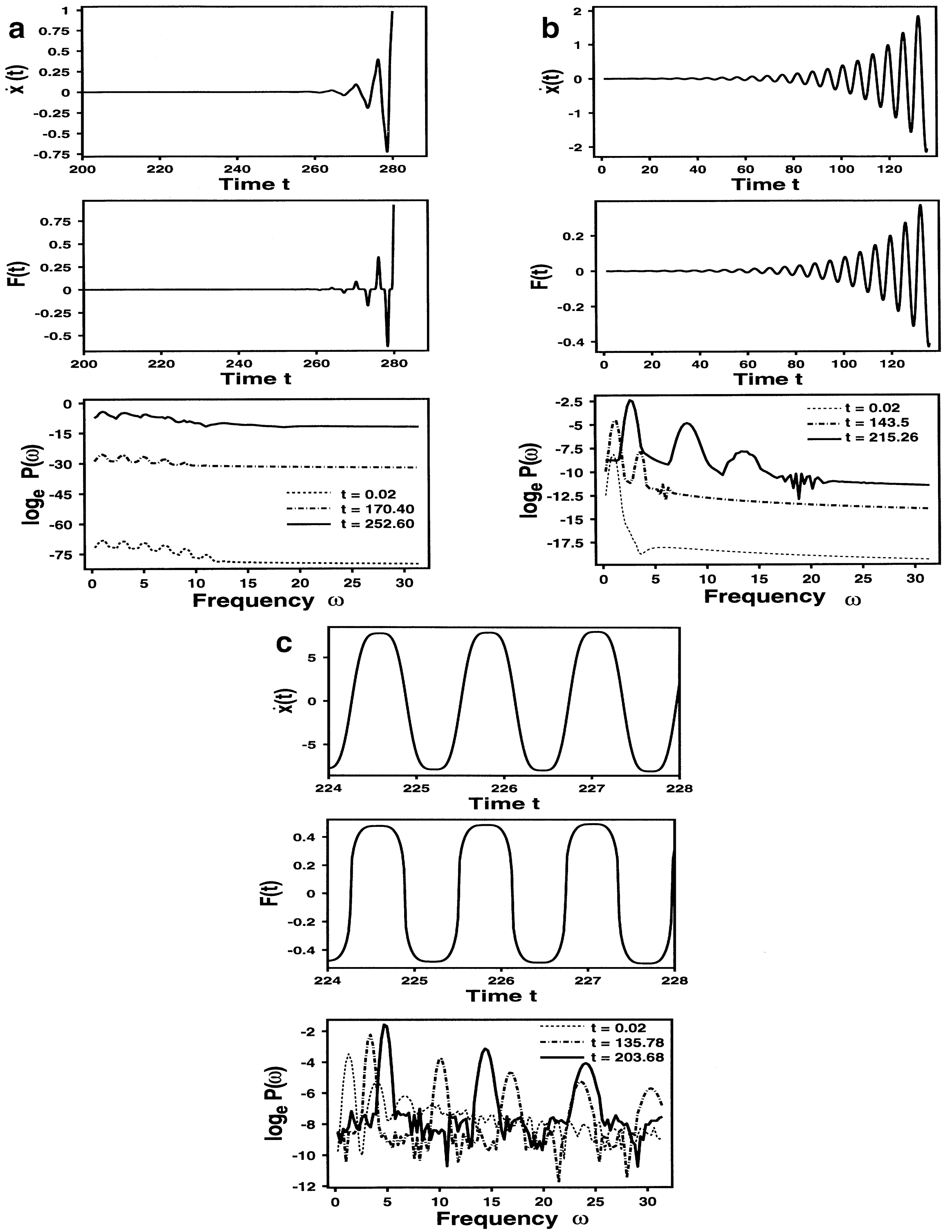

FIG. 7. Dynamics of driving force $F(t)$ and driven system $\dot{x}(t)$ and Fourier-power spectra $P(\omega)$ of $F(t)$ for different β. The driving effort is $\bar{F}^2 = 1.0$ and the considered friction coefficient $\eta = 0.05$. (a) $\beta = 1.1$, (b) $\beta = 2.0$, (c) $\beta = 4.0$.

merical simulations show that in most cases, the control is stable for all friction coefficients η and driving efforts $\bar{F}^2$. For linear oscillators, it is possible to estimate the stability analytically. With (2.13) and (2.1), one obtains

$$\ddot{\epsilon}(t) + \eta\dot{\epsilon}(t) + \omega^2\epsilon(t) = 0. \quad (5.4)$$

This differential equation can be solved by

$$\epsilon(t) = \epsilon_i e^{-\frac{\eta t}{2}}\left[\cos\left(\sqrt{\omega^2 - \frac{\eta^2}{4}}\, t\right) + \frac{1}{\sqrt{\omega^2 - \eta^2/4}} \sin\left(\sqrt{\omega^2 - \frac{\eta^2}{4}} t\right)\right], \quad (5.5)$$

where we assume $\epsilon_i = \epsilon(0)$ and $\dot{\epsilon}(0) = 0$. The stability we now define as

$$S^* = \ln\left(\frac{\dot{\epsilon}_i^*}{\dot{\epsilon}_f^*}\right), \quad (5.6)$$

where

$$\dot{\epsilon}_i^* = \max_{0<t<\Delta t_0}\left\{\epsilon(t)^2\right\}, \quad (5.7)$$

$$\dot{\epsilon}_f^* = \max_{T-\Delta t_T<t<T}\left\{\epsilon(t)^2\right\}. \quad (5.8)$$

The control is stable since $S^* \geq 0$. With

$$\bar{F}^2 \approx \frac{32\eta E(0)}{(4\omega^2 - \eta^2)}(e^{\eta T} - 1), \quad (5.9)$$

and Eq. (2.28), we find that terminal time T approximately is

$$T = \frac{1}{\eta}\ln\left(\frac{(4\omega^2 - \eta^2)(\frac{\bar{F}^2}{4\eta})}{8E_0\omega^2} + 1\right). \quad (5.10)$$

Therefore, we find $S^* = \ln\left(\frac{(4\omega^2 - \eta^2)(\frac{\bar{F}^2}{4\eta})}{8E_0\omega^2} + 1\right)^2$. This estimation is close to numerical data for small friction coefficients η (Fig. 6).

VI. OTHER NORMS

We now generalize the measure for the driving force. So far, the forcing function has been an element of the normed $\mathcal{L}_2 - $ *Banach space* [20]. Now the forcing function is allowed to be an element of the general $\mathcal{L}_\beta - $ *Banach spaces*. Each function $f(t)$ of these spaces are Lebesgue measurable and have to satisfy the norm condition,

$$||f|| = \int_0^T |f(t)|^\beta(t)dt, \quad (6.1)$$

where $||f||$ is a positive real number.

For strong resonance, the terminal energy $E(T)$ again is fixed, whereas the general driving effort $\bar{F}_G = (\bar{F}^2)^{\frac{1}{\beta}} = (\int_0^T |F|^\beta(t)dt)^{\frac{1}{\beta}}$ is to be minimized. In addition, we assume the initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ are fixed.

This sets up the Lagrange problem

$$\delta \int_0^T |F|^\beta + \mu(t)\left(\ddot{x} + \eta\dot{x} + \frac{\partial V}{\partial x} - F(t)\right) + \lambda\left(\frac{1}{2}\dot{x}^2 + V(x) - E\right)\delta_D(t-T)dt = 0, \quad (6.2)$$

where λ and μ are real Lagrange multipliers, whereas $\mu(t)$ is time dependent. The associated Euler-Lagrange equations yield the conditions,

$$\ddot{\mu}(t) - \eta\dot{\mu}(t) + \frac{\partial^2 V}{\partial x(t)^2}\mu(t) = 0, \quad (6.3)$$

$$F = \left(\frac{|\mu(t)|}{\beta}\right)^{\frac{1}{\beta-1}} \mathrm{sgn}(\mu(t)), \quad (6.4)$$

$$F(T)^2 - F(T)\eta\dot{x}(T) - \frac{|F(T)|^2}{\beta} = 0, \quad (6.5)$$

$$\dot{F}(T) = \frac{F(T)(\eta - \frac{\partial V}{\partial x}\frac{1}{\dot{x}})}{((\beta-2)\frac{F(T)^2}{|F(T)|^2} + 1)}. \quad (6.6)$$

Equation (6.3) is closely related to the time-reflected unperturbed dynamics of the oscillator [Eq. (2.13)]. This indicates that it might be possible to generalize the paradigm of the dynamical key. The dynamics of the driving force and oscillator that results from this solution is shown for various norms in Fig. 7. For large norm exponents $\beta > 2$, the driving force has the form of a square wave function and is similar to bing-bang controls. The corresponding power spectrum shows large higher harmonics (Fig. 7). The Fourier power P in Fig. 7 is defined by

$$P(k) = \sqrt{\frac{A(k)^2 + B(k)^2}{A_{\max}^2 + B_{\max}^2}}, \quad (6.7)$$

where the Fourier coefficients $A(k), B(k)$ are given by

$$A(k) = \frac{L}{2}\left[\sum_{k=1}^{120}\sum_{n=\frac{t_i}{dt}}^{L+\frac{t_i}{dt}} \sin\left(nk\frac{2\pi}{L}\right) F(n) \exp^{\frac{-16(\frac{L}{2}-n)^2}{L^2}}\right], \quad (6.8)$$

$$B(k) = \frac{L}{2}\left[\sum_{k=1}^{120}\sum_{n=\frac{t_i}{dt}}^{L+\frac{t_i}{dt}} \cos\left(nk\frac{2\pi}{L}\right) F(n) \exp^{\frac{-16(\frac{L}{2}-n)^2}{L^2}}\right], \quad (6.9)$$

and

$$A_{\max} = \max_k\{A(k)\}, \quad B_{\max} = \max_k\{B(k)\}. \quad (6.10)$$

The window length for the Fourier transformation was $L = 1200$. The transformation was calculated for three different initial times t_i. For the numerical calculation of $F(t)$, the time step was $dt = \frac{1}{50}$. Therefore, the time is defined by $t = ndt$ and the frequency is $\omega(k) = \frac{2\pi k}{Ldt}$.

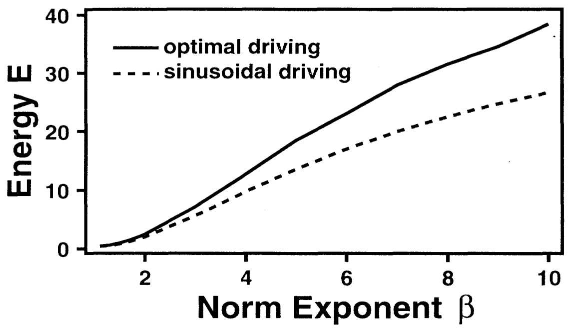

FIG. 8. Energies a nonlinear oscillator can achieve for different norms. The potential of the oscillator is $V = \frac{x^2}{2} + \frac{5}{4}x^4$.

For large norm exponents, β, large forces are penalized by the norm. This leads to a pruning of the peaks. For the small norm exponent $\beta < 2$, large forces are emphasized and the driving forces have the shape of δ pulses. The power spectrum reaches a homogeneous frequency distribution for decreasing β. Close to terminal time T, the size of these peaks sharply increases. A comparison of the terminal energies that a linear oscillator can reach under the strong resonance condition for optimal driving and sinusoidal driving for different norm exponents β is given in Fig. 8. The initial values of the simulation are $x(0) = \dot{x}(0) = \dot{y}(0) = 0$, $y(0) = 0.001$. The driving effort is $\bar{F}^2 = 1.0$. For small norm exponents β, the energies of optimal and sinusoidal driven oscillator are close. For increasing norm exponent the optimal driving force becomes more and more effective compared to sinusoidal driving.

Figure 7 shows a comparison of the power spectrum of the forcing functions for various β. For technical applications, the norm $\beta = 2$ may be advantageous if a small content of higher harmonics of applied driving forces is desirable. For all investigated norms, the shift of the basic frequency $w_1 = \omega(1)$ of the driving force at a certain amplitude of the oscillation is the same (Fig. 9) and is equal to the frequency of the unperturbed oscillation at this amplitude. This finding may be considered as a generalization of the principle of the dynamical key.

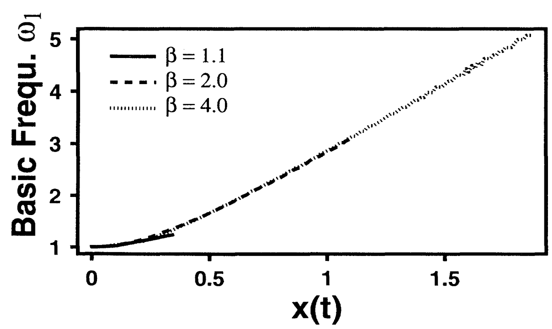

FIG. 9. Comparison of the power spectrum of various driving forces for different norm exponents. The potential of the oscillator is $V = \frac{x^2}{2} + \frac{5}{4}x^4$. The initial values of the simulation are $x(0) = \dot{x}(0) = \dot{y}(0) = 0$, $y(0) = 0.001$. The driving effort is $\bar{F}^2 = 1.0$.

VII. CONCLUSION

In general, there are two types of resonances: (i) resonances of a driven oscillator, (ii) resonances of two coupled oscillators. In this paper, resonances of the first type have been considered. We show that for a special driving force, which is determined through a time reflection of the unperturbed natural dynamics of the oscillator, the energy transfer is maximal, i.e., the system reacts most sensitively to its own transient dynamics. This is called the "principle of the dynamical key." The energy gain of the optimal driving forces compared to sinusoidal forcing is especially large for a large nonlinearity and weak damping. For other norms such as the quadratic norm or other types of resonances, the optimal forcing function matches with the period of the unperturbed dynamics. The resulting Euler-Lagrange equation for a general norm [Eq. (6.3)] is closely related to the time-reflected natural dynamics [Eq. (2.11)]. In particular, the frequency shift appears to be a key quantity. The relation between shift of the basic frequency of the forcing function and the amplitude of the oscillation is the same for all norms and is equal to the frequency shift of the unperturbed dynamics (Fig. 7).

This would suggest that resonances of coupled oscillators occur if the condition $\omega_1(H_0^1 - \Delta H) = \omega_2(H_0^2 + \Delta H)$ is satisfied. ω_1 and ω_2 are the frequencies of the oscillators, H_0^1 and H_0^2 are the initial energies of the oscillators, and ΔH is the energy exchange. This means the energy exchange between two conservative oscillators should be large if the frequencies of both oscillators match and the amplitude frequency coupling is opposite, i.e., $\omega_1 = \omega_2$ and $\frac{\partial \omega_1}{\partial \Delta H} = -\frac{\partial \omega_2}{\partial \Delta H}$.

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APPENDIX: VARIATIONAL PRINCIPLE WITH FREE TERMINAL TIME

We introduce an approach to calculate analytically Lagrange problems with free terminal time T and natural boundary conditions. We seek to extremalize the functional,

$$I = \int_0^T L(x^1, \dot{x}^1, \ddot{x}^1, \ldots, x^i, \dot{x}^i, \ddot{x}^i, \ldots, x^N, \dot{x}^N, \ddot{x}^N, t)dt, \tag{A1}$$

where T is not fixed and where $x^i(T)$ and $\dot{x}^i(T)$ have to lie on a given manifold

$K(x^1,\dot{x}^1,\ldots,x^i,\dot{x}^i,\ldots,x^N,\dot{x}^N,t) = 0$. The function L is assumed to be a function on N variables $x^1,x^2,\ldots,x^N$ and their time derivatives up to second order.

This problem can be solved by a variation of the functional $S = \int_0^T L_g dt$, where L_g is the Lagrange function $L_g = L + \lambda K \delta_D(t-T)$, e.g., for the case of strong resonance L is given by Eq. (2.3) and K by Eq. (2.4), where the variables are given by $x^1(t) = x(t)$, $x^2(t) = F(t)$ and $N = 2$. To obtain a stationary solution for the $x^i(t)$, $i = 1,\ldots,N$ the necessary condition $\delta S = 0$ has to be satisfied,

$$\delta S = \delta \int_0^T L(x^1,\dot{x}^1,\ddot{x}^1,\ldots,x^i,\dot{x}^i,\ddot{x}^i,\ldots,x^N,\dot{x}^N,\ddot{x}^N,t) + \lambda K(x^1,\dot{x}^1,\ldots,x^i,\dot{x}^i,\ldots,x^N,\dot{x}^N,t)\, \delta_D(t-T)dt = 0. \tag{A2}$$

Since the terminal time T is variable, we choose a parametric representation of the problem [22–24] and replace x^i, $\dot{x}^i$, $\ddot{x}^i$, and t with the following substitution rules:

$$t = t(p) \quad \text{with} \quad t(0) = 0 \quad \text{and} \quad t(1) = T, \tag{A3}$$

$$x^i(t) = x^i(t(p)) = x^i(p), \tag{A4}$$

$$\dot{x}^i(t) = \frac{x^i_p}{t_p}, \tag{A5}$$

$$\ddot{x}^i(t) = \frac{x^i_{pp}}{t_p^2} - \frac{x^i_p t_{pp}}{t_p^3}, \tag{A6}$$

where a subscripted p denotes the partial derivatives with respect to the parameter p. With $\int_0^1 K t_p \delta_D[t(p)-T]dp = \int_0^1 K\delta_D(p-1)dp$ the functional then assumes the form

$$S = \int_0^1 L\left(\ldots,x^i,\frac{x^i_p}{t_p},\frac{x^i_{pp}}{t_p^2} - \frac{x^i_p t_{pp}}{t_p^3},\ldots,t\right)t_p + \lambda K\left(\ldots,x^i,\frac{x^i_p}{t_p},\ldots,t\right)\delta_D(p-1)dp. \tag{A7}$$

Then we execute the variations for each variable,

$$\delta S = \sum_{i=1}^N \left(\int_0^1 dp \left[\frac{\partial L}{\partial x^i}t_p + \lambda\frac{\partial K}{\partial x^i}\delta_D(p-1)\right]\delta x^i + \left[\frac{\partial L}{\partial x^i_p}t_p + \lambda\frac{\partial K}{\partial x^i_p}\delta_D(p-1)\right]\delta x^i_p + \left[\frac{\partial L}{\partial x^i_{pp}}t_p + \lambda\frac{\partial K}{\partial x^i_{pp}}\delta_D(p-1)\right]\delta x^i_{pp}\right)$$
$$+ \left[\frac{\partial L}{\partial t}t_p + \lambda\frac{\partial K}{\partial t}\delta_D(p-1)\right]\delta t + \left[\frac{\partial L}{\partial t_p}t_p + \lambda\frac{\partial K}{\partial t_p}\delta_D(p-1)\right]\delta t_p + \left[\frac{\partial L}{\partial t_{pp}}t_p + \lambda\frac{\partial K}{\partial t_{pp}}\delta_D(p-1)\right]\delta t_{pp} = 0. \tag{A8}$$

Then we eliminate δx^i_p, δx^i_{pp}, δt_p, δt_{pp} by partial integration and evaluate the δ functions. We obtain from Eq. (A8):

$$\delta S = \sum_{i=1}^N \left\{\left(\lambda\frac{\partial K}{\partial x^i}\delta x^i\right)\bigg|_{p=1} + \int_0^1\left(\frac{\partial L}{\partial x^i}t_p\right)\delta x^i dp + \left(\lambda\frac{\partial K}{\partial x^i_p}\delta x^i_p\right)\bigg|_{p=1} + \left[\left(\frac{\partial L}{\partial x^i_p}t_p\right)\delta x^i\right]_0^1 - \int_0^1\frac{d}{dp}\left(\frac{\partial L}{\partial x^i_p}t_p\right)\delta x^i dp\right.$$
$$\left. + \left(\lambda\frac{\partial K}{\partial x^i_{pp}}\delta x^i_{pp}\right)\bigg|_{p=1} + \left[\left(\frac{\partial L}{\partial x^i_{pp}}t_p\right)\delta x^i_p\right]_0^1 - \left[\frac{d}{dp}\left(\frac{\partial L}{\partial x^i_{pp}}t_p\right)\delta x^i\right]_0^1 + \int_0^1\frac{d^2}{dp^2}\left(\frac{\partial L}{\partial x^i_{pp}}t_p\right)\delta x^i dp\right\}$$
$$+ \left(\lambda\frac{\partial K}{\partial t}\delta t\right)\bigg|_{p=1} + \int_0^1\left(\frac{\partial L}{\partial t}t_p\right)\delta t dp + \left(\lambda\frac{\partial K}{\partial t_p}\delta t_p\right)\bigg|_{p=1} + \left[\left(L + \frac{\partial L}{\partial t_p}t_p\right)\delta t\right]_0^1$$
$$- \int_0^1\frac{d}{dp}\left(L + \frac{\partial L}{\partial t_p}t_p\right)\delta t dp + \left(\lambda\frac{\partial K}{\partial t_{pp}}\delta t_{pp}\right)\bigg|_{p=1} + \left[\left(\frac{\partial L}{\partial t_{pp}}t_p\right)\delta t_p\right]_0^1$$
$$- \left[\frac{d}{dp}\left(\frac{\partial L}{\partial t_{pp}}t_p\right)\delta t\right]_0^1 + \int_0^1\frac{d^2}{dp^2}\left(\frac{\partial L}{\partial t_{pp}}t_p\right)\delta t dp = 0. \tag{A9}$$

This yields to the following Euler-Lagrange equations for $0 < p < 1$:

$$t_p\frac{\partial L}{\partial x^i} - \frac{d}{dp}\left(t_p\frac{\partial L}{\partial x^i_p}\right) + \frac{d^2}{dp^2}\left(t_p\frac{\partial L}{\partial x^i_{pp}}\right) = 0, \tag{A10}$$

$$t_p\frac{\partial L}{\partial t} - \frac{d}{dp}\left(L + t_p\frac{\partial L}{\partial t_p}\right) + \frac{d^2}{dp^2}\left(t_p\frac{\partial L}{\partial t_{pp}}\right) = 0, \tag{A11}$$

for all $i = 1,\ldots,N$. At the upper boundary, i.e., for $p = 1$, we have

$$\lambda\frac{\partial K}{\partial t_{pp}} = 0, \tag{A12}$$

$$\lambda\frac{\partial K}{\partial t_p} + \frac{\partial L}{\partial t_{pp}}t_p = 0, \tag{A13}$$

$$\lambda\frac{\partial K}{\partial t} + L + t_p\frac{\partial L}{\partial t_p} - \frac{d}{dp}\left(t_p\frac{\partial L}{\partial t_{pp}}\right) = 0, \tag{A14}$$

$$\lambda\frac{\partial K}{\partial x^i_{pp}} = 0, \tag{A15}$$

$$\lambda \frac{\partial K}{\partial x_p^i} + \frac{\partial L}{\partial x_{pp}^i} t_p = 0, \quad (A16)$$

$$\lambda \frac{\partial K}{\partial x^i} + t_p \frac{\partial L}{\partial x_p^i} - \frac{d}{dp}\left(t_p \frac{\partial L}{x_{pp}^i}\right) = 0, \quad (A17)$$

for all $i = 1,\ldots,N$. Since K is independent of x_{pp}^i and t_{pp}, Eqs. (A15), (A12) are always fulfilled and are not considered in the further calculation. At the lower boundary, i.e., for $p = 0$, we obtain, for these variables where the initial conditions $x^i(0)$ and $\dot{x}^i(0)$ are not fixed, the conditions:

$$\frac{\partial L}{\partial x_{pp}^i} t_p = 0, \quad (A18)$$

$$t_p \frac{\partial L}{\partial x_p^i} - \frac{\partial}{\partial p}\left(t_p \frac{\partial L}{x_{pp}^i}\right) = 0. \quad (A19)$$

Now we transform the Euler-Lagrange equations, Eqs. (A10), (A11) and the boundary conditions Eqs. (A13), (A14), (A16), (A17), (A18), (A19), back to a parameter-free representation with the following substitution rules:

$$x^i(p) = x^i(t(p)) = x^i(t), \quad (A20)$$

$$x_p^i = \dot{x}^i(t(p))t_p, \quad (A21)$$

$$x_{pp}^i = \ddot{x}^i t_p^2 + \dot{x}^i t_{pp}, \quad (A22)$$

for all $i = 1,\ldots,N$. The equations of motion Eqs. (A10), (A11) then read

$$\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}^i}\right) = 0, \quad (A23)$$

$$\sum_{i=1}^{N}\left\{-\dot{x}^i\left[\frac{\partial L}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}^i}\right)\right]\right\} = 0, \quad (A24)$$

for all $i = 1,\ldots,N$. At the upper boundary, i.e., for $t = T$, we obtain from the Eqs. (A13), (A14), (A16), (A17),

$$\sum_{i=1}^{N}\left[\dot{x}^i\left(\lambda \frac{\partial K}{\partial \dot{x}^i} + \frac{\partial L}{\partial \ddot{x}^i}\right)\right] = 0, \quad (A25)$$

$$\lambda \frac{\partial K}{\partial t} + L + \sum_{i=1}^{N}\left[-\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - \ddot{x}^i \frac{\partial L}{\partial \ddot{x}^i} + \dot{x}^i \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}^i}\right)\right] = 0, \quad (A26)$$

$$\lambda \frac{\partial K}{\partial \dot{x}^i} + \frac{\partial L}{\partial \ddot{x}^i} = 0, \quad (A27)$$

$$\lambda \frac{\partial K}{\partial x^i} + \frac{\partial L}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}^i}\right) = 0, \quad (A28)$$

for all $i = 1,\ldots,N$. For these variables where the initial conditions $x^i(0)$ and $\dot{x}^i(0)$ are not fixed, we obtain from the Eqs. (A18), (A19) at the lower boundary, i.e., for $t = 0$:

$$\frac{\partial L}{\partial \ddot{x}^i} = 0, \quad (A29)$$

$$\frac{\partial L}{\partial \dot{x}^i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}^i}\right) = 0\ . \quad (A30)$$

Then, we use the special Lagrange function [Eq. (2.2)] for strong resonance and substitute the variables x^i in Eqs. (A23)–(A30) by $x^1(t) = x(t)$, $x^2(t) = F(t)$ and $N = 2$.

Since the initial conditions for $x(0)$ and $\dot{x}(0)$ are fixed the Eqs. (A29), (A30) apply only for the variable F. For F, Eqs. (A29), (A30) as well as Eqs. (A27), (A28) are always satisfied because L [Eq. (2.3)] does not contain time derivatives of $F(t)$. When Eq. (A27) is evaluated for the variable $x^1 = x$ it is equal to Eq. (A25). Therefore, we drop Eq. (A25). In the following, we consider only Eq. (A26):

$$F^2(T) + \mu(T)\left.\frac{\partial V}{\partial x}\right|_T - \mu(T)F(T) + \dot{\mu}(T)\dot{x}(T) = 0, \quad (A31)$$

and evaluate Eqs. (A27), (A28) only for the variable $x^1 = x$:

$$\lambda \dot{x}(T) + \mu(T) = 0, \quad (A32)$$

$$\lambda \left.\frac{\partial V}{\partial x}\right|_T + \mu(T)\eta - \dot{\mu}(T) = 0. \quad (A33)$$

The equations of motion Eqs. (A23), (A24) read

$$\ddot{\mu}(t) - \dot{\mu}(t)\eta + \mu(t)\frac{\partial^2 V}{\partial x^2} = 0, \quad (A34)$$

$$2F(t) - \mu(t) = 0. \quad (A35)$$

The full set of equations of motion is given by Eqs. (2.1), (A34), (A35). The initial conditions for Eq. (2.1) are given by x_0 and $\dot{x}_0$. The initial conditions for Eq. (A34) [$\mu(0)$ and $\dot{\mu}(0)$] and the value of λ and T can be determined with Eqs. (2.4), (A31)–(A33). Since Eq. (A35) contains no time derivatives, the time dependence of F is given by Eq. (A34).

A trial solution for equations Eqs. (2.1), (A34), (A35) is given in the form of $F(t) = \alpha\eta\dot{x}(t)$. For this trial solution, the boundary conditions [Eqs. (A31)–(A33)] are only fulfilled if $\alpha = 2$ or $\alpha = 0$.

Therefore, a solution for the corresponding Euler-Lagrange equations Eqs. (A34), (A35) is given by

$$F(t) = 2\eta\dot{x}(t), \quad (A36)$$

$$\mu(t) = 4\eta\dot{x}(t), \quad (A37)$$

$$\lambda = -4\eta\ , \quad (A38)$$

if $E(T) > E(0)$ and

$$F(t) = 0, \quad (A39)$$

$$\mu(t) = 0, \quad (A40)$$

$$\lambda = 0\ , \quad (A41)$$

if $E(T) < E(0)$.

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