PHYSICAL REVIEW E VOLUME 51, NUMBER 2 FEBRUARY 1995

Averaged equations for Josephson junction series arrays

Kurt Wiesenfeld*
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332

James W. Swift†
Department of Mathematics, Northern Arizona University, Flagstaff, Arizona 86011-5717
(Received 22 August 1994)

We derive the averaged equations describing a series array of Josephson junctions shunted by a parallel inductor-resistor-capacitor load. We assume that the junctions have negligible capacitance ($\beta = 0$), and derive averaged equations that turn out to be completely tractable: in particular, the stability of both in-phase and splay states depends on a single parameter, δ. We find an explicit expression for δ in terms of the load parameters and the bias current. We recover (and refine) a common claim found in the technical literature, that the in-phase state is stable for inductive loads and unstable for capacitive loads.

PACS number(s): 05.45.+b, 74.50.+r

I. INTRODUCTION

Josephson junction arrays are perhaps the most widely studied class of coupled nonlinear oscillator systems. This stems in large part from their relevance in a number of applications, including their use as voltage standards [1] and their potential as submillimeter wave generators [2] and parametric amplifiers [3]. They also serve as prime examples of nonlinear dynamical systems with many degrees of freedom. Particularly good progress has been made for a subclass of this category, namely, globally coupled oscillators. Examples of this type — where each oscillator is coupled with equal strength to all others — arise not only in the context of electrical circuits, but in the fields of laser physics and classical mechanics as well.

Recent theoretical work has shown that some Josephson arrays have remarkable dynamical properties. The most striking discovery was made by Watanabe and Strogatz [4] for the class of arrays depicted in Fig. 1, namely, one-dimensional series arrays of N identical zero-capacitance junctions, driven by a constant current and shunted by a parallel load. Using a clever change of coordinates they showed that the differential equations admit $N-3$ independent constants of motion, for any $N > 3$. Furthermore, they found a rigorous reduction of the problem to a five-dimensional system of differential equations, independent of N.

The same technique allowed Watanabe and Strogatz to completely analyze the dynamics of the N-oscillator system

$$\dot{\varphi}_i = 1 + \frac{\kappa}{N}\sum_{j=1}^{N}\cos(\varphi_j - \varphi_i - \delta)\ , \quad i = 1, ..., N. \qquad (1)$$

They observed in particular that the central issue — whether the attracting dynamics is the in-phase (i.e., synchronized) oscillation or an incoherent state — depends only on the sign of $\kappa \sin\delta$.

The purpose of the present paper is to derive Eq. (1) as the averaged version of the Josephson junction array shown in Fig. 1, and to obtain an explicit expression for the key parameters κ and δ. Our approach follows closely that of Ref. [5], which treated the special case of a pure resistive load. Starting from the full circuit equations, we apply a first-order averaging method which is valid in the weakly coupled limit, but holds for a general inductor-resistor-capacitor (LRC) load. We also add some observations about the behavior of the averaged system. A nice feature of the present analysis is that the results admit a direct physical interpretation: The combined current of the Josephson junction oscillators acts as a periodic driving voltage for the LRC circuit, and *the*

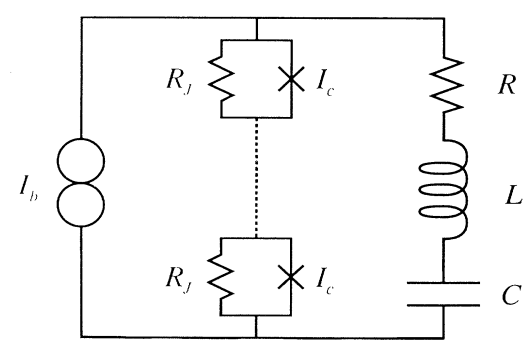

FIG. 1. Circuit schematic for the shunted Josephson array.

*Electronic mail address: kw2@prism.gatech.edu
†Electronic mail address: James.Swift@nau.edu

in-phase oscillation is stable when the Josephson junction frequency is larger than the resonant frequency of the LRC circuit. Conversely, if the junction frequency is smaller than the resonant frequency of the load, then the manifold of incoherent states is stable. Thus we recover (and refine) an oft quoted piece of conventional wisdom found in the Josephson array literature, that the in-phase state is stable for inductive loads, but unstable for capacitive loads. Furthermore, we observe that the way to extract the most energy from the Josephson junctions is to tune their frequency to match the resonant frequency of the LRC circuit.

Strogatz and Mirollo [6] computed analytically the stability of the splay-phase states (the most symmetric of the incoherent states) of the unaveraged system in the $N \to \infty$ limit. Surprisingly, their results showed excellent agreement with numerical calculations [7] even for N as small as 4. Though Strogatz and Mirollo did not assume weak coupling, their stability results do not easily allow a physical interpretation. We show that our results derived from the averaged system (for any N) agree with their results (for $N \to \infty$) in the weak coupling limit. The weak coupling limit also provides us with a simple physical interpretation of these results.

II. DERIVATION OF THE AVERAGED EQUATIONS

Consider the array depicted in Fig. 1. The goal of this section is to show that, in the limit of large shunt impedance, the circuit dynamics is governed by differential equations of the form (1), and to evaluate κ and δ in terms of the physical parameters of the system.

Our starting point is the Kirchhoff equations for the circuit. We assume that the N junctions are identical and have negligible capacitance ($\beta = 0$ in the common notation). The governing circuit equations are

$$\frac{\hbar}{2eR_J}\dot{\phi}_k + I_c \sin\phi_k + \dot{Q} = I_b, \tag{2}$$

$$L\ddot{Q} + R\dot{Q} + \frac{Q}{C} = \frac{\hbar}{2e}\sum_{j=1}^{N}\dot{\phi}_j, \tag{3}$$

where $k = 1, 2, \cdots, N$. Here, ϕ_k is the quantum phase difference across the kth Josephson junction, R_J is the junction resistance, I_c is the junction critical current, Q is the charge on the load capacitor, I_b is the applied bias current, L, R, and C are the load inductance, resistance, and capacitance, respectively, $\hbar$ is Planck's constant divided by 2π, e is the electron charge, and the overdot denotes differentiation with respect to time t. Substitution of Eq. (2) into Eq. (3) yields

$$L\ddot{Q} + (R + NR_J)\dot{Q} + \frac{Q}{C} = NR_J I_b - R_J I_c \sum_{j=1}^{N}\sin\phi_j. \tag{4}$$

It is convenient to shift the load variable Q by a constant

$$\frac{Q}{C} - NR_J I_b \to \frac{Q}{C} \tag{5}$$

so that this becomes

$$L\ddot{Q} + (R + NR_J)\dot{Q} + \frac{Q}{C} = -R_J I_c \sum_{j=1}^{N}\sin\phi_j. \tag{6}$$

In order to compare arrays having different numbers of junctions, it is natural to define scaled load parameters:

$$l = L/N, \quad r = R/N, \quad c = NC. \tag{7}$$

Introducing the dimensionless time τ and charge q defined by

$$\tau = \frac{2e}{\hbar}R_J I_c t, \tag{8}$$

$$q = lR_J I_c \left(\frac{2e}{\hbar}\right)^2 Q, \tag{9}$$

the circuit equations (2) and (6) become

$$\dot{\phi}_k + \sin\phi_k + \epsilon\dot{q} = \alpha, \tag{10}$$

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = -\frac{1}{N}\sum_{j=1}^{N}\sin\phi_j, \tag{11}$$

where the overdot now denotes differentiation with respect to dimensionless time τ, and where

$$\epsilon = \frac{\hbar}{2eI_c l}, \tag{12}$$

$$\alpha = I_b/I_c, \tag{13}$$

$$\gamma = \frac{(r + R_J)\hbar}{2eR_J l I_c}, \tag{14}$$

$$\omega_0^2 = \frac{1}{lc}\left(\frac{\hbar}{2eR_J I_c}\right)^2. \tag{15}$$

Note that γ includes the effect of both the junction resistance and the load resistance. Thus γ is never zero: in fact $\gamma \geq \epsilon$ with equality when the load resistance is zero.

Up to this point, Eqs. (10) and (11) are merely scaled versions of the exact circuit equations (2) and (3). To get things in a form suitable for averaging, we transform from the variables ϕ_k to natural angles ψ_k [5]. The latter are "natural" in the sense that, in the uncoupled limit, the angular velocity $\dot{\phi}_k$ is nonuniform, while $\dot{\psi}_k$ is a constant. This is accomplished by the transformation [5]

$$\psi(\phi) = 2\arctan\left[\sqrt{\frac{\alpha - 1}{\alpha + 1}}\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right], \tag{16}$$

$$\phi(\psi) = 2\arctan\left[\sqrt{\frac{\alpha + 1}{\alpha - 1}}\tan\left(\frac{\psi}{2}\right)\right] - \frac{\pi}{2}. \tag{17}$$

Equation (10) becomes

$$\frac{1}{\sqrt{\alpha^2-1}}\dot{\psi}_k = 1 - \frac{\epsilon\dot{q}}{\alpha - \sin(\phi_k)}. \quad (18)$$

Note that $\omega = \sqrt{\alpha^2-1}$ is the frequency of an uncoupled junction. It is convenient to rescale time to units where this frequency is unity, by taking $\tau\sqrt{\alpha^2-1} \to \tau$, so that Eqs. (18) and (11) become

$$\dot{\psi}_k = 1 - \frac{\epsilon\omega\dot{q}}{\alpha - \sin(\phi_k)} \quad (19)$$

and

$$\omega^2\ddot{q} + \gamma\omega\dot{q} + \omega_0^2 q = -\frac{1}{N}\sum_{j=1}^{N} \sin\phi(\psi_j). \quad (20)$$

The system of equations (19) and (20) is exact, and so far we have not made an assumption of weak coupling. Notice that Eq. (19) is of the form $\dot{\psi}_k = 1 + O(\epsilon)$. For small ϵ, we can obtain the drift of ψ_k by averaging Eq. (19) over one period of the oscillation to obtain

$$\langle\dot{\psi}_k\rangle = 1 - \frac{1}{2\pi}\int_0^{2\pi} \frac{\epsilon\omega\dot{q}}{\alpha - \sin\phi(\psi_k)} d\tau. \quad (21)$$

We can proceed by using the function $q(\tau)$ obtained by solving Eqs. (19) and (20) with $\epsilon = 0$. In other words, we assume that each Josephson junction's quantum phase difference is $\phi_k(\tau) = \phi(\tau + c_k)$, with the functional form given by Eq. (17). This is the solution to Eq. (10), in terms of the new τ, when $\epsilon\dot{q} = 0$. The c_k are arbitrary constants. We refer to $\epsilon\dot{q} \to 0$ as the *weak coupling limit.* Physically, this occurs whenever the load impedance (per junction) goes to infinity, and very little current flows through the load. Mathematically, the weak coupling limit can occur for three distinct reasons: (1) $\epsilon \to 0$, with γ and ω_0 of order 1, (2) $\gamma \to \infty$, with ϵ and ω_0 of order 1, or (3) $\omega_0 \to \infty$, with ϵ and γ of order 1. If either of the last two conditions hold, then Eq. (11) shows that $\dot{q}$ is vanishingly small, and the analysis to follow holds even though ϵ is of order 1.

In terms of the original load parameters (see Fig.1), these three weak coupling limits are (1) $L/N \to \infty$, (2) $R/N \to \infty$, or (3) $NC \to 0$.

The LC load ($R = 0$) is a special case, because the impedance of the LC load is zero at its resonant frequency. Thus the LC load system is *strongly coupled* near $\omega = \omega_0$, even when $\epsilon \to 0$. Note from Eqs. (12) and (14) that $\epsilon = \gamma$ for an LC load ($R = r = 0$). The positive resistance of the Josephson junction (R_J) makes γ positive and keeps the current finite at resonance.

To find the appropriate expression for $q(\tau)$, we begin with the $\epsilon = 0$ solution to Eq. (19):

$$\psi_k(\tau) = \tau + c_k, \quad (22)$$

where the c_k are arbitrary initial conditions. From Eq. (17) there follows the useful trigonometric identity

$$\sin\phi(\psi) = \alpha - \frac{\alpha^2-1}{\alpha - \cos\psi}. \quad (23)$$

It is convenient to write this even function of ψ in terms of its Fourier series

$$\sin\phi(\psi) = \sum_{n=0}^{\infty} A_n \cos(n\psi), \quad (24)$$

where, in particular,

$$A_1 = 2(\alpha^2 - 1 - \alpha\sqrt{\alpha^2-1}). \quad (25)$$

Note that A_1 is a decreasing function of α with $0 \geq A_1 > -1$ for $\alpha \geq 1$.

Combining Eqs. (20) and (22)–(25) yields

$$\omega^2\ddot{q} + \gamma\omega\dot{q} + \omega_0^2 q = -\frac{1}{N}\sum_{j=1}^{N}\sum_{n=0}^{\infty} A_n \cos n(\tau + c_j). \quad (26)$$

This equation has the steady state solution

$$q(\tau) = -\frac{1}{N}\sum_{j=1}^{N}\sum_{n=0}^{\infty} B_n \cos[n(\tau + c_j) + \beta_n], \quad (27)$$

where

$$B_n^2 = \frac{A_n^2}{(n^2\omega^2 - \omega_0^2)^2 + (\gamma n\omega)^2}, \quad (28)$$

$$\beta_n = \arctan\left[\frac{\gamma n\omega}{n^2\omega^2 - \omega_0^2}\right]. \quad (29)$$

Note that B_n and β_n are just the well-known amplitude and phase shift response of a linear damped oscillator driven at frequency $n\omega$. The relative sign between A_n and B_n determines the correct branch of the inverse tangent. We will choose B_n to be positive, and since A_1 is negative, we have $0 \leq \beta_1 \leq \pi$.

The next step is to substitute this expression for $q(\tau)$ back into Eq. (21). Note that the identity Eq. (23) allows us to rewrite Eq. (21) as

$$\langle\dot{\psi}_k\rangle = 1 - \frac{\epsilon\omega}{2\pi}\int_0^{2\pi} \dot{q}(\tau)\frac{\alpha - \cos(\tau + c_k)}{\alpha^2 - 1} d\tau. \quad (30)$$

Now, since $q(\tau)$ is 2π periodic, it is evident from Eq. (30) that only the fundamental Fourier component of $\dot{q}(\tau)$ contributes to the integral, with the result

$$\langle\dot{\psi}_k\rangle = 1 + \frac{\epsilon B_1}{2N\omega}\sum_{j=1}^{N} \sin(c_j - c_k - \beta_1). \quad (31)$$

The final step is to replace the "initial values" c_k by their slowly evolving counterparts $\langle\psi_k(\tau)\rangle$, and drop the angular brackets to get the first-order averaged equations

$$\dot{\psi}_k = 1 + \frac{\epsilon B_1}{2N\omega}\sum_{j=1}^{N} \cos(\psi_j - \psi_k - \delta), \quad (32)$$

where $\delta = \frac{\pi}{2} + \beta_1$ is given by

$$\sin\delta = \frac{\omega^2 - \omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}} \tag{33}$$

with $\frac{\pi}{2} \leq \delta \leq \frac{3\pi}{2}$. This is the main result of the paper. Note that δ is simply related to β_1, which is the phase shift of the linear *LRC* circuit [Eq. (10)] driven at the frequency (ω) of the uncoupled Josephson junctions [Eq. (11)].

In terms of the parameters in the original equations, (2) and (3), the important dimensionless frequencies are

$$\omega_0^2 = \frac{1}{\mathrm{LC}}\left(\frac{\hbar}{2eR_JI_c}\right)^2, \qquad \omega^2 = \left(\frac{I_b}{I_c}\right)^2 - 1. \tag{34}$$

III. IN-PHASE AND INCOHERENT STATES

We start this section by recalling some results of Refs. [8], [5], and [4]. Equation (32) admits two types of solutions: the *in-phase* (or coherent) solution, where all of the angles are equal, and *incoherent* solutions, where the "center of mass" of the N angles ψ_i, when they are placed on a circle, is at the center of the circle. The incoherent solutions rotate rigidly and have period exactly 2π. [It is easy to see that the coupling terms in Eq. (32) cancel out for incoherent solutions.]

The in-phase solution is unique: In geometric terms, this periodic orbit is a circle (a one-dimensional manifold) in the $(N+2)$-dimensional phase space. It is natural to ask "How many incoherent solutions are there?" There is a unique incoherent solution if $N = 2$ or 3, but an infinite number of incoherent solutions if $N > 3$. In fact the set of incoherent solutions is an $(N-2)$-dimensional manifold for any $N \geq 3$, called the *incoherent manifold* by Watanabe and Strogatz [4]. The incoherent manifold is foliated by circles, which are the the incoherent solutions. Every incoherent solution is neutrally stable to the $N-2$ perturbations which leave it in the incoherent manifold. Hence every incoherent solution has $N-2$ unit Floquet multipliers.

We can compute the dimension of the incoherent manifold as follows. Place $N-2$ oscillators on the circle of radius 1, so that their center of mass is not at the origin. (This gives the $N-2$ dimensions of the incoherent manifold.) Then the position of the last two oscillators is uniquely determined since the center of mass of all N oscillators is at the origin. Note that, if the center of mass of the first $N-2$ oscillators is farther than $2/(N-2)$ from the origin, then it is not possible to place the last two oscillators so that the center of mass of all N is at the origin. (A slick mathematical argument gives the same result: The requirement that the center of mass of the N oscillators be at the origin is a codimension-2 constraint.)

A major result of Ref. [4] is that *unaveraged* systems with a "sinusoidal" nonlinearity, including Eqs. (2) and (3), have an incoherent manifold foliated by periodic orbits. In other words, any initial condition in the incoherent manifold is part of a periodic orbit. Thus the splay solutions in the unaveraged equations have $N-2$ unit Floquet multipliers, just as they did in the averaged equations. Incoherent solutions can be defined for unaveraged systems in terms of time delays [8,9].

The most symmetric of the incoherent solutions, with the angles all equally spaced in time, is called the *splay-phase* solution. The stability of the in-phase and splay-phase solutions in Eq. (32) is easy to calculate, following Ref. [8] (Sec. 6). We give the stability of these periodic solutions in terms of the Floquet exponents, which are analogous to the eigenvalues of the linearization about a fixed point. A given periodic orbit has as many exponents as there are phase space dimensions; if any of the exponents has a positive real part, then a typical perturbation will grow exponentially, and the periodic orbit is unstable. All periodic orbits have at least one zero Floquet exponent corresponding to a perturbation along the orbit.

The in-phase solution has a single Floquet exponent equal to zero, and $N-1$ exponents equal to

$$\lambda_{\text{in phase}} = \frac{-\epsilon B_1}{2\omega}\sin\delta \tag{35}$$

where δ is given by Eq. (33) and B_1 is

$$B_1 = \frac{2\omega(\sqrt{\omega^2+1} - \omega)}{\sqrt{(\omega^2 - \omega_0^2)^2 + (\gamma\omega)^2}} \tag{36}$$

from Eqs. (25) and (28). The splay-phase solution has $N-2$ Floquet exponents equal to zero and a complex conjugate pair

$$\lambda_{\text{splay}} = \frac{\epsilon B_1}{4\omega}(\sin\delta \pm i\cos\delta). \tag{37}$$

We see the crucial role of $\sin\delta$ in the local stability analysis (ϵ and ω are positive). If $\sin\delta > 0$ then the in-phase solution is stable and the splay phase is unstable. If $\sin\delta < 0$ then the in-phase solution is unstable and the spay-phase solution is neutrally stable. Watanabe and Strogatz [4] showed that these local results hold globally, due to the existence of a Lyapunov function for Eq. (1): If $\sin\delta < 0$ then almost every initial condition converges to the incoherent manifold, and we say that the incoherent manifold is stable. Thus these three statements are all equivalent, provided $\sin\delta \neq 0$: (1) the incoherent manifold is stable; (2) the splay solution is neutrally stable; (3) the in-phase solution is unstable. In other words, the incoherent manifold is stable exactly when the splay-phase state is neutrally stable.

From Eq. (33) we see that the in-phase solution is stable when $\omega > \omega_0$, and the incoherent manifold is stable when $\omega < \omega_0$. There is no bistability in the averaged equations. We note that numerical simulations of the unaveraged array equations show bistability in an *LC*-shunted Josephson array [10,11] [i.e., Eqs. (2) and (3) with $R = 0$]. Therefore the averaged equations have somewhat different dynamics than the original system, though both have an incoherent manifold of solutions.

Strogatz and Mirollo [6] computed the Floquet exponents of the splay state of Eqs. (2) and (3) in the limit $N \to \infty$. They found that all but four of the Floquet exponents are zero (the fact that Watanabe and Strogatz later proved is true for finite N), while the other four are the eigenvalues of a 4×4 matrix. In our notation, the fourth-order characteristic equation is

$$\lambda^4 + \gamma\lambda^3 + (\omega_0^2 + \omega^2)\lambda^2 + [\epsilon\omega(\sqrt{\omega^2+1} - \omega) + \gamma\omega^2]\lambda + \omega_0^2\omega^2 = 0. \quad (38)$$

These eigenvalues derived for the case $N \to \infty$ are in excellent agreement with numerical calculations [7] even for $N = 4$, and it is conceivable that the result is exactly true for any N. In any event, it makes sense to compare this formula with the eigenvalues derived from the averaged system, as we now do.

If we set $\epsilon = 0$, the characteristic equation, Eq. (38) factors,

$$(\lambda^2 + \gamma\lambda + \omega_0^2)(\lambda^2 + \omega^2) = 0, \quad (39)$$

which has a natural physical interpretation, namely, the first factor corresponds to the decay of the current in the LRC branch of the circuit, while the pure imaginary eigenvalues $\pm i\omega$ from the second factor correspond to the oscillation frequency of a single junction in the absence of a load. If one computes from Eq. (38) the order ϵ correction to the Floquet exponents $\pm i\omega$, one finds precisely the result for the averaged system, namely, Eq. (37). Physically, then, the averaged result captures to lowest order the effect of interactions between the load and the junctions.

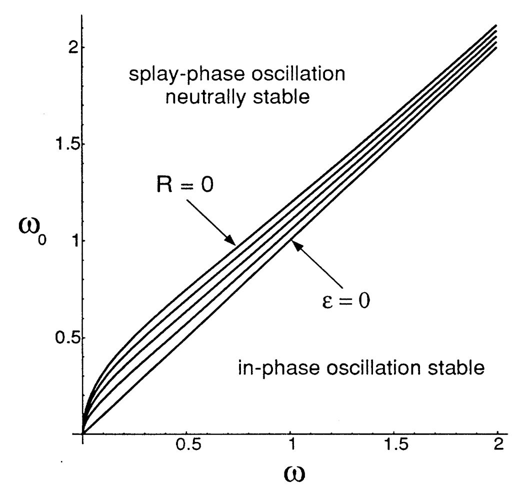

FIG. 2. Bifurcation of the splay state as given by Eq. (40) for $\epsilon/\gamma = 0$, 0.25, 0.5, 0.75, and 1. The dimensionless parameters ω and ω_0 measure the frequencies of the Josephson junction oscillators and the LRC resonant circuit, respectively. The curve with $\epsilon = 0$ is the result for the averaged system. The uppermost curve, with $\epsilon/\gamma = 1$, corresponds to an array with LC load ($R = 0$) which is strongly coupled near resonance.

Finally, we can see how accurately the averaged equations capture the transition where the splay states go unstable. In terms of our own parameters, Eq. (14) of Ref. [6] gives the transition curve

$$\omega_0^2 = \omega^2 + \frac{\epsilon}{\gamma}\omega(\sqrt{\omega^2+1} - \omega). \quad (40)$$

Recalling that the averaged system has the corresponding transition at $\omega_0 = \omega$, we see there is exact agreement only for $\epsilon = 0$. However, for $\epsilon > 0$ the transition curve determined from Eq. (40) never gets very far from the diagonal when plotted on the ω-ω_0 plane, as shown in Fig. 2. Note from Eqs. (12) and (14) that $\epsilon/\gamma = R_J/(r + R_J)$ so that this ratio never gets too big: $0 < \epsilon/\gamma \leq 1$. Moreover, for large ω (i.e., the limit of large bias current I_b) the transition curve always approaches the line $\omega = \omega_0$. Thus, in each of the three weak coupling limits ($\epsilon \to 0$, $\gamma \to \infty$ or $\omega_0 \to \infty$) our averaging agrees with the results of Mirollo and Strogatz obtained for $N \to \infty$.

IV. DISCUSSION

Our main result is the derivation of the averaged system (32) from the original Eqs. (2) and (3). Watanabe and Strogatz have shown that, while the original equations are unusually tractable owing to the existence of a great many constants of motion, the averaged equations are completely solvable [4]. In this paper we derived an explicit formula for the coupling-phase δ, which is the key parameter governing the stability of both the in-phase and splay states.

An advantage of the averaged equations is that they admit a fairly direct physical interpretation of the main features of the array dynamics. Usually, for a series LRC combination one identifies two resonance frequencies, which we can call the natural resonance frequency ω_0 and the shifted resonance frequency ω_1. In terms of our dimensionless units, we have

$$\omega_1 = \omega_0 - \gamma^2/2, \quad (41)$$

where ω_0 and γ were defined earlier via Eqs. (15) and (14), respectively. In the presence of a periodic driving voltage at a frequency ω, the current oscillations (and thus the power dissipated in the load) are greatest when $\omega = \omega_0$, while the capacitor's charge oscillations are greatest when $\omega = \omega_1$. Of course, for "high-Q" circuits, the current- and charge-response curves are sharply peaked and ω_0 is very nearly equal to ω_1.

Now, we noted in the last section that the averaged Josephson array equations display a single dynamical transition at $\omega = \omega_0$. It follows that *the maximum power delivered to the LRC load is attained at the transition point.* We also recover an old result found in the Josephson array literature [2,12], namely, that stable in-phase operation of a series array requires that the load "looks inductive:" an *LRC* load is said to look inductive if its impedance has a positive imaginary part, which is equivalent to the condition $\omega > \omega_0$. We note that the averaged equations admit an analogous stability principle for the splay state: the splay state is stable if the load impedance has a negative imaginary part (i.e., if the load looks capacitive).

ACKNOWLEDGMENTS

We would like to thank Steve Strogatz for helpful comments on this paper. K.W. is supported by the Office of Naval Research under Contract No. N00014-91-J-1257. J.W.S is supported by an Organized Research Grant from Northern Arizona University.

[1] C.A. Hamilton, F.L. Lloyd, K. Chieh, and W.C. Goeke, IEEE Trans. Instrum. Meas. **38** 314 (1989); R.L. Steiner and B.F. Field, *ibid.* **38** 296 (1989).
[2] A.K. Jain, K.K. Likharev, J.E. Lukens, and J.E. Sauvageau, Phys. Rep. **109**, 310 (1984).
[3] S. Wahlstein, S. Rudner, and T. Claeson, J. Appl. Phys. **49**, 4248 (1978).
[4] S. Watanabe and S.H. Strogatz, Physica D **74**, 197 (1994).
[5] J.W. Swift, S.H. Strogatz, and K. Wiesenfeld Physica D **55**, 239 (1992).
[6] S.H. Strogatz and R.E. Mirollo, Phys. Rev. E **47**, 220 (1993).
[7] S. Nichols and K. Wiesenfeld, Phys. Rev. A **45**, 8430 (1992).
[8] P. Ashwin and J.W. Swift, J. Nonlinear Sci. **2,** 69 (1992).
[9] P. Ashwin and J.W. Swift, J. Nonlinear Sci. **3**, 459 (1993).
[10] P. Hadley and M.R. Beasley, J. Appl. Phys. **50**, 621 (1987).
[11] K.Y. Tsang and I.B. Schwartz, Phys. Rev. Lett. **68**, 2265 (1992).
[12] P. Hadley, Ph.D. thesis, Stanford University, 1989.