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Corner spontaneous magnetization

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The spontaneous magnetization of a corner spin on a square planar Ising ferromagnet with free boundary conditions is obtained exactly, confirming conformal predictions.

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About a decade ago, conformal field theoretical ideas [1] were applied to predict the critical behavior of the spontaneous magnetization m_c of a spin in the corner of a wedge-shaped lattice with free boundaries (i.e., no applied fields or modified bonds) for systems in the same universality class as the planar Ising ferromagnet. For an opening angle of α, the result is [2] $m_c \propto t^{\pi/2\alpha}$ where $t=(T_c-T)/T_c$. This is obtained from the critical correlation between an apical spin and an edge spin as given conformally, followed by an application of scaling, With $\alpha=\pi$, the historically important exact result of McCoy and Wu [3] is recaptured. This was an exact result indicating that special critical behavior might be found at the boundaries of lattices.

The case of $\alpha=\pi/2$ has received much attention, none of it entirely successful, beginning with the work of Barber, Peschel, and Pearce [4]. The difficulty is that, although the spectrum of the free-edge transfer matrix is known [5], standard methods [6] reduce the correlation function between an apical spin and one in the edge a distance n away to an $n\times n$ determinant which, because of intrinsic lack of translational symmetry, does not have Töplitz structure [7], unlike in [3].

In [4], an alternative approach [8] was used, in the special case of the Hamiltonian limit, to generate equations for certain matrix elements which the authors did not solve. Subsequently, Kaiser and Peschel [9] conjectured an analytic expression for m_c by numerical analysis of these equations. In this work, we shall confirm this conjecture by an exact calculation, and also obtain the spontaneous magnetization $m_e(j)$ at any distance j along the edge from the corner, as well as the scaling function which interpolates between m_c and the edge magnetization $m_e(\infty)$.

Using standard transfer matrix ideas [6] [with spins $\sigma(m,n)=\pm 1$ at Cartesian coordinates (m,n) with $1\leqslant m\leqslant M$ and $1\leqslant n\leqslant N$ and vertical (horizontal) interactions K_1 (K_2) in units of kT], we have

$$m_e(j)=\langle\sigma(j,1)\sigma(j,N)\rangle=\frac{\langle 0|\sigma_j^x(V_2V_1)^{N-1}V_2\sigma_j^x|0\rangle}{\langle 0|(V_2V_1)^{N-1}V_2|0\rangle} \tag{1}$$

with

$$V_1=\exp\left(-K_1^*\sum_{j=1}^{M}\sigma_j^z\right),\quad V_2=\exp\left(K_2\sum_{j=1}^{M-1}\sigma_j^x\sigma_{j+1}^x\right), \tag{2}$$

where $e^{-2K_1^*}=\tanh K_1$ is the usual dual variable and $\sigma_j^z|0\rangle=-|0\rangle$ for any $1\leqslant j\leqslant M$. This is appropriate to sum over all states of a free edge with the implied equal weight. We shall work with the symmetrized transfer matrix $V'=V_1^{1/2}V_2V_1^{1/2}$, which can be written as

$$V'=\exp\left(-\sum_k \gamma(k)(X_k^\dagger X_k-\tfrac{1}{2})\right), \tag{3}$$

where the Fermi operators X_k are given by

$$X_k=\sum_{j=1}^{2M} y_{j,k}\Gamma_j \tag{4}$$

in terms of the spinors $\Gamma_{2j-1}=f_j^\dagger+f_j$, $\Gamma_{2j}=-i(f_j^\dagger-f_j)$ with $f_j^\dagger=P_{j-1}\sigma_j^+$ where $P_0=1$ and $P_j=\Pi_{k=1}^{j}(-\sigma_k^z)$ for $j\geqslant 1$. This last step is the Jordan-Wigner transformation to fermions $f_j^\dagger$ and f_j. Evidently $|0\rangle$ is the f_j vacuum. The function $\gamma(k)$ was given by Onsager [10]:

$$\cosh\gamma(k)=\cosh 2K_1^*\cosh 2K_2-\sinh 2K_1^*\sinh 2K_2\cos k \tag{5}$$

with $\gamma(k)\geqslant 0$ for $k\in\mathbb{R}$.

The values of $y_{j,k}$ are given from an eigenvalue problem [5] as

$$\begin{aligned} y_{2j,k}&=N(k)\sin[kj-\varphi_0(k)],\\ y_{2j-1,k}&=iN(k)\sin[kj-\varphi_1(k)], \end{aligned} \tag{6}$$

where $N(k)$ is taken real positive and fixed by $\|y\|_2=1/\sqrt{2}$. The phase angles are

$$e^{i\varphi_0(k)}=\left(\frac{e^{-ik}-A^{-1}}{e^{ik}-A^{-1}}\right)^{1/2},\quad e^{i\varphi_1(k)}=-\left(\frac{e^{-ik}-B}{e^{ik}-B}\right)^{1/2}, \tag{7}$$

with $\varphi_0(0)=0$ and $\varphi_1(0)=\pi$ for $B>1$, where $A=\exp[2(K_1+K_2^*)]$ and $B=\exp[2(K_1-K_2^*)]$. Finally, the wave numbers are quantized on the finite lattice by

$$e^{iMk}=-i\alpha(k)e^{i\delta^*(k)}, \tag{8}$$

where $\delta^*(k)=\pi+\varphi_0(k)+\varphi_1(k)-k$ is an element of Onsager's hyperbolic triangle and $\alpha(k)=\pm i$, this being related to reflection invariance by $y_{2M-2j,k}=-\alpha(k)y_{2j+1,k}$. It is crucial to note that with $B>1$ ($T<T_c$), (8) has an imaginary wave number solution with $\alpha=i$ given by $e^{ik}=B^{-1}+O(B^{-M})$, for which $\gamma(k)=O(B^{-M})$, evidently a

source of asymptotic degeneracy in the spectrum of V'. The operator for this mode is denoted X_c and the associated $y_{j,c}$ are (to order B^{-M})

$$y_{2j-1,c}=i(e^{2v_0}-1)^{1/2}e^{-jv_0},\quad y_{2j,c}=(1-e^{-2v_0})^{1/2}e^{-(M-j)v_0}, \tag{9}$$

with $v_0=\ln B$. We find that

$$m_e(j)=e^{K_1^*}\left|\lim_{M\to\infty}\frac{\langle 0|f_jX_c^\dagger|\Phi\rangle}{\langle 0|\Phi\rangle}\right|, \tag{10}$$

where $|\Phi\rangle$ is the X_k vacuum and maximum eigenvector of V'. To derive this result, note that both $|0\rangle$ and $|\Phi\rangle$ are eigenvectors of P_M with eigenvalue 1.

The matrix element is determined by noting that $\langle 0|f_j^\dagger X_c^\dagger|\Phi\rangle=0$ because $|0\rangle$ is the f_j vacuum. Inverting (4) by using the canonicality of the transform gives $f_j^\dagger$ as a linear combination of X_k and $X_k^\dagger$, including X_c which anticommutes with $X_c^\dagger$ leaving $\langle 0|\Phi\rangle$. We get for $1\leq j\leq M$

$$\frac{1}{2}\sum_{\substack{k\in(-\pi,\pi)\\ \alpha(k)=-i}} N(k)^2e^{ikj}K(k)=i(e^{2v_0}-1)^{1/2}\times(e^{-v_0j}-e^{-v_0}e^{-(M-j)v_0}), \tag{11}$$

where

$$K(k)=\frac{e^{-i\varphi_0(k)}+e^{-i\varphi_1(k)}}{N(k)}\frac{\langle 0|X_k^\dagger X_c^\dagger|\Phi\rangle}{\langle 0|\Phi\rangle}. \tag{12}$$

Multiplying by e^{-iqj}, with $e^{iMq}=-i\alpha(q)e^{i\delta^*(q)}$, $\alpha(q)=\pm i$, summing over $j=1,\ldots,M$, and finally taking $M\to\infty$ gives two equations for K:

$$\frac{P}{\pi}\int_{-\pi}^{\pi}dk\frac{e^{i(k-q)}}{e^{i(k-q)}-1}(1+e^{i[\delta^*(k)-\delta^*(q)]})K(k)=-4i(e^{2v_0}-1)^{1/2}\left(\frac{B^{-1}}{e^{iq}-B^{-1}}+\frac{e^{-i\delta^*(q)}}{e^{iq}-B}\right), \tag{13}$$

$$2K(q)+\frac{P}{\pi}\int_{-\pi}^{\pi}dk\frac{e^{i(k-q)}}{e^{i(k-q)}-1}(1-e^{i[\delta^*(k)-\delta^*(q)]})K(k)=-4i(e^{2v_0}-1)^{1/2}\left(\frac{B^{-1}}{e^{iq}-B^{-1}}-\frac{e^{-i\delta^*(q)}}{e^{iq}-B}\right). \tag{14}$$

The integral operator on the left-hand side (lhs) of (13) was encountered by Yang [11] in his derivation of the bulk spontaneous magnetization. Yang's equation has $\delta'(k)=\varphi_0(k)-\varphi_1(k)$ in place of $\delta^*(k)$, but this change involves replacing B by B^{-1}. The spectrum of Yang's operator for $T>T_c$ [11] allows inversion of (13) for $T<T_c$ using the theory of Jacobi elliptic functions [12], to give

$$K(k)=-4i\frac{e^{K_2^*}(2\sinh 2K_1)^{1/2}\sinh(v_0/2)(1+e^{ik})}{[(e^{ik}-A)(e^{ik}-B)]^{1/2}(e^{ik}-B^{-1})}, \tag{15}$$

which may be checked by direct substitution. An alternative and more direct route to derive (15) is to take the difference of (14) and (13) giving

$$K(k)-(\mathscr{H}K)(k)=4i(1-e^{-2v_0})^{1/2}(e^{ik}-e^{-v_0})^{-1}, \tag{16}$$

where $\mathscr{H}$ denotes the Hilbert transform. This means that

$$g(e^{ik})=K(k)-2i(1-e^{-2v_0})^{1/2}(e^{ik}-e^{-v_0})^{-1} \tag{17}$$

has no singularity inside the unit circle. We now recall that $y_{j,k}$ is an odd function of k, which gives a symmetry requirement for the matrix element leading to

$$K(-k)=e^{i[k+\delta^*(k)]}K(k). \tag{18}$$

Substituting (17) into this equation gives a relation between $g(z^{-1})$ and $g(z)$

$$z\theta(z)g(z)-\theta(z^{-1})g(z^{-1})=-\frac{Bz\theta(z^{-1})}{z-B}-\frac{z\theta(z)}{z-B^{-1}}, \tag{19}$$

where

$$\theta(z)=\left(\frac{z-A}{z-B}\right)^{1/2}\quad\text{so that}\quad e^{i\delta^*(k)}=\frac{\theta(e^{ik})}{\theta(e^{-ik})}. \tag{20}$$

Using a Wiener-Hopf technique on the right-hand side (rhs) of (19) allows us to identify each of the two terms on the lhs up to some constant, since the first term is analytic for z inside the unit circle and the second one is analytic for z outside. The constant is obtained by imposing that $g(z)$ has no pole in $z=0$.

Applying the inversion of (4) to the magnetization formula (10) gives

$$m_e(j)=m_e-e^{K_1^*+K_2^*}(2\sinh 2K_1^*)^{1/2}\sinh(v_0/2)I(j), \tag{21}$$

where

$$I(j)=\frac{1}{\pi}\int_{A^{-1}}^{B^{-1}}dx\frac{x^{j-1}}{1-x}\left(\frac{x-A^{-1}}{B^{-1}-x}\right)^{1/2} \tag{22}$$

and [3] $m_e=e^{K_1^*}[\sinh 2(K_2-K_1^*)/\sinh 2K_2]^{1/2}$.

For $j=1$ the integral is elementary and gives

$$m_c=e^{K_2^*}(e^{4K_1^*}-1)^{1/2}\sinh(v_0/2) \tag{23}$$

as conjectured in [9]. For large j, we have

$$m_e(j) \sim m_e - e^{K_1^*} e^{-v_0(j-1)} (\pi j \sinh 2K_1 \sinh 2K_2)^{-1/2} \quad (24)$$

in accordance with a simple bubble picture of the generic dependence of magnetization [13,14]: the edge magnetization near a corner (but sufficiently far away on a scale of the correlation length) deviates from the $j \to \infty$ because lines which separate oppositely magnetized phases and which surround the point $(j,0)$ can intercept the vertical line $x=0$. Assuming such bubbles have no overhangs with respect to the $(0,1)$ direction and the solid-on-solid weight $\exp(-bL_\perp + ax)$ where $L_\perp$ is the length of the vertical lines (assumed continuous), then a straightforward transfer-integral calculation gives

$$\frac{m_e(j)}{m_e} = 1 - e^{-ax} \int_0^\infty dy \, \frac{2}{\pi} \int d\omega \, \sin\theta(\omega) \sin[\theta(\omega) + \omega y] \times \left(\frac{2b}{b^2+\omega^2} \right)^j , \quad (25)$$

where $\tan\theta(\omega) = \omega/b$.

For j large, this gives the asymptotic contribution

$$m_e(j) = m_e \left(1 - \frac{1}{2\sqrt{\pi x}} e^{-[a - \ln(2/b)]x} \right) \quad (26)$$

in qualitative agreement with the exact result.

Finally, we have the scaling function

$$F(x) = s - \lim \frac{m_e(x/v_0)}{m_e} = \frac{2}{\pi} \int_0^\infty \frac{1 - e^{-x(1+u^2)}}{1+u^2} du. \quad (27)$$

The small x behavior is $F(x) \sim 2(x/\pi)^{1/2}$ which means that a correction to scaling contributes to the corner magnetization near T_c.

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