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Moving solitons in the damped Ablowitz-Ladik model driven by a standing wave

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We predict theoretically that, via a resonance mechanism, stable moving solitons exist in a discrete (1+1)-dimensional nonlinear Schrödinger (Ablowitz-Ladik) equation with dissipation and an ac driving term in the form of a standing wave. Agreement between the predicted threshold (minimum) values of the strength of the drive which is able to sustain the moving solitons and those measured in direct numerical simulations is excellent. Our results show an example of multistability in damped, standing-wave-driven systems. The dynamical instability for the motion of solitons in the unstable regimes is also analyzed.

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As is well known, solitons play an important role as nonlinear collective excitations in various physical systems, especially for those which can be considered as effectively one dimensional. In many cases, solitons may be relatively easily created by application of a sufficiently strong localized initial disturbance. However, in any real physical system solitons are attenuated by dissipation. It is necessary to devise special means to sustain a soliton in the dissipative medium. A simple example is the driven magnetic flux quantum (fluxon) in a damped long Josephson junction [1]. A dc bias current uniformly distributed along the junction gives rise to a constant driving force which compensates the friction and forces the fluxon to move with an equilibrium velocity. It is well documented that an ac drive may support nontranslating envelope solitons [2] or breathers [3] in damped continuum systems. Generally, *translation* of a soliton cannot be sustained by a spatially uniform ac drive in homogeneous continuum systems. However, it is possible to sustain a translating soliton by a *traveling* component of the external field, e.g., moving breathers in the sine-Gordon chain driven by a traveling wave [4], moving kinks driven by rotating magnetic field in an annular long Josephson junction [5], or driven by a traveling electromagnetic or acoustic wave in a charge-density-wave conductor [6].

Recently, a great deal of attention has been attracted to solitons in discrete nonlinear systems, especially after the discovery of strongly nonlinear localized intrinsic modes in dynamical lattices [7]. In Ref. [8], it was proposed to drive solitons in discrete systems by an ac external field with the following mechanism: a soliton moving in a lattice with spacing a at a mean velocity V passes sites of the lattice periodically with the frequency $2\pi V/a$. If ω is the frequency of the external ac field, the moving soliton may be in resonance with the ac field provided that

$$V=\frac{a}{2\pi}\,\omega \tag{1}$$

for the lowest order of the resonance. This idea was developed [8] for the lattice models without internal degrees of freedom, i.e., described by a single-component real variable. As one of the specific examples, the Toda lattice (TL) model was employed. The analysis there was based only on considerations of energy balance for the translating soliton. However, this, in general, is insufficient to guarantee that the predicted motion of a soliton will exist because the momentum balance condition must also hold [9]. The TL model is known in two different forms which are equivalent in the absence of perturbing terms: the usual form and the so-called dual one. While the energy balance analyzed in Ref. [8] pertains to both, it is only the dual form that makes the additional consideration of the momentum balance unnecessary, as in this formulation of the TL the momentum of a soliton is always zero [9]. Thus, the effect predicted in Ref. [8] can occur only in the dual version of the damped ac-driven TL. This was confirmed in direct numerical experiments [10] and was observed experimentally in an electric transmission line which is described by the dual TL model [11]. In the following, we will show that, for a damped ac-driven nonlinear Schrödinger (NLS) equation with the Ablowitz-Ladik discretization, this simple resonance mechanism fails to support the motion of a soliton precisely due to the violation of momentum balance. Furthermore, we will consider drivings of the standing-wave type for which a different resonance mechanism exists to support a moving envelope soliton in the damped discrete system. The motivation of this study is to understand the energy transfer mechanism which may be applicable to various one-dimensional physical systems. Also, this problem is of obvious general interest as a new dynamical effect.

We consider the Ablowitz-Ladik (AL) system [12] with additional terms accounting for dissipation and external ac drive,

$$\begin{aligned} i\dot{\psi}_n &= -(\psi_{n+1}+\psi_{n-1})(1+|\psi_n|^2)-i\Gamma\psi_n \\ &\quad +\epsilon\cos(k_m n)e^{-i\Omega t}\,, \end{aligned} \tag{2}$$

subject to the periodic boundary condition $\psi_n \equiv \psi_{N+n}$ with n the site index. Here Γ is the dissipation coefficient, while Ω and ϵ are the frequency and amplitude of the ac drive, respectively. The wave number k_m of the drive is determined by the periodic boundary condition, $k_m=(2\pi m)/N$, with m

being an integer, $-N/2 \leq m \leq N/2$. Notice that the driving term in Eq. (2) has the form of a *standing* wave. The principal difference between this system and that considered in Refs. [5] and [6] is that a soliton here has internal degrees of freedom.

In the absence of the perturbations, the AL system is completely integrable [12]. It has the one-soliton solution,

$$\psi_n = \sinh\beta \, \text{sech}\{\beta[n - X(t)]\} e^{-i(\omega t - \alpha n + \sigma)}, \quad (3)$$

where $X(t) = Vt + \xi$, and the soliton's frequency ω and velocity V are expressed in terms of two real parameters $\alpha \in [-\pi, \pi]$ and $\beta \in (0, +\infty)$:

$$\omega = -2 \cosh\beta \cos\alpha, \quad (4)$$

$$V = 2 \sin\alpha \frac{\sinh\beta}{\beta}, \quad (5)$$

ξ and σ being trivial parameters describing the initial position and phase of the soliton.

The simplest analytical technique which allows us to attack the problem, provided that the perturbing terms are small enough, is based on the so-called balance equations for the quantities which are integrals of motion in the unperturbed system [3]. Although the unperturbed AL system for the infinite lattice has an infinite number of conservation laws, here we will explicitly utilize the conservation of the energy, E, and momentum P,

$$E = -\sum_n \psi_n \psi_{n+1}^* + \psi_n^* \psi_{n+1}, \quad (6)$$

$$P = i \sum_n \psi_n \psi_{n+1}^* - \psi_n^* \psi_{n+1}. \quad (7)$$

For the one-soliton solution, we have

$$E = -4 \sinh\beta \cos\alpha = 2\omega \tanh\beta, \quad (8)$$

$$P = 4 \sinh\beta \sin\alpha = 2\beta V. \quad (9)$$

Notice that, for $k_m \equiv 0$, the system (2) is the damped AL system driven by a spatially uniform ac force. It is easy to show that, for given β, α exists such that the velocity of the soliton and the resonant frequency of the drive (i.e., $\Omega = \omega$) satisfy the resonance condition, Eq. (1) (where $a = 1$). However, the momentum for this purely ac-driven system decays according to $\dot{P} = -2\Gamma P$. Although, by increasing the driving strength ϵ, a sufficient amount of energy can be pumped into the system to sustain the amplitude of the soliton, the decay of the momentum [see Eq. (9)] leads to the decay of the velocity of the soliton. Thus the resonance mechanism summarized by Eq. (1) is not sufficient to support the translating motion of the purely ac-driven soliton.

Next we turn to the case of general k_m. If the uniformly translating motion of the soliton can be supported by the drive, it is necessary to have balance conditions for the energy and momentum. By assuming that the form of the soliton solution (3) is robust under the perturbation and the excited oscillating background is small, we have

$$\langle \dot{E} \rangle = -2\Gamma \langle E \rangle - 4\epsilon (\sinh\beta)(\cos k_m) \, I_E, \quad (10)$$

$$\langle \dot{P} \rangle = -2\Gamma \langle P \rangle + 4\epsilon (\sinh\beta)(\sin k_m) \, I_P, \quad (11)$$

where

$$I_E = \left\langle \int_{-\infty}^{+\infty} dx \, \text{sech}(\beta x) \cos(k_m y) \sin[(\omega - \Omega)t - \alpha y + \sigma] \left[1 + 2\sum_{s=1}^{\infty} \cos(2\pi s y) \right] \right\rangle,$$

$$I_P = \left\langle \int_{-\infty}^{+\infty} dx \, \text{sech}(\beta x) \sin(k_m y) \cos[(\omega - \Omega)t - \alpha y + \sigma] \left[1 + 2\sum_{s=1}^{\infty} \cos(2\pi s y) \right] \right\rangle,$$

with $y = x + Vt$, $\langle \ \rangle$ being the time average. In the following we ignore the higher resonances with $s \geq 1$. It can be shown that, if the resonance condition

$$(\omega - \Omega) = (\alpha \pm k_m) V \quad (12)$$

does not hold, I_E and I_P vanish. From Eqs. (10) and (11) it follows that the motion of the soliton cannot be sustained. However, if the resonance condition holds, we have $I_E = I_P = [(\pi \sin\sigma)/(2\beta)] \text{sech}[(\alpha \pm k_m)\pi/(2\beta)]$ for $\alpha \neq 0$. Thus the balances of energy and momentum, i.e., $\langle \dot{E} \rangle = 0$ and $\langle \dot{P} \rangle = 0$ *both* lead to

$$\epsilon = \frac{4\beta\Gamma}{\pi \sin\sigma} \cosh\left[\frac{(\alpha \pm k_m)\pi}{2\beta} \right]. \quad (13)$$

Therefore, the threshold for driving strength above which a stable moving soliton exists is

$$\epsilon_{\text{thr}} = \frac{4\beta}{\pi} \Gamma, \quad (14)$$

when the resonance condition (12) is

$$\omega = \Omega, \qquad \alpha = \pm k_m \quad (15)$$

and the corresponding phase delay is $\sigma=\pi/2$. For $\alpha=0$, following the same steps as the above analysis, using the energy balance condition, leads to the threshold for the stationary soliton [13]

$$\epsilon_{\mathrm{thr},\alpha=0}=\frac{2\beta}{\pi}\Gamma, \tag{16}$$

which is half of the threshold estimate for a moving soliton. This can easily be explained by the form of the drive. Because the external drive can be decomposed into left and right traveling waves for nonvanishing k_m, a moving soliton only resonates with one of the components. Whereas, for $k_m=0$, the nonmoving soliton resonates with the whole drive. Hence the factor of 2 between the thresholds. The resonance condition (15) implies that the *phase velocity* of the carrier wave of the soliton is equal to the phase velocity of one of the two traveling wave components of the drive while the velocity of the soliton is determined by Eq. (5). This resonance mechanism is, obviously, different from the one described by Eq. (1). In the limit $\beta\to 0$, the AL equation approaches the continuum NLS equation. It is easy to check that, in this limit, the threshold estimate (16) goes over to the familiar expression for the threshold amplitude of the ac drive capable of supporting a soliton in the damped NLS equation [2].

Under the resonance condition (15), in the regime where $2\pi/(k_m V)\gg(2\pi)/\omega$, and $2\pi/k_m\gg 2/\beta$, the soliton can be viewed as a damped point particle in a slowly varying external potential. The analysis above has then to be modified. In this regime an adiabatic approximation is valid, leading to

$$\dot{E}\approx -2\Gamma E+\epsilon E\cos[k_m X(t)]\sin[k_m X(t)-\sigma]\times\sum_n \operatorname{sech}\{\beta[n-X(t)]\},$$

$$\dot{P}\approx -2\Gamma P+\epsilon P\sin[k_m X(t)]\cos[k_m X(t)-\sigma]\times\sum_n \operatorname{sech}\{\beta[n-X(t)]\},$$

where $X(t)$ is the slowly varying position of the soliton. Approaching the points $X(t)$ such that $k_m X(t)=l\pi$, l being an integer, the translating motion of the soliton is damped and the energy of the soliton is drained. This indicates that, for the parameters in this regime, it is impossible for the drive to lock a translating soliton motion. Note that this regime corresponds to the small k_m regime. The reason for this absence of stable solitons is similar to the one observed for breathers in the spatially nonuniformly driven sine-Gordon (SG) systems [14].

We have so far not considered the effect of the excited oscillating background. It is straightforward to show that, by linearizing Eq. (2), under the resonance condition (15), a spatially extended wave is excited:

$$\psi_n=A\cos(k_m n)\exp(-i\omega t) \tag{17}$$

with the magnitude of the amplitude,

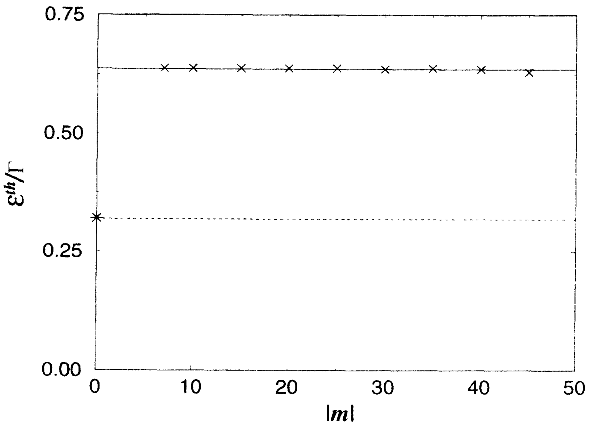

FIG. 1. Thresholds scaled by Γ for different values of m. The solid line is the theoretical prediction for $m\neq 0$. For comparison we have also plotted (dotted line) the theoretical prediction for $m=0$. The crosses are the thresholds measured in numerical simulations for $m\neq 0$; and the star, for $m=0$ (see text).

$$|A|^2=\frac{\epsilon^2}{4(1-\cosh\beta)^2\cos^2 k_m+\Gamma^2}. \tag{18}$$

For $k_m\sim\pi/2$, with the driving strength ϵ at the threshold (14), the background amplitude is of the order of β which is about as large as the amplitude of a soliton for, e.g., $\beta\sim 1$. Therefore, in this regime, i.e., $k_m\sim\pi/2$, it can be expected that a translating soliton is strongly coupled with the background wave and will be swamped by a large background excitation.

We have performed direct numerical simulations of Eq. (2) with periodic boundary conditions. We chose the parameter $\Gamma=0.005$, and the length of the system $N=200$. As for the initial state, the soliton form (3) was used with $\beta=0.5$, $\sigma=\pi/2$. It is easy to confirm that the system (2) has the following symmetries: (i) it is invariant under $k_m\leftrightarrow -k_m$, and (ii) the time evolution of Eq. (2) for $0<k_m<\pi/2$ is the same as that for $k_m'=\pi-k_m$, modulo a constant phase, for the initial condition (3) with the parameters chosen under the resonance condition (15). Thus, for our simulations, it is only necessary to consider $0\leqslant m\leqslant 50$, where m is the integer which determines the wave number k_m. Our numerical results have confirmed the features predicted in our analysis and are summarized as follows: (a) For $7\leqslant m\leqslant 45$, the steady translating solitons riding on the background of a spatially extended wave (17) were observed. A typical set of threshold measured in the simulations for the driving strengths for different values of m is shown in Fig. 1, along with the theoretical predictions given by Eq. (16) for $m=0$ (the nontranslating soliton), and by Eq. (14) for $m\neq 0$. To obtain this set of data, we adjusted the value of α and the driving frequency Ω to each value of m so that the resonance condition (15) is satisfied for a constant $\beta=0.5$. For all cases, the final velocity of the soliton at $t=1000$ (time units) and the initial velocity were measured and compared with the theoretical value of Eq. (5). They differ by less than 3% from the theoretical values of the velocity. Clearly, the translating motion is well maintained. As can be seen, the agreement between the measured thresholds and the theoretical predictions are excellent, with the relative error in all the cases less than 0.2% except for $m=45$ where it is 1%. (b) To demonstrate that this locking phenomenon is indeed a resonance with the external drive, we started with a soliton with α,ω $(=\Omega)$ tuned to the resonance condition (15) for $m=30$, the threshold was found to be ϵ=0.003 18. Now keeping all the parameters for the initial soliton and the external driving frequency fixed except for k_m, which is changed from k_{30} to k_{31} or to k_{29} in the external drive, we have observed that the soliton still decays into the oscillating background, even by increasing ϵ to 0.01. This example clearly confirms that the resonance mechanism is working. (c) For small m, i.e., $1\leq m\leq 6$, as predicted above we have failed to lock the translating motion of a soliton. (d) For $45<m\leq 50$, where $k_m\sim\pi/2$, the background oscillations (17) were excited according to Eq. (18), and their amplitudes become comparable to the amplitude of the soliton, so that it becomes hard to identify any localized object as the vestige of the soliton.

In conclusion, we have analyzed the motion of solitons in the standing-wave-driven, damped discrete (1+1)-dimensional NLS equation in the Ablowitz-Ladik discretization for different stability regimes. The theoretical prediction for the threshold for the driving strength, above which the steady motion of a soliton exits, is in excellent agreement with the numerical results. The general stability features were confirmed by the numerical simulations. It can be expected that a similar mechanism should be operating in the corresponding damped continuum NLS equations driven by a standing wave. We note that this resonance phenomenon is also exhibited in the system considered in Ref. [4], where the governing equation for the perturbed SG chain is

$$U_{tt}-U_{xx}+\sin U=-\Gamma U_t+\epsilon\cos(\Omega t-k_n x). \qquad (19)$$

Here $k_n=2\pi n/L$, with L being the length of the system. In the low-amplitude, NLS limit, considering the scaling regime in which, $\delta\ll 1$, $X=\delta x$, $T=(\delta^2/2)t$, $\Gamma=\delta^2\gamma$, and $\epsilon=4\delta^3\tilde{\epsilon}$, for $U=2\delta\phi(X,T)\exp(-it)$+c.c., we have $i\phi_T+\phi_{XX}+2|\phi|^2\phi=-i\gamma\phi-\tilde{\epsilon}(X,T)$, where $\tilde{\epsilon}(X,T)=\tilde{\epsilon}\exp\{i[2(1-\Omega)\delta^{-2}T+k_n\delta^{-1}X]\}$. According to the resonance mechanism discussed above, a resonating NLS soliton has the velocity $v=(2k_n)\delta^{-1}$, and frequency $\omega=2(\Omega-1)\delta^{-2}$. Consequently, the corresponding resonating small-amplitude SG breather has the frequency Ω and velocity $V=k_n$ in the original coordinates (x,t):

$$U_b=4\eta\,\mathrm{sech}[\eta(x-k_n t)]\cos(\Omega t-k_n x), \qquad (20)$$

where $\eta=\sqrt{2(1-\Omega)+k_n^2}$. This is precisely the breather which was generated by the drive in numerical simulations [4]. Thus the resonance mechanism here further clarifies the velocity selection mechanism for the moving breather obtained in Ref. [4].

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[13] If the higher harmonics $s\geq 1$ are taken into account, the threshold for $\alpha=0$ is more precisely expressed as $\epsilon_{\mathrm{thr},\alpha=0}=(2\beta\Gamma)/S$, where $S\equiv\beta\Sigma_n\mathrm{sech}(\beta n)$. Note that $\lim_{\beta\to 0}S=\pi$. For the parameter values of β which we use, Eq. (16) is an excellent approximation, e.g., for $\beta=0.5$, $|S-\pi|/\pi\sim 10^{-8}$.

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where

$$I_E=\left\langle \int_{-\infty}^{+\infty} dx\ \mathrm{sech}(\beta x)\cos(k_m y)\sin[(\omega-\Omega)t-\alpha y+\sigma]\left[1+2\sum_{s=1}^{\infty}\cos(2\pi s y)\right]\right\rangle,$$

$$I_P=\left\langle \int_{-\infty}^{+\infty} dx\ \mathrm{sech}(\beta x)\sin(k_m y)\cos[(\omega-\Omega)t-\alpha y+\sigma]\left[1+2\sum_{s=1}^{\infty}\cos(2\pi s y)\right]\right\rangle,$$

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does not hold, I_E and I_P vanish. From Eqs. (10) and (11) it follows that the motion of the soliton cannot be sustained. However, if the resonance condition holds, we have $I_E=I_P=[(\pi\sin\sigma)/(2\beta)]\mathrm{sech}[(\alpha\pm k_m)\pi/(2\beta)]$ for $\alpha\neq0$. Thus the balances of energy and momentum, i.e., $\langle\dot{E}\rangle=0$ and $\langle\dot{P}\rangle=0$ *both* lead to

$$\epsilon=\frac{4\beta\Gamma}{\pi\sin\sigma}\cosh\left[\frac{(\alpha\pm k_m)\pi}{2\beta}\right]. \quad (13)$$

Therefore, the threshold for driving strength above which a stable moving soliton exists is

$$\epsilon_{\mathrm{thr}}=\frac{4\beta}{\pi}\Gamma, \quad (14)$$

when the resonance condition (12) is

$$\omega=\Omega, \qquad \alpha=\pm k_m \quad (15)$$