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Hydrodynamic stability of compressible plane Couette flow

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The evolution of two-dimensional spatial Fourier harmonics in a compressible plane Couette flow is considered. A new mechanism of energy exchange between the mean flow and sound-type perturbations is discovered.

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Usually the stability of hydrodynamic flows is studied by common modal analyses—perturbed quantities are expanded in full Fourier integrals. The modes are assumed to have an exponential time behavior, and if an imaginary part of a frequency is positive, the modes will grow in time, leading (in the sense of the classical linear theory) to an instability. However, this conclusion is correct for the operators with orthogonal eigenfunctions (normal modes). But the operator arising in the Couette flow is exponentially far from normal [1,2] and the eigenfunctions are not orthogonal to each other. Therefore, in accordance with linear algebraic theory [2], the linear system may be amplified by arbitrarily large factors, even if the imaginary parts of all eigenfrequencies are negative. Hence, the usage of the full spectral (Fourier or Laplace) expansion may be misleading in this context.

The nonmodal approach to the studying of the hydrodynamic instabilities in free shear flows, originated from Lord Kelvin [3], becomes recently well established and extensively used [2,4–7]. In the Kelvin formalism one considers the temporal evolution of *spatial Fourier harmonics* ("Kelvin modes" [4]) of the perturbations without any spectral expansion in time. The wave number of each spatial Fourier harmonic (SFH) along the flow shear varies in time: in the linear approximation there exists a "drift" of SFH in the plane of wave numbers (**k**-plane) [4–6]. The method establishes itself as an effective and convenient "tool" in the study of the wide range of physical processes occurring in shear flows including anomalous processes of energy exchange between the mean flow and the perturbations—the shear energy extraction by SFH in different kinds of hydrodynamic and magnetohydrodynamic (MHD) shear flows [4–6].

Until now, this approach was predominantly applied to the study of incompressible flows. In the present paper we consider one rather simple kind of *compressible* shear flow—the two-dimensional (2D) Couette flow of the continuous, compressible medium. It is well known [8] that in such a case the medium between the planes should move with the velocity $\vec{U}_0=(U_{0x}=Ay;0;0)$. Here we choose the direction of the X axis along the regular velocity vector, while the Y axis is directed along the shear. Note that the constant positive parameter $A\equiv V_0/L>0$, where V_0 is the velocity of the moving plane and L is the distance between the planes [8].

The basic system of linearized equations governing the evolution of the small-scale perturbations in this flow is

$$(\partial_t+Ay\partial_x)d+\partial_x u_x+\partial_y u_y=0, \tag{1}$$

$$(\partial_t+Ay\partial_x)u_x+Au_y=-c_s^2\partial_x d, \tag{2}$$

$$(\partial_t+Ay\partial_x)u_y=-c_s^2\partial_y d, \tag{3}$$

where $d\equiv\rho'/\rho_0$. Note that in Eqs. (2) and (3) we have used the polytropic equation of state $p=K\rho^\gamma$ to express the pressure perturbation by means of the density perturbation. Making the following substitution of variables [5], $x_1=x-Ayt$, $y_1=y$, $t_1=t$, we can rewrite Eqs. (1)–(3) in the following form:

$$\partial_{t_1}d+\partial_{x_1}u_x+(\partial_{y_1}-At_1\partial_{x_1})u_y=0, \tag{4}$$

$$\partial_{t_1}u_x+Au_y=-c_s^2\partial_{x_1}d, \tag{5}$$

$$\partial_{t_1}u_y=-c_s^2(\partial_{y_1}-At_1\partial_{x_1})d. \tag{6}$$

The coefficients of the initial system were spatially inhomogeneous—they depend on the spatial coordinate y. In new variables this inhomogeneity turns into a temporal inhomogeneity. This circumstance allows us to perform the Fourier analyses of (4)–(6), expanding unknown functions with respect to *only* spatial variables x_1 and y_1,

$$\left\{\begin{array}{c} u_x \\ u_y \\ d \end{array}\right\} = \int dk_{x_1}dk_{y_1}\left\{\begin{array}{c} \hat{u}_x(k_{x_1},k_{y_1},t_1) \\ \hat{u}_y(k_{x_1},k_{y_1},t_1) \\ \hat{d}(k_{x_1},k_{y_1},t_1) \end{array}\right\} \times\exp[i(k_{x_1}x_1+k_{y_1}y_1)], \tag{7}$$

and reduce these equations to the following ones:

$$D^{(1)}=v_x+\beta(\tau)v_y, \tag{8}$$

$$v_x^{(1)}=-Rv_y-D, \tag{9}$$

$$v_y^{(1)}=-\beta(\tau)D, \tag{10}$$

where hereafter $F^{(n)}$ will denote the nth order time derivative of F and $D\equiv i\hat{d}$, $R\equiv A/(c_s k_x)$, $\tau\equiv c_s k_x t_1$, $\beta(\tau)\equiv k_y(0)/k_x-R\tau\equiv\beta_0-R\tau$, $v_x\equiv\hat{u}_x/c_s$, and $v_y\equiv\hat{u}_y/c_s$.

Note that the wave number of a SFH along the flow shear $[k_y(\tau)\equiv k_y(0)-Rk_x\tau]$ varies in time. This process of the linear drift of SFH in the **k** space below will be referred as "the linear drift."

Evaluating the expression for the R parameter we find that $R=(V_0/c_s)(l_x/L)$. Since we are considering only small-scale perturbations $(l_x\equiv 1/k_x\ll L)$, it is clear that if we consider the *subsonic* flow $(V_0<c_s)$, then $R\ll 1$.

It is interesting to note that (8)–(10) may be rearranged in such a way as to get an important *algebraic* relation between the perturbation functions: $v_y-\beta(\tau)v_x=RD+\text{const}$, where const is some constant of integration. Using this expression we are able to write [basing upon the system (8)–(10)] *one* second-order differential equation for $v_x(\tau)$,

$$v_x^{(2)}+\omega^2(\tau)v_x=-\beta(\tau)\times\text{const}, \tag{11}$$

and relations, expressing v_y and D through v_x and its first derivative $v_x^{(1)}$,

$$v_y=[\text{const}+\beta(\tau)v_x-Rv_x^{(1)}]/(1+R^2), \tag{12}$$

$$D=-\{R[\text{const}+\beta(\tau)v_x]+v_x^{(1)}\}/(1+R^2), \tag{13}$$

where $\omega^2(\tau)\equiv 1+\beta^2(\tau)=[k_x^2+k_y^2(\tau)]/k_x^2\equiv k^2(\tau)/k_x^2$.

The total energy density of the perturbations in the **k** space we define as

$$E\equiv(|v_x|^2+|v_y|^2)/2+|D|^2/2. \tag{14}$$

The crucial point of the solving process should be the solution of Eq. (11). Its general solution is the sum of the *special* solution of this equation and the *general* solution of the corresponding homogeneous equation

$$v_x^{(2)}+\omega^2(\tau)v_x=0. \tag{15}$$

This equation is rather well known in mathematical physics. It describes the linear oscillations of the mathematical pendulum with variable length [9]. The quantity $\omega(\tau)$ has the meaning of the angular frequency of the oscillations. Equation (15) is solved approximately when $\omega(\tau)$ depends on τ adiabatically [9]. Mathematically this condition may be written simply as $|\omega(\tau)^{(1)}|\ll\omega^2(\tau)$ or, taking into account the definition of $\omega(\tau)$, as

$$R|\beta(\tau)|\ll\omega^3(\tau)=[1+\beta^2(\tau)]^{3/2}. \tag{16}$$

For subsonic Couette flows, $R\ll 1$ and the condition (16) holds for all possible values of $|\beta(\tau)|$. In other words, since the temporal variability of $|\beta(\tau)|$ is determined by the "linear drift" of SFH, (16) is valid at all stages of the evolution of the SFH.

When condition (16) holds, the approximate solution of the homogeneous Eq. (15) may be written as

$$\hat{v}_x(\tau)=a(\tau)\sin[\varphi(\tau)+\varphi_0], \tag{17}$$

$$\varphi(\tau)=\int\omega(\tau)d\tau=-\frac{1}{2R}[\beta(\tau)\omega(\tau)+\ln|\beta(\tau)+\omega(\tau)|]. \tag{18}$$

Since not only $\omega(\tau)$ but also $\omega^{(1)}(\tau)$ varies adiabatically, the same is true for $a(\tau)$ and $a^{(1)}(\tau)$ [9]. That is why there exists an *adiabatic invariant* $a^2(\tau)\omega(\tau)=C$, and the amplitude $a(\tau)$ is simply expressed through the angular frequency $\omega(\tau)$ [9]. The C parameter manages, as we shall see below, the "weight" of the homogeneous solution in the general solution of Eq. (11).

The approximate *special* solution of the inhomogeneous equation (11) may also be derived owing to the smallness of the R parameter. In particular, the solution may be expressed by the following series [10]:

$$\bar{v}_x(\tau)=\text{const}\times\sum_{n=0}^{\infty}R^{2n}y_n(\tau), \tag{19a}$$

$$y_0(\tau)=-\beta(\tau)/\omega^2(\tau), \tag{19b}$$

$$y_n(\tau)=-\frac{1}{\omega^2(\tau)}\frac{\partial^2 y_{n-1}}{\partial\beta^2}. \tag{19c}$$

Since $R\ll 1$, the terms with higher powers of R are negligible and the general solution of the inhomogeneous equation (11) may be written approximately as

$$v_x(\tau)\equiv\hat{v}_x(\tau)+\bar{v}_x(\tau)$$
$$\approx\frac{C}{\sqrt{\omega(\tau)}}\sin[\varphi(\tau)+\varphi_0]-\frac{\beta(\tau)\times\text{const}}{\omega^2(\tau)}. \tag{20}$$

When $C/\text{const}\ll 1$ the SFH may be treated as a mainly incompressible and vortical perturbation, while when $C/\text{const}\gg 1$ it is mainly of the sound type. It must be noted that though (20) is the *approximate* solution, the actual accuracy of the solution is extraordinarily high. In purpose to check the accuracy we have performed a numerical solution of Eq. (11) and compare the results with the ones obtained through (20). The coincidence was excellent even for the $R=0.1$ case and was even better for more small R's. It means that the adiabatic approximation is fully relevant to the solution of the problem under consideration.

Having the expression for $v_x(\tau)$, we can certainly calculate its derivative $v_x^{(1)}(\tau)$ and then, in turn, we can find all characteristics of the perturbation by using Eqs. (12)–(14). The most interesting and important result appears for the total energy density of the perturbations $E(\tau)$. Using (14), and dropping the negligible terms, we get the following simple result:

$$E(\tau)\approx \frac{1}{2}\left[C\omega(\tau)+\left(\frac{\text{const}}{\omega(\tau)}\right)^2\right]. \tag{21}$$

According to (21) there exists an unusual nonexponential enhancement of the energy in the 2D Couette flow. When the perturbation is incompressible, $C=0$ and (21) reduces to the well-known expression, describing the "transient" growth of the energy of SFH [3,4,6]. For the sound-type ($C/\text{const}\gg 1$) perturbation, $E\sim\omega(\tau)=\sqrt{1+(\beta_0-R\tau)^2}$. Initially, for $k_y(0)k_x>0$ ($\beta_0>0$), at $0<\tau<\tau_*\equiv|\beta_0|/R$, the energy decreases and reaches its minimum at $\tau=\tau_*$. Afterwards, it begins to increase at $\tau_*<\tau<\infty$, when the SFH "emerges" into the area of **k** plane in which $k_y(\tau)k_x<0$ (the area may be called the "growth area" for the sound-type perturbations). If the SFH is in the growth area from the beginning ($\beta_0<0$), its energy increases monotonically. In the general case the "transient growth" and the "sound-type" evolution are superimposed on each other. Thus we see that the compressible 2D perturbations can extract the energy of the mean (regular) compressible shear flow.

The possibility of the shear energy extraction by SFH may have important and far-reaching consequences. According to the concept outlined in [6] the unusual linear mechanism of shear energy extraction, which exists for the incompressible perturbations, may be a base for the transition to turbulent state and its maintenance in free shear flows. For sound-type perturbations we find another, also quite effective "channel" of the shear energy extraction, which may be responsible energetically for the onset of compressible turbulence. The further, more detailed study of the phenomena will be presented elsewhere, in a more detailed and extensive publication.

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[1] S. C. Reddy, P. J. Schmid, and D. S. Hennigson, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **53**, 15 (1993).

[2] L. N. Trefethen, A. E. Trefethen, S. C. Reddy, and T. A. Driscoll, Science **261**, 578 (1993).

[3] Lord Kelvin (W. Thomson), Philos. Mag. **24**, Ser. **5**, 188 (1887).

[4] P. Marcus and W. H. Press, J. Fluid Mech. **79**, 525 (1977).

[5] G. D. Chagelishvili, R. G. Chanishvili, T. S. Khristov, and J. G. Lominadze, Phys. Rev. E **47**, 366 (1993).

[6] G. D. Chagelishvili, R. G. Chanishvili, J. G. Lominadze, and I. N. Segal, in *Proceedings of the Fourth International Conference on Plasma Physics and Controlled Nuclear Fusion*, edited by T. D. Guyenne (ESA, Noordwijk, 1992), p. 23.

[7] K. M. Butler and B. F. Farrell, Phys. Fluids A **4**, 1637 (1992).

[8] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics* (Pergamon Press, London, 1959), Vol. 6.

[9] Ya. B. Zel'dovich and A. D. Mishkis, *Elementi Prikladnoi Matematiki* (Nauka, Moscow, 1972) (in Russian).

[10] K. Magnus, *Schwingungen* (B. G. Teubner, Stuttgart, 1976).