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Ring dark solitons

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Self-defocusing nonlinear media can support dark solitary waves with ring symmetry that are robust but slowly change their parameters. This alternative type of optical dark soliton is investigated numerically and analytically, and it is shown that in the small-amplitude limit such solitons are described by the so-called cylindrical Korteweg–de Vries equation known, e.g., from plasma physics. Ring dark solitons may coexist with other types of dark solitons, displaying almost an elastic interaction.

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Dark spatial solitons are known to exist as low-intensity dips on a background field that do not diffract as the beam propagates [1] (see also the review paper [2] and references therein). These solitons are the reflectionless modes of the optical waveguide they induce whereas bright spatial solitons are the bound modes [3]. In the case of two transverse degrees of freedom such solitons may be observed as dark-soliton strips or grids with the properties similar to those of two-dimensional dark solitons [4]. Dark solitons of circular symmetry (optical vortex solitons) have been predicted and shown to be stable [5,6], and they have already been observed experimentally in self-defocusing materials [7]. On the other hand, dark soliton strips are unstable to transverse long-wavelength modulations [8]. Such instabilities are caused by the unbounded soliton as can be appreciated from simple physics [9]. Numerical calculations show that due to that instability a dark strip decays into a sequence of optical vortex solitons of opposite polarities [10,11]. The instability band is characterized by the maximum wave number Q_{cr} of transverse perturbations [8], so that a soliton strip is stable to transverse modulations $Q_\perp$ having a short period, i.e., for $Q_\perp > Q_{cr}$. Let us consider a loop formed by a quasi-two-dimensional dark soliton of length L. Then, in the first-order approximation we come to the conclusion that *transverse instabilities can be suppressed* provided the condition $L < 2\pi/Q_{cr}$ holds. The soliton of the lowest energy is naturally expected to have a circular symmetry, therefore one may observe stable dynamics of *ring dark solitary waves* in bulk self-defocusing materials. The present paper aims to introduce this type of optical dark soliton and to describe its properties analytically and numerically. As follows from our analysis, in the small-amplitude approximation such solitons are described by the cylindrical Korteweg–de Vries (KdV) equation that resembles famous cylindrical (or ring) solitons in collisionless plasmas investigated more than 15 years ago, both theoretically [12,13] and experimentally [14,15]. We derive also a general formula describing the internal dynamics of ring dark solitons.

As is well known, propagation of monochromatic transverse electric field $E(x,y,z)$ in a nonlinear self-defocusing medium with the intensity-dependent refractive index $n = n_0 - n_2|E|^2$ ($n_2 > 0$) is described by the nonlinear Schrödinger (NLS) equation which may be written in the form

$$i\frac{\partial u}{\partial z} + \frac{1}{2}\,\Delta_\perp u - |u|^2 u = 0, \tag{1}$$

where $\Delta_\perp$ is the transverse Laplacian which can also include the effects of positive temporal dispersion, $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2 + (D-2)\partial^2/\partial t^2$, where $D=2,3$ is the dimension parameter (the case $D=3$ includes dispersion-induced effects through the second-order time derivative which is important for such phemomena as light bullets; see [16]). Here we use the following notations: $u = E/E_0$ is the dimensionless electric field normalized by the background amplitude E_0, the dimensionless longitudinal (z) and transverse (x and y) coordinates are normalized by the spatial scale $L_0 = (n_0/n_2)^{1/2}/(kE_0)$, k being the wave number.

We note that for the solutions with radial symmetry which depend only on the radius r the transverse Laplacian may be written in the form

$$\Delta_\perp = \frac{\partial^2}{\partial r^2} + \frac{(D-1)}{r}\,\frac{\partial}{\partial r}\,, \tag{2}$$

so that Eq. (1) looks like a standard two-dimensional NLS equation (which obviously supports dark solitons) provided the term $\sim r^{-1}$ is small. Therefore we may look for dark-soliton solutions of circular symmetry in the form of a dark-soliton ring with slowly varying parameters,

$$u(z,r) = e^{-iz}(\cos\phi \tanh Z + i\sin\phi), \tag{3}$$

$$Z = \cos(\phi)[r - R(z)], \tag{4}$$

where $\phi = \phi(z)$ ($|\phi| < \pi/2$) and $R(z)$ are the slowly varying soliton angle and the coordinate of its center, respectively. The physical meaning of these parameters is rather simple: the soliton angle ϕ describes the contrast of a dark soliton, $\cos^2\phi$, and it is connected with the phase jump across the soliton, 2ϕ (see Ref. [17] for more details), and $R(z)$ is the soliton ring radius on the distance z. Evolution of the soliton parameters may be analyzed in the framework of the so-

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called adiabatic approximation of the perturbation theory for dark solitons [17], considering the term $\sim r^{-1}$ as a perturbation. The resulting evolution equations derived from the perturbation-induced dynamics of the system Hamiltonian take the form

$$\frac{d\phi}{dz}=\frac{(D-1)}{3R}\cos\phi,\quad \frac{dR}{dz}=\sin\phi. \tag{5}$$

Combining these equations, we find the radial velocity of the ring dark soliton as a function of its radius R,

$$W\equiv\frac{dR}{dz}=\sigma\left[1-\cos^2\phi(0)\left(\frac{R(0)}{R}\right)^{2(D-1)/3}\right]^{1/2}, \tag{6}$$

where $\sigma=\text{sgn}[\sin\phi(0)]=\pm 1$, $\phi(0)$ and $R(0)$ being the initial values of the parameters. Equation (6) shows that the minimum radius of the ring dark soliton is

$$R_{min}=R(0)[\cos\phi(0)]^{3/(D-1)}, \tag{7}$$

and at $R=R_{min}$ the dark soliton has the maximum contrast. Depending on the initial value $\phi(0)$ of the soliton phase ϕ, the dark soliton can collapse to reach R_{min}, or it diverges decreasing its contrast.

The linear stability analysis [8] predicts that the dark soliton strip is stable when the condition (in our notations)

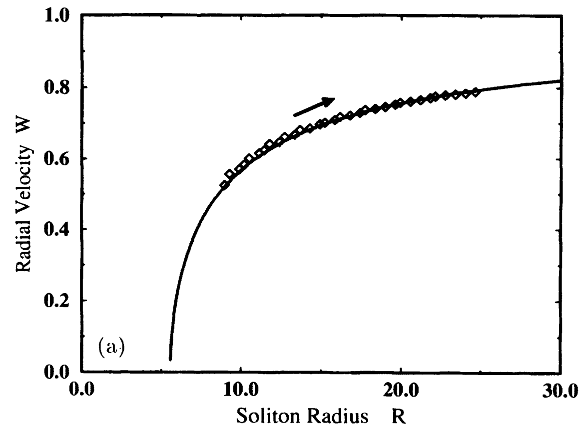
$$Q_{\perp}>Q_{cr}(\phi)=[2\sqrt{\sin^4\phi+\cos^2\phi}-(1+\sin^2\phi)]^{1/2} \tag{8}$$

is satisfied, $Q_{\perp}$ being the wave number of the transverse perturbations. The result (8) shows that the instability band vanishes for small-amplitude dark solitons when $\cos\phi\rightarrow 0$. Thus the dark soliton is expected to be always stable, even expanding, when the limit length of the ring $2\pi R_{min}$ is smaller than the minimum wavelength $2\pi/Q_{cr}(0)$ for the instability region, i.e., provided

$$R_{min}Q_{cr}(0)<1. \tag{9}$$

We have made numerical simulations in the framework of Eq. (1) for both the cases mentioned above, and the results at $D=2$ are presented in Figs. 1 and 2. Figures 1(a) and 1(b) show the evolution of the absolute value of the radial soliton velocity W vs the current value of the soliton radius $R(z)$. We present here two particular cases, $\sin\phi(0)>0$, when the ring soliton simply diverges [Fig. 1(a)], and $\sin\phi(0)<0$, when it first collapses to reach the minimum value R_{min} [Fig. 1(b)]. Solid lines in Figs. 1(a) and 1(b) represent the results given by our approximate analytical solution (6), which seems to be in excellent agreement with numerical simulations. When the dark soliton collapses, at the turning point the validity of our adiabatic approximation is destroyed and the dark ring slightly changes its radial velocity, and subsequently it expands along an effectively shifted theoretical dependence [see Fig. 1(b)]. Nevertheless, this solitary wave is highly robust and it perfectly conserves its radial symmetry as is shown in Fig. 2 where the dependence of the soliton contrast $\cos^2\phi$ vs propagation distance is also presented.

We have also tried to analyze the robustness of the ring dark solitons in collisions with other ring dark solitons, dark-soliton strips, and vortex solitons. Figure 3 shows, as an example, interaction of two dark-soliton strips and a ring dark soliton. Dark-soliton strips were generated by an even initial condition similar to the purely two-dimensional case, and they always displayed the plane symmetry. The interaction looks almost elastic, however, the cross points produce a phase shift (see Fig. 3) characterizing, e.g., interaction of stable bright solitons on a plane.



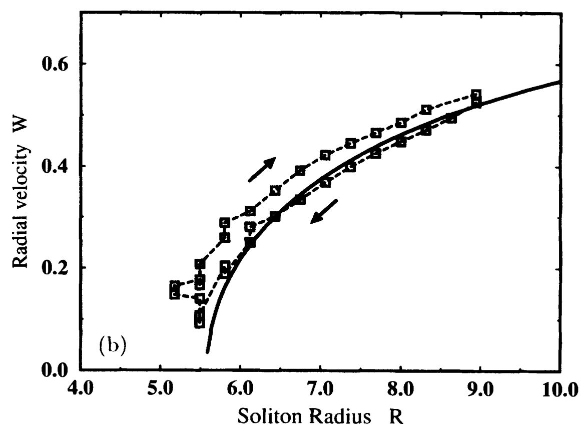


FIG. 1. Radial velocity $|W|=|\sin\phi|$ of a ring dark solitary wave for (a) $\sin\phi(0)>0$ and (b) $\sin\phi(0)<0$ vs the soliton radius R. Solid lines are results of the analytical approach given by Eq. (6) and the marks indicate results of numerical simulations.

It is interesting to compare the ring dark solitons described here with the other types of quasiplane solitons of radial symmetry known in nonlinear physics. As a matter of fact, the so-called cylindrical acoustic solitons were shown to propagate in collisionless plasmas [12], and they were intensively investigated experimentally (see, e.g., [14,15]). To get a deeper insight into the physics of ring dark solitons, we consider the so-called small-amplitude approximation [18] to Eq. (1), looking for solutions in the form

$$u(z,r)=[1+a(z,r)]\exp[-iz+i\theta(z,r)], \tag{10}$$

where $a(z,r)$ and $\theta(z,r)$ are assumed to be presented in the form of formal asymptotic series,

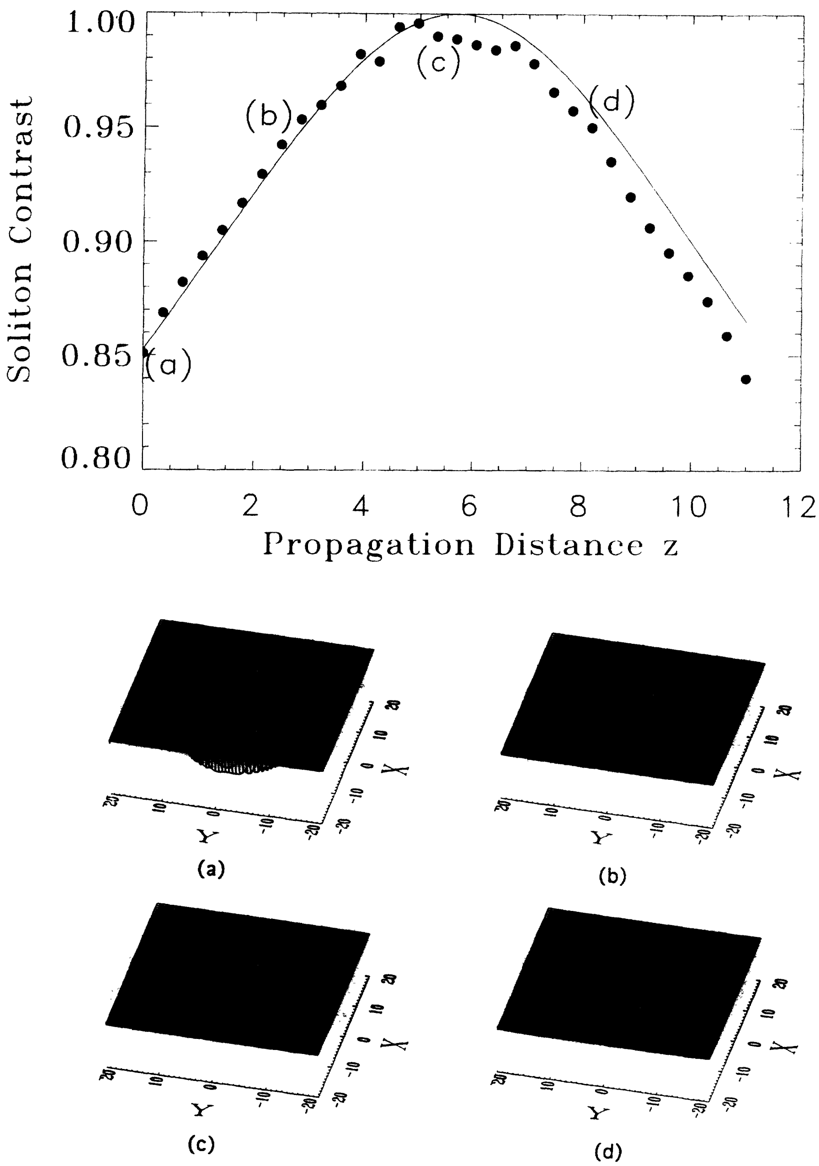


FIG. 2. Evolution of a ring dark soliton for the case presented in Fig. 1(b). The upper figure shows the soliton contrast $\cos^2\phi$ vs propagation distance z, and the four plots below display a ring dark soliton on the unit-intensity background at different distances marked on the upper figure. The plots (a) and (b) correspond to the lower numerical dependence in Fig. 1(b) (collapsing soliton) whereas (c) and (d), to the upper one (divergent soliton).

$$a=\epsilon^2 a_0+\epsilon^4 a_1+\cdots, \quad \theta=\epsilon\theta_0+\epsilon^3\theta_1+\cdots, \tag{11}$$

ϵ being a small parameter. Substituting (10), (11) into Eq. (1), and using new (slow) variables $\tau=\epsilon(r-Cz)$ and $\xi=\epsilon^3 z$, where $C^2=1$ is the dimensionless sound velocity of linear waves on a cw background, in the lowest-order approximation (in ϵ) of the reductive perturbation technique we obtain the relation for the phase, $\partial\theta_0/\partial\tau=(2/C)a_0$, and derive the equation for the soliton amplitude,

$$8C\frac{\partial a_0}{\partial\xi}+24a_0\frac{\partial a_0}{\partial\tau}-\frac{\partial^3 a_0}{\partial\tau^3}+\frac{4C}{\xi}a_0=0. \tag{12}$$

Equation (12) is the famous cylindrical KdV equation which is known to be the basic nonlinear equation describing cylindrical and spherical pulse solitons in plasmas [12], electric lattices [19], as well as tsunami-type waves in shallow water [20]. The properties of its soliton solutions are also well understood (see, e.g., Ref. [13]). Therefore we establish a

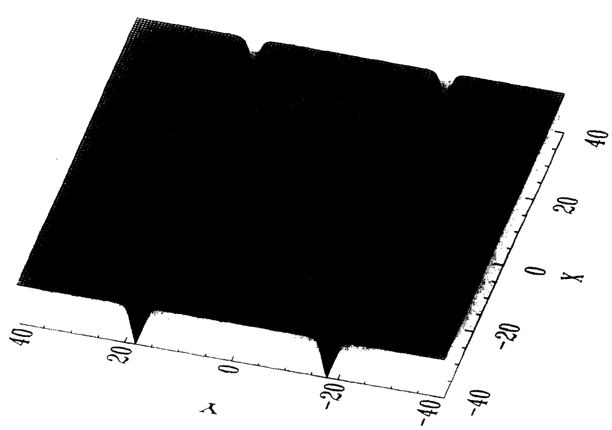


FIG. 3. Interaction of a ring dark solitary wave with a pair of dark-soliton strips. Note a phase shift at the crossing points.

similarity between the type of dark solitons described in the present paper and a special class of ring bright solitons known, e.g., in plasmas. This allows us to expect an experimental verification of our results similar to the experiments [14,15] which confirmed the existence of cylindrical plasma-wave solitons.

Lastly, we would like to note that we do not see at the moment any serious restriction on the conditions to observe such solitary waves experimentally. In fact, it seems one should simply make a phase mask of circular symmetry in the experiments which reported earlier discovery of dark soliton strips [4].

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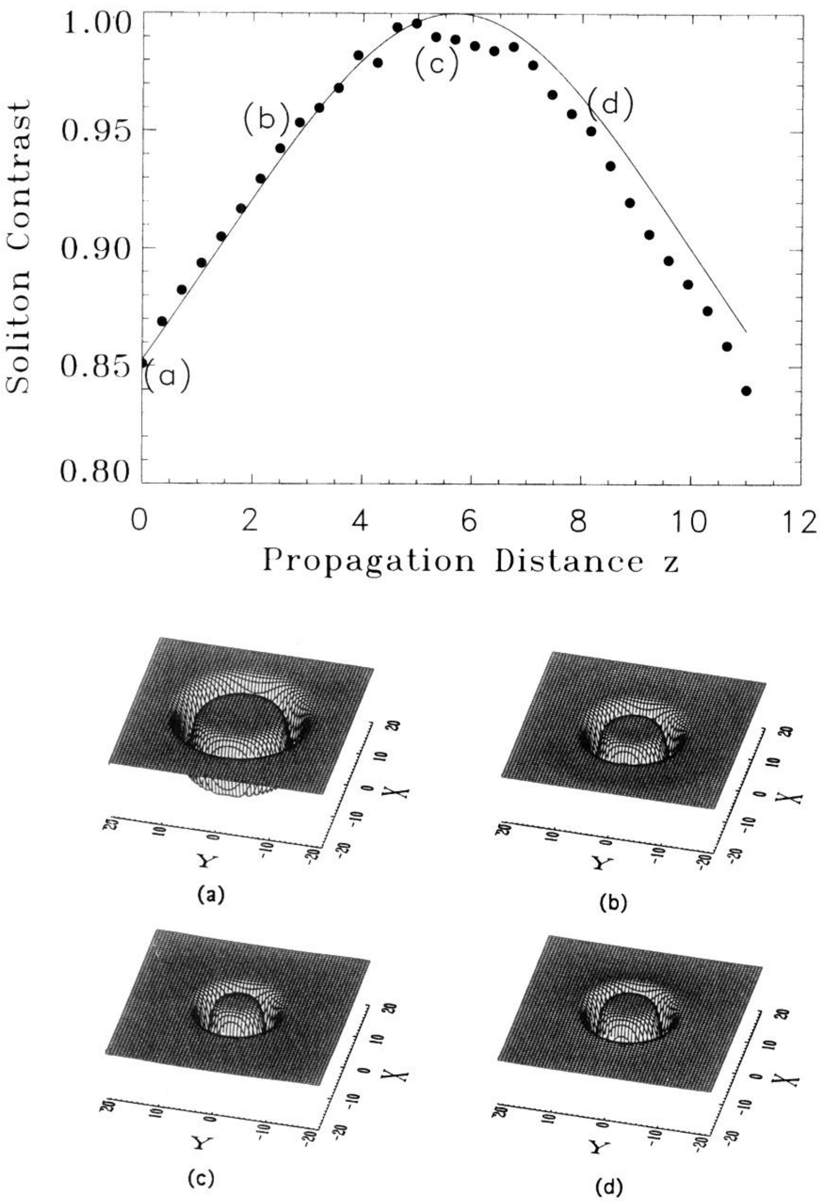

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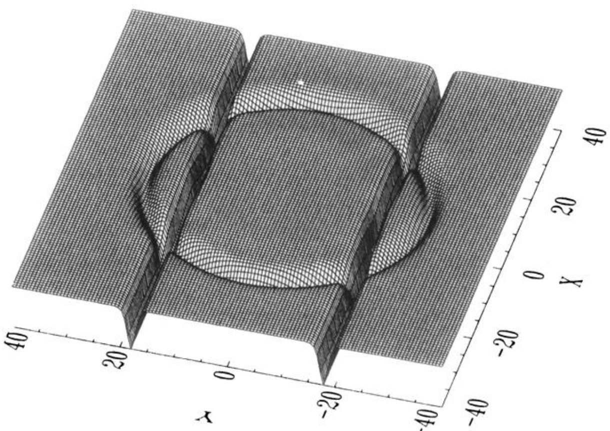

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