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Surface shape of a spinning bucket of sand

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The free surface shape of a bucket of sand rotating about its cylindrical axis is studied both theoretically and experimentally. The experimental surface shape is found to depend on the material's angle of internal friction and the imposed rotation rate, and provides a way of experimentally measuring the angle of internal friction. The results may be approximated by a theory which assumes that the surface deforms according to the Coulomb yield condition. The results are compared with the surface of a spinning bucket of water.

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INTRODUCTION

A key ingredient in understanding the flow of granular media is understanding when such a medium will make the transition from rigid to free-flowing behavior. This transition, fundamental to the onset of surface avalanches [1–3], has attracted considerable interest in the 200 years since Coulomb first studied it in the 1790s [4]. Coulomb proposed, in what is known as the Coulomb yield condition (CYC), that a granular material will first yield along an internal surface on which the shearing and normal forces, F_S and F_N, are related by $F_S \geqslant F_N \tan\theta_f$. The material parameter θ_f is called the angle of internal friction. Since the angle of internal friction is the angle of inclination of the steepest possible theoretical pile of grains, θ_f is associated with the critical state of a granular medium [1–3]. The angle of internal friction also occurs in many other problems including granular Faraday experiments and segregation experiments. Unfortunately, real sandpiles have a range of possible slopes [2,3,5]; and, as a result, the angle of internal friction is ill-defined. This suggests, despite its simplicity and success, that the Coulomb yield condition is only an approximation to the actual mechanism determining the rigid-flowing transition.

We have chosen to study these issues by studying the surface shape of a bucket of sand spinning about its cylindrical axis. The similar problem involving the free surface shape of a spinning bucket of *water* is a popular exercise for graduate students in physics [6]. Water can support no shearing forces without flowing; as a result, the nonrotating equilibrium surface is flat and $\theta_f=0$. The surface shape of water in a cylindrical container of radius R is a simple paraboloid of revolution at any angular velocity ω with a dimensionless surface height

$$h(\xi)=\frac{z(r)}{R}=\frac{\Omega^2}{2}\left(\xi^2-\tfrac{1}{3}\right), \tag{1}$$

where $\xi=r/R$, $h=z/R$, and $\Omega=\omega\sqrt{R/g}$ are the dimensionless radius, surface height, and rotational frequency, respectively.

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THEORETICAL SURFACE SHAPE

The theoretical surface shape for sand is more challenging to calculate than that of water. Unlike water, a granular material can support a nonzero shearing force before deforming. The details of when and how this deformation occurs are described by the constitutive relation, the relationship between applied stress and the resulting strain. This relation combined with conservation of mass and momentum would provide a set of continuum equations for granular materials equivalent to the Navier-Stokes equations of incompressible fluid mechanics [7]. Without a constitutive relation, the correct continuum equations cannot be formulated. Consequently, the mathematical description of the constitutive relation for a granular material is a subject of considerable past work and current interest [8,9].

In the absence of an agreed-upon constitutive relation, we assume that sand obeys the simplest possible constitutive relation, a rigid-plastic relation, in which the material is rigid at stresses below a critical value and perfectly plastic above this value. We further assume that the transition point is given by the Coulomb yield condition.

We make the following assumptions.

(1) The sand obeys the Coulomb yield condition.

(2) The surface is everywhere at its critical state (i.e., the surface is exactly at the rigid-plastic transition, just about to deform).

(3) The sand density is uniform and constant.

Balancing forces on a material element located at the surface leads to an expression for the radial slope,

$$s_{\pm}=\frac{\xi\Omega^2\mp\tan\theta_f}{1\pm\xi\Omega^2\tan\theta_f}=\frac{dh}{d\xi}, \tag{2}$$

where ξ, h, and Ω are the dimensionless radius, height, and angular velocity defined previously. The two solutions correspond to the two directions that the frictional force can point. s_+ corresponds to friction toward the center while the eventual motion is toward the wall. s_- corresponds to friction toward the wall while the grains move toward the center. Note that at large radius or rotation rate, where $\xi\Omega^2 \gg \tan(\theta_f)$, the slope approaches a constant value, $s_{\pm}\rightarrow 1/\tan(\pm\theta_f)=\tan(90°\pm\theta_f)$.

Integrating the slope to find the surface shape yields

$$h_{\pm}(\xi)=\frac{\xi}{\tan(\pm\theta_f)}-\frac{1}{\Omega^2}\left(\frac{1+\tan^2(\theta_f)}{\tan^2(\theta_f)}\right)\ln[1+\xi\Omega^2\tan(\pm\theta_f)]+h_0. \tag{3}$$

This result can be compared with that of water, Eq. (1), and in the limit $\theta_f\rightarrow 0$ it reduces to Eq. (1), as it must. The integration constant h_0 is found by requiring conservation of mass, i.e.,

$$0=\int_0^{2\pi}\int_0^1 h_{\pm}(\xi)\xi d\xi d\theta. \tag{4}$$

We now restrict our attention to solutions that can occur when we begin with a flat upper surface at $\Omega=0$ and ramp the rotation rate gradually upward. As long as the rotation rate never decreases, the friction force will always be toward the center of the bucket and only the solutions $h=h_+$ will occur.

Unlike water, which yields as soon as a shear is applied, sand can support a nonzero shear as shown in the CYC. As a result, beginning with a flat upper surface at $h=0$ and ramping the rotation rate up from zero, the surface first deforms when the centripetal force cannot be provided by friction alone. This occurs first near the outer wall leaving an undeformed center region of radius ξ_c. As the rotation rate increases, the center region shrinks and $\xi_c\rightarrow 0$. While the center is undeformed, Eq. (4) effectively becomes

$$0=\int_0^{2\pi}\int_{\xi_c}^1 h(\xi)\xi d\xi d\theta, \tag{5}$$

with the additional condition

$$h(\xi_c)=0. \tag{6}$$

The transition between the two regimes occurs when $\xi_c=0$. Beyond this transition at higher rotation rates,

$$h_0=\frac{-2}{3\tan(\theta_f)}+\frac{1}{\Omega^2}\frac{1+\tan^2(\theta_f)}{\tan^2(\theta_f)}\left[\left(1-\frac{1}{\tan(\theta_f)^2\Omega^4}\right)\ln[1+\tan(\theta_f)\Omega^2]-\frac{1}{2}+\frac{1}{\tan(\theta_f)\Omega^2}\right]. \tag{7}$$

For $\Omega<\Omega_c$, Eq. (5) and Eq. (6) result in a transcendental equation for ξ_c and h_0.

The resulting surface shapes are shown in Fig. 1. Note the flat region near $\xi=0$ for the lowest frequencies in Fig. 1 where the sand has not deformed. As the rotation frequency increases, this region shrinks to zero leaving behind a cusp at $\xi=0$ which gradually disappears as the rotation rate is increased. At the highest rotation rates, the surface slope becomes a constant value at large radii.

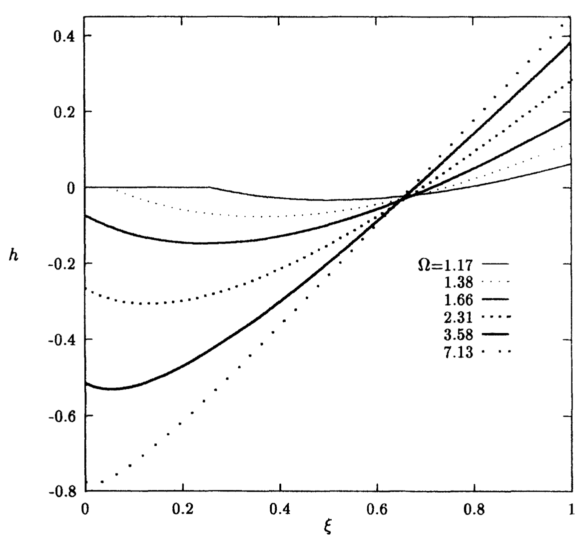

FIG. 1. Predicted surface shape $h(\xi)$ as a function of rotation frequency for a system beginning with a flat upper surface and $\theta_f=34.6$. Note the nondeforming central region at the lowest rotation rates. At intermediate values of Ω, this leads to a cusp at $\xi=0$ which flattens out at higher Ω. At these highest rotation frequencies, the slope becomes nearly constant at large ξ.

EXPERIMENTAL SURFACE SHAPE FOR SAND

The experimental apparatus is a cylindrical container of polyvinyl chloride (PVC) plastic and steel with an inner radius of 12.5 cm, a wall thickness of 2 cm, and a height of 30.48 cm. The container is mounted on a large dc motor so that its cylindrical axis is the axis of rotation. The container is partly filled with common sand with angle of internal friction $\theta_f\approx 34\pm 1°$ and bulk density $\rho=1.56\pm 0.04$ g/cm^3. The apparatus is much more massive than the sand within it. The large moment of inertia minimizes fluctuations in the angular velocity. Incremental changes in the smaller moment of inertia of the sand, due to changes in the surface shape, have a negligible effect on the rotation rate of the container and its contents. A photogate timer measures the rotation frequency which is constant to better than 1%. Although the current apparatus allows us to accurately measure the rotation rate, it does not allow us to accurately set the rotation rate to a desired value.

The measurement is made using a screw-driven stylus with a finely tapered tip which is lowered axially down at a particular radius as the bucket rotates below it. The stylus casts shadows from two lights placed off axis. As the stylus tip approaches the surface, the distance between each shadow and the tip diminishes until tip and shadow converge at the sand surface. This enables us to measure the height of

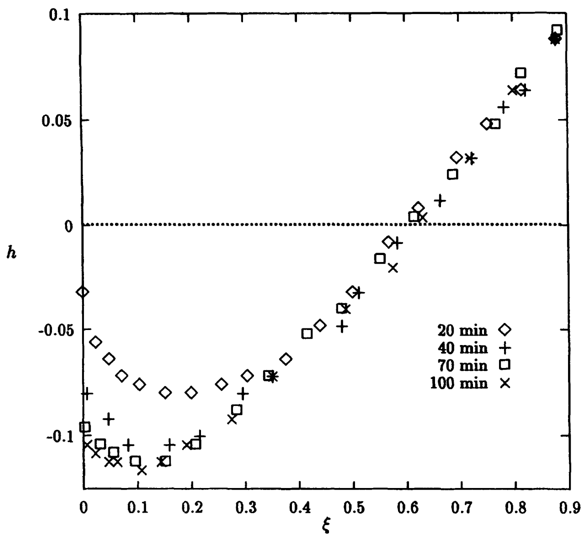

FIG. 2. Complete surface shapes measured at a constant rotation rate, $\Omega=1.22$ using measurement method (i) as described in the text. Note the continuing evolution of the surface. The error in each measurement is on the order of the symbol size.

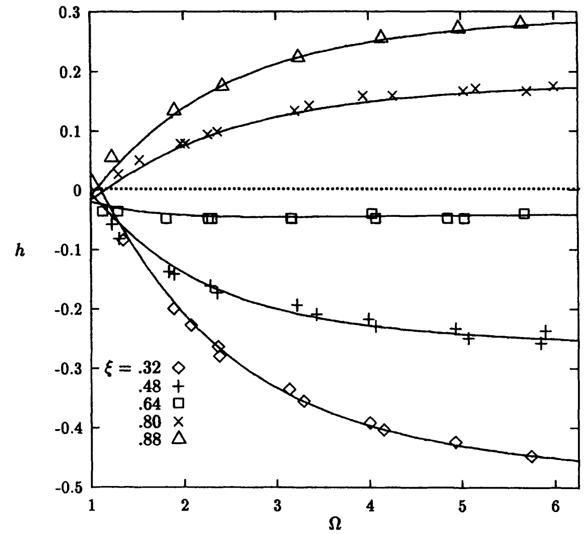

FIG. 3. Surface shapes mapped while ramping the rotation rate, measurement method (ii) as described in the text. The error in each measurement is on the order of the symbol size. Also shown are the theoretical predictions with $\theta_f=34.6$.

the surface within ± 0.5 mm without actually touching the sand.

The surface shape is mapped in one of the following two ways.

(i) Beginning with an initially flat upper surface at $\Omega=0$, the simplest procedure is to slowly increase the rotation rate to the desired final value and then hold the rotation rate constant while measuring the height of the upper surface. Sample results are shown in Fig. 2. Note that the surfaces are azimuthally symmetric and the error in each measurement is on the order of the size of the symbol. With our apparatus, a measurement of the entire surface typically takes 5–20 min depending on the desired resolution. Figure 2 also shows a series of surface measurements at varying times after the final rotation rate is achieved. From this data it is apparent that the surface shape slowly evolves even though the rotation rate is held fixed. Such behavior is not described by the above theoretical description.

(ii) The alternative method of mapping the surface is to fix the radial position, ξ, and measure the surface height as a function of Ω. In this way, the measurements can be performed quickly. Sample results are shown in Fig. 3. The data fit the model very well. Moreover, using θ_f as the variable parameter, at nearly every radial position we find $\theta_f=34.6\pm 0.7$. The exception is near $\xi=0.7$ at which there is little or no variation in the surface height as a function of Ω. This apparatus can therefore be used to easily measure the angle of internal friction θ_f.

At the highest rotation rates, $\Omega>7$, an additional effect becomes important. A steep depression appears around $\xi=0$. This is caused by compaction of the sand under centrifugal effects. When this depression forms, grains in this region slide back toward the center. In this region, the slope s_- should occur with the corresponding change in the local surface shape to h_-. The observed slope is in the range of s_-, but insufficient experimental resolution in the depression prevents us from a comparison of theoretical and experimental surface shapes. When this depression forms, the complete surface shape cannot be described by Eq. (3) since our derivation assumed the density of the sand was always constant throughout the evolution of the surface.

CONCLUSIONS

Experimentally, at large $\xi\Omega^2$, the surface looks linear and its slope changes slowly with increasing frequency. This agrees well with the theoretical prediction that for large $\xi\Omega^2$, the surface slope approaches a constant value determined by θ_f (for water, the slope approaches 90° from the horizontal). The predicted cusp at $\xi=0$ is visible in both the experimental data of Fig. 2 and the theoretical plots of Fig. 1.

The time evolution of the surface with Ω constant, shown in Fig. 2, is due to phenomena not included in this simple theory. The most likely cause is granular convection [10,11]. The fact that the surface drops near the center and rises slightly near the walls would support this conclusion. This motion occurs on a much slower time scale than the variation of surface shape with rotation rate. As a result, we were able to use this separation of time scales to measure the surface in Fig. 3 with good agreement with the prediction.

The rotating bucket apparatus provides a simple geometry in which to test and compare theories of granular materials. The simple theory used here is in good agreement with the experiment as long as the rotation rate Ω is not too high and the measurements are quickly made before convection can occur. Under these circumstances, an angle of internal friction can be measured accurately and easily. For other regimes, a more sophisticated theory accounting for convective and density changing compaction effects will be required.

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