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Landau-Ginzburg equation for the cyclotron resonance maser

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A set of collective variable equations are derived that accurately describe the electron and field dynamics in a low-efficiency cyclotron resonance maser amplifier up to the saturation of the field amplitude. Using this collective variable description, it is shown that the evolution of the slowly varying electromagnetic field can be described qualitatively by a Landau-Ginzburg equation with complex coefficients. This is a model that also describes the field evolution in an atomic laser. The electromagnetic field evolution can now be described by an analytical solution far into the nonlinear regime, where numerical integration of the evolution equations was previously necessary.

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INTRODUCTION

Gyrotrons and cyclotron auto resonance masers (CARMs) are a class of cyclotron resonance maser (CRM). Such devices are important sources of coherent high power microwave radiation.

The radiation source of these devices is a relativistic electron beam gyrating as it propagates along a uniform magnetic field. The radiation emitted by the electrons is contained within a waveguide structure. The interaction between the gyrating electrons and radiation can give rise to a phase bunching of the beam and to a coherent, exponentially increasing, radiation field amplitude. A small signal radiation field can be injected at the beginning of the device to form an amplifier.

In a low-efficiency cyclotron resonance maser amplifier, it is possible to describe the coupled dynamics of the electrons and the electromagnetic field of a single TE_{mn} or TM_{mn} waveguide mode using the following set of scaled evolution equations [1]:

$$\frac{d\phi_j}{d\bar{z}}=p_j-i\frac{\mu}{\sqrt{1+2\mu(p_j-\delta_j)}}(Ae^{i\phi_j}-\text{c.c.}),\qquad(1)$$

$$\frac{dp_j}{d\bar{z}}=-\sqrt{1+2\mu(p_j-\delta_j)}(Ae^{i\phi_j}+\text{c.c.}),\qquad(2)$$

$$\frac{dA}{d\bar{z}}=\langle\sqrt{1+2\mu(p-\delta)}e^{-i\phi}\rangle,\qquad(3)$$

where

$$j=1,\ldots,N_e,\quad \bar{z}=\rho\frac{k_\perp^2}{k_\parallel}z,\quad \phi=\omega t-k_\parallel z+\tan^{-1}\left(\frac{u_y}{u_x}\right)-(m-1)\theta_0-\frac{\pi}{2},\quad p=\frac{k_\parallel}{\rho k_\perp^2}\frac{\gamma}{u_\parallel}\left(\omega-\frac{\omega_c}{\gamma}-\frac{k_\parallel u_\parallel}{\gamma}\right),$$

$$u_\perp=\gamma v_\perp,\quad u_\parallel=\gamma v_\parallel,\quad |A|^2=\frac{1}{\rho}\frac{v_g^2}{v_{\parallel 0}^2}\frac{U_w}{U_{b0}},\quad \omega_c=\frac{eB_0}{m_e},\quad k_{c0}=\frac{\omega_c}{u_{\parallel 0}},\quad \alpha=\frac{u_\perp}{u_\parallel},\quad \nu=\frac{k_\parallel}{k_{c0}}\alpha_0^2,\quad \langle\cdots\rangle=\frac{1}{N_e}\sum_{j=1}^{N_e}\cdots,$$

$$\rho_{\rm TE}=\left(\frac{e\mu_0}{8m_e}\frac{k_\parallel^2u_{\perp 0}^2}{k_\perp^2u_{\parallel 0}^3}ID_{\rm TE}^2J_{m-1}^2(k_\perp R_0)\right)^{1/3},\quad \rho_{\rm TM}=\left[\frac{e\mu_0}{8m_ec^2}\frac{\omega^2k_\parallel^2u_{\perp 0}^2}{k_\perp^2k_{c0}^2u_{\parallel 0}^3}I\left(\frac{v_g\gamma_0}{u_{\parallel 0}}-1\right)^2D_{\rm TM}^2J_{m-1}^2(k_\perp R_0)\right]^{1/3},$$

$$D_{\rm TE}=[J_m(\chi'_{mn})\sqrt{\pi(\chi'^2_{mn}-m^2)}\,]^{-1},\quad D_{\rm TM}=[J'_m(\chi_{mn})\sqrt{\pi}\chi_{mn}]^{-1}.$$

j is the electron index number, N_e is the total number of (macro) electrons, subscripts $\perp$ and $\parallel$ represent vector components perpendicular and parallel to the waveguide axis, ω is the radiation frequency, $(k_\perp, k_\parallel)$ are the wave vector components of the mode, γ is the electron relativistic factor, (R_0,θ_0) are the polar coordinates with respect to the waveguide axis of the electron guiding centers, $(v_\perp, v_\parallel)$ are the electron velocity components, v_g is the radiation group velocity, B_0 is the axial magnetic guiding field, A is a scaled complex radiation field strength, U_w is the energy per unit waveguide length of the radiation, U_b is the energy (including rest mass) per unit waveguide length of the electron beam, I is the electron beam current, χ'_{mn} is the nth root of $J'_m(k_\perp R_w)=0$, χ_{mn} is the nth root of $J_m(k_\perp R_w)=0$, R_w is the waveguide radius, δ is the value of p at $\bar{z}=0$, and subscripts 0 indicate initial values at $\bar{z}=0$. Other symbols have their usual meaning. The set of equations (1)–(3) are valid in the limits $\rho\ll 1$, the low-efficiency limit, and $k_\perp r_L\ll 1$, which allows us to neglect space charge effects and the inhomogeneity of the transverse mode profile over an electron gyro-orbit. For clarity, an initially monoenergetic electron beam with constant α_0 has been assumed.

The scaling parameters ρ and ν are the "fundamental CRM parameter" and the "free energy parameter," respectively. The linear growth rate of the exponential CRM instability is proportional to ρ and the energy available for conversion from the electron beam to the radiation is proportional to ν. These parameters are combined to define the "depletion parameter," $\mu \equiv \rho/\nu$. For μ sufficiently large free energy depletion effects are important in the electron-radiation evolution. When $\mu \ll 1$ then free energy depletion is negligible and Eqs. (1)–(3) reduce to those of the Compton regime free electron laser (FEL) [1].

A COLLECTIVE VARIABLE DESCRIPTION OF THE CRM

Using the variables

$$\bar\phi = \phi - \delta\bar z, \quad \bar p = p - \delta, \quad \bar A = A e^{i\delta\bar z}, \tag{4}$$

Eqs. (1)–(3) can be written in the more compact form

$$\frac{d\bar\phi_j}{d\bar z} = \bar p_j - i\frac{\mu}{\sqrt{1+2\mu\bar p_j}}(\bar A e^{i\bar\phi_j} - \text{c.c.}), \tag{5}$$

$$\frac{d\bar p_j}{d\bar z} = -\sqrt{1+2\mu\bar p_j}(\bar A e^{i\bar\phi_j} + \text{c.c.}), \tag{6}$$

$$\frac{d\bar A}{d\bar z} = \langle \sqrt{1+2\mu\bar p}\; e^{-i\bar\phi}\rangle + i\delta\bar A. \tag{7}$$

Equations (5)–(7) have two constants of evolution:

$$\langle \bar p\rangle + |\bar A|^2 = |\bar A_0|^2, \tag{8}$$

$$\frac{\langle \bar p^2\rangle}{2} - i(\bar A\langle \sqrt{1+2\mu\bar p}\; e^{i\bar\phi}\rangle - \text{c.c.}) - \delta|\bar A|^2$$

$$= -i(\bar A_0\langle e^{i\bar\phi_0}\rangle - \text{c.c.}) - \delta|\bar A_0|^2, \tag{9}$$

which correspond to conservation of energy and the Hamiltonian of the system, respectively.

This set of equations can be further reduced by defining a complex variable

$$u_j = \sqrt{1+2\mu\bar p_j}\; e^{i\bar\phi_j}, \tag{10}$$

which enables the set of evolution equations (5)–(7) to be written as a set of two complex equations

$$\frac{du_j}{d\bar z} = \frac{i}{2\mu} u_j(|u_j|^2 - 1) - 2\mu\bar A^*, \tag{11}$$

$$\frac{d\bar A}{d\bar z} = \langle u^*\rangle + i\delta\bar A. \tag{12}$$

Defining the variables

$$b = \langle u^*\rangle, \quad \mathcal{P} = \langle |u|^2 u^*\rangle, \tag{13}$$

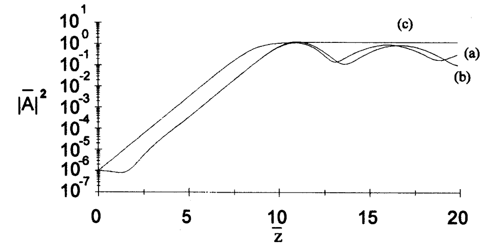

FIG. 1. Plot of $|\bar A|^2$ vs $\bar z$ for $\delta = 0$ and $\mu = 0.1$. (a) Numerical solution of Eqs. (11) and (12). (b) Numerical solution of collective variable equations (17)–(19). (c) Analytical solution of Eq. (27).

it is easily shown that

$$\frac{db}{d\bar z} = -\frac{i}{2\mu}(\mathcal{P} - b) - 2\mu\bar A, \tag{14}$$

$$\frac{d\mathcal{P}}{d\bar z} = -\frac{i}{2\mu}\langle u^*|u|^2(|u|^2-1)\rangle - 4\mu\bar A\langle |u|^2\rangle - 2\mu\bar A^*\langle u^{*2}\rangle. \tag{15}$$

The variables b and $\mathcal{P}$ are analogous to the "bunching parameter" and "energy modulation parameter" of FEL theory [2]. The last term in (15) is proportional to $\langle e^{-2i\bar\phi}\rangle$ and, for reasons outlined in a similar analysis of the FEL [2], will be neglected from now on. In order to close the set of equations it is necessary to use the following ansatz:

$$\langle (\bar p - \langle \bar p\rangle)^2 u^*\rangle \approx \langle (\bar p - \langle \bar p\rangle)^2\rangle\langle u^*\rangle, \tag{16}$$

which is exact if the $\bar p$ have a Gaussian distribution. It can be seen that if $\mu = 0$, i.e., $u = e^{i\bar\phi}$, then (16) reduces to the ansatz used in [2] to facilitate a collective variable description of the FEL. Using the definition of u, the constants of motion (8), (9), and ansatz (16) then it is possible to write $d\mathcal{P}/d\bar z$ in terms of b, $\mathcal{P}$, and $\bar A$ only, so closing the system of equations (12), (14) and (15) to

$$\frac{db}{d\bar z} = -\frac{i}{2\mu}(\mathcal{P} - b) - 2\mu\bar A, \tag{17}$$

$$\frac{d\mathcal{P}}{d\bar z} = -\frac{i}{2\mu}(1 - 4\mu|\bar A|^2)(\mathcal{P} - b) + 4\mu(\bar A|b|^2 - \bar A^* b^2) + 4\mu i|\bar A|^4 b - 4\mu\bar A + 8\mu^2|\bar A|^2\bar A - 4\mu i\delta(|\bar A|^2 - |\bar A_0|^2)b - 4\mu(\bar A_0 b_0^* - \text{c.c.})b, \tag{18}$$

$$\frac{d\bar A}{d\bar z} = b + i\delta\bar A, \tag{19}$$

where $b_0 = b(\bar z = 0)$ and $\bar A_0 = \bar A(\bar z = 0)$. Figures 1 and 2 show how $|\bar A|^2$ varies with $\bar z$ from a numerical integration of the collective variable equations (17)–(19) compared with a similar integration of Eqs. (11) and (12) for small and large values of μ, respectively. It can be seen that the agreement between these results up to saturation is excellent for the case where $\mu = 0.1$ and good for the case where $\mu = 0.8$. At worst, the field intensities at saturation predicted by the two methods differ by a factor of 2. The use of (16) and the neglect of

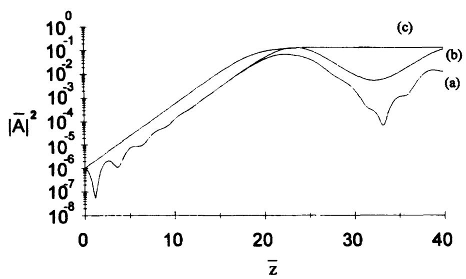

FIG. 2. Plot of $|\bar A|^2$ vs $\bar z$ for $\delta=0$ and $\mu=0.8$. (a) Numerical solution of Eqs. (11) and (12). (b) Numerical solution of collective variable equations (17)–(19). (c) Analytical solution of Eq. (27).

terms varying as $\langle e^{-2i\phi}\rangle$ is therefore justified as in FEL theory when describing the evolution of the system up to saturation.

DERIVATION OF A LANDAU-GINZBURG EQUATION

Successive differentiation of (19) and the use of (17) and (18) transforms the set of collective variable equations to a single third-order nonlinear differential equation describing the electromagnetic field evolution,

$$\frac{d^3\bar A}{d\bar z^3} - i\delta\frac{d^2\bar A}{d\bar z^2} + 2\mu\frac{d\bar A}{d\bar z} - i\bar A = -2i\bar A\left|\frac{d\bar A}{d\bar z} - i\delta\bar A\right|^2 + 2i\bar A^*\left(\frac{d\bar A}{d\bar z} - i\delta\bar A\right)^2 + 2i|\bar A|^2\frac{d^2\bar A}{d\bar z^2} + 2|\bar A|^4\left(\frac{d\bar A}{d\bar z} - i\delta\bar A\right) + 2i\delta^2|\bar A|^2\bar A. \tag{20}$$

It has been assumed that $\bar A_0$ is negligibly small and $b_0=0$. It will be shown here that this equation can be reduced to the form of a Landau-Ginzburg equation with complex coefficients. The methods used will be similar to those used in [3].

Neglecting all the nonlinear terms in (20) it is found that the field evolves as

$$\bar A(z) \approx a\,\exp(-ik\bar z), \tag{21}$$

where k is that root of the dispersion relation

$$k^3+\delta k^2-2\mu k-1=0 \tag{22}$$

such that $k=k_r+ik_i$, $k_i>0$. It is easily checked using the variable $\lambda=-k$ that this dispersion relation is identical to that obtained in [1] from a linear analysis of the original set of equations (1)–(3) in the limit $\rho\ll 1$.

Using this linear form to evaluate the nonlinear terms in (20) and using the further assumption that the amplitude a is a slowly varying function of the variable $\bar z$, an approximate evolution equation for a can be obtained from (20),

$$\frac{d^3a}{d\bar z^3} - i(3k+\delta)\frac{d^2a}{d\bar z^2} - (3k^2+2k\delta-2\mu)\frac{da}{d\bar z} = -2i(k+\delta)(2k+k^*+\delta)a|a|^2e^{2k_i\bar z} - 2i(k+\delta)a|a|^4e^{4k_i\bar z}. \tag{23}$$

Neglecting all derivatives of a higher than the first and considering only the cubic nonlinear term it is possible to write the equation for a as

$$\frac{da}{d\bar z} \approx -ce^{2k_i\bar z}a|a|^2, \tag{24}$$

where

$$c=c_r+ic_i=-2i\,\frac{(k+\delta)(2k+k^*+\delta)}{3k^2+2\delta k-2\mu}. \tag{25}$$

This allows an equation for the original field variable $\bar A$ to be written using (21) as

$$\frac{d\bar A}{d\bar z} = -ik\bar A - c\bar A|\bar A|^2, \tag{26}$$

which is a Landau-Ginzburg equation with complex coefficients and has the same form as the corresponding equations describing the saturation process and the phase transition in atomic lasers [4]. It can be shown from (26) that the corresponding equations for the field intensity $I=|\bar A|^2$ and phase ξ where $\bar A=|\bar A|e^{i\xi}$ are

$$\frac{dI}{d\bar z} = 2I(k_i-c_rI), \tag{27}$$

$$\frac{d\xi}{d\bar z} = -k_r-c_iI. \tag{28}$$

From the form of (27) it is apparent that c_r is analogous to the self-saturation coefficient of atomic laser theory [4]. The solutions for the field intensity and phase are

$$I(\bar z)=\frac{k_iI_0e^{2k_i\bar z}}{(k_i-c_rI_0)\left(1+\dfrac{c_rI_0}{(k_i-c_rI_0)}e^{2k_i\bar z}\right)}, \tag{29}$$

$$\xi(\bar z) = -k_r\bar z - \frac{c_i}{2c_r}\ln\left(1+\frac{c_rI_0}{k_i}(e^{2k_i\bar z}-1)\right)+\xi_0, \tag{30}$$

where $I_0=I(\bar z=0)$ and $\xi_0=\xi(\bar z=0)$. It is clear that as $\bar z\to\infty$ then

$$I\to\frac{k_i}{c_r},\quad \frac{d\xi}{d\bar z}\to -k_r-\frac{k_ic_i}{c_r}. \tag{31}$$

The solution of (27) is shown in Figs. 1 and 2 for $\mu=0.1$ and $\mu=0.8$, respectively. Note that because the Landau-Ginzburg equation has only been used to model the evolution of the exponentially growing wave, its solution does not display the region of "lethargy" at the beginning of the interaction, as this is caused by interference of the growing, de-

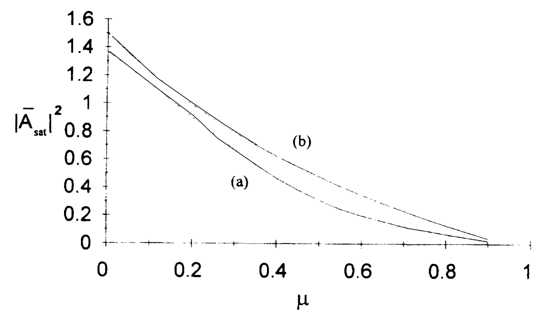

FIG. 3. Plot of $|\bar{A}_{\text{sat}}|^2$ vs μ for $\delta=0$. (a) Numerical solution of Eqs. (1)–(3). (b) Analytical solution of Eq. (27) in the limit $\bar{z}\rightarrow\infty$.

caying, and oscillatory waves corresponding to the three roots of the dispersion relation (22). This is most pronounced for cases of large μ for which the linear growth rate of the amplified wave is smallest. Figure 3 shows the variation of the saturation intensity predicted by (31) with μ for the case of exact resonance ($\delta=0$). This is therefore an *analytical* prediction of the effect of free energy depletion [1] on the saturation intensity of a cyclotron resonance maser. The intensity at the first saturation peak, $|\bar{A}_{\text{sat}}|^2$, is also plotted in Fig. 3 as calculated from a numerical solution of Eqs. (11) and (12). The variation of the saturation intensity with μ is very similar in each case, and validates the use of the Landau-Ginzburg equation. Note that as $\mu\rightarrow 0$ the saturation intensity predicted by the analytical result (29) approaches 1.5. The linear instability threshold for μ of $(27/32)^{1/3}\approx 0.95$ [1] can also be observed. The elegance of the Landau-Ginzburg model is evident given the simplicity of the equations and their excellent agreement of results with those from the full numerical integration of Eqs. (11) and (12).

CONCLUSION

It has been shown that it is possible to describe the evolution of the electron and field dynamics of a low-efficiency cyclotron resonance maser amplifier up to saturation of the field amplitude using collective variables. Using this description, it is possible to reduce the equation for the evolution of the slowly varying complex field amplitude to a Landau-Ginzburg equation with complex coefficients. Analytical predictions of the effect of free energy depletion on the saturated field intensity show good agreement with corresponding numerical calculations. The Landau-Ginzburg form obtained here shows that the evolution of the electromagnetic field up to saturation in a cyclotron resonance maser can be described by the same equation used to model the field evolution in an atomic laser.

The results presented in this paper are remarkable from two points of view: Firstly, the simplicity of the Landau-Ginzburg equations when compared with the complicated set from which they were derived, and secondly the similarity between atomic lasers and cyclotron resonance masers through the Landau-Ginzburg equation given the obvious differences between them.

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