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Domain wall between traveling waves

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A boundary (wall) separating two domains filled with traveling waves is considered within the framework of coupled Ginzburg-Landau (GL) equations with complex coefficients and group-velocity terms. The domain wall may be realized as a boundary produced in two dimensions by a collision of waves traveling in different directions, or as a sink or source of left- and right-traveling waves in one dimension. In the latter case the configuration is always symmetric, while in the former case it may be asymmetric. Under the assumption that the group velocities and imaginary parts of the coefficients in the GL equations are small, while the nonlinear coupling coefficient is close to 1, both symmetric and asymmetric solutions are obtained analytically. In particular, it is found that the sink must be broader than the source, which seems to agree with the recently reported experimental observations [P. Kolodner, Phys. Rev. A **46**, 6431 (1992)] of the sinks and sources in traveling-wave convection, and that the wall uniquely selects the wave numbers of the colliding waves. In the asymmetric case, it is demonstrated that the boundary is moving at a certain velocity, which is also found.

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Since the pioneering works of Cross [1] and Manneville and Pomeau [2], linear defects in the form of domain walls (DW's) in two-dimensional (2D) patterns have attracted considerable attention from theorists [3–6]. In experiments with convection in pure and binary fluids, which is the most typical example of a physical system able to produce well-ordered 2D patterns, the DW's have not been in a focus, but they were observed as a "byproduct" in many works (see, e.g., Refs. [7]). Another well-controllable system allowing accurate experiments with various patterns was the so-called Faraday ripples, i.e., a liquid layer subject to vertical oscillations. In this system, a linear defect that could be certainly identified as a stable DW was observed recently [8]. As concerns the theory, the DW solutions were analyzed in rather full detail in the framework of the Ginzburg-Landau (GL) equations with real coefficients [1–6], which corresponds to the classical Rayleigh-Bénard convection in pure liquids (in systems of this type, a DW separating the trivial phase and a nontrivial one can also be generated by the so-called ramp, i.e., a spatially inhomogeneous region sandwiched between sub- and supercritical domains, which was analyzed in detail in terms of convection [9] and the Couette-Taylor flow [10]).

The objective of this work is to consider the DW's in models of wave (oscillatory) media, which can be realized, first of all, as the traveling-wave convection in binary liquids (see, e.g., Ref. [11]). In these systems, a DW may be produced in two different ways: as a result of a collision between two traveling waves propagating under different angles in 2D geometry, or as a result of a collision between left- and right-traveling waves in 1D. In the latter case, one is dealing with local defects of the sink and source type (a sink absorbs the colliding waves, while a source emits waves), which may be regarded as *spatiotemporal* DW's. Generally, the 2D case is more complicated, and it was not studied systematically. Contrary to this, interaction of the left- and right-traveling waves in 1D was considered in a number of experimental works; e.g., in Ref. [12], as well as theoretically [13,14]. Results of a very thorough experimental investigation of the source and sink defects in 1D were recently reported by Kolodner [15].

It is noteworthy that structures that are qualitatively similar to the spatiotemporal DW's in the 1D pattern-forming media may as well occur in absolutely different physical contexts. An interesting example is an exact solution of this type found in Ref. [16] for the system of two coupled equations for the pump and Stokes waves in a nonlinear optical fiber, the coupling being produced by the stimulated Raman scattering.

The theoretical description of the DW in the nonoscillatory systems is based on a pair of coupled GL equations for the amplitudes of the basic spatial harmonics which constitute the 2D pattern [3,5]. It is natural to expect that the DW in the 2D wave system may also be described by coupled GL equations. However, the GL equations for the wave system must include complex coefficients (which combine the dissipative and dispersive properties of the medium, corresponding, respectively, to the real and imaginary parts of the complex coefficients), as well as additional group-velocity terms. Similar coupled GL equations are well known to govern interaction of the left- and right-traveling waves in 1D. The coupled GL equations with the group-velocity terms but with purely real coefficients were introduced by Cross [17] (earlier, a description of the system of the same type based on a single fourth-order GL equation, which is, in fact, equivalent to a pair of the coupled second-order equations, was put forward in Ref. [18]). In the present work, the following system of the GL equations will be considered, which is similar to the equation system recently employed by Coullet

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et al. [14] for analysis of the sources and sinks in 1D:

$$(A_1)_t+c_1(A_1)_x=\gamma A_1+D_1(A_1)_{xx}+i\beta D_1(A_1)_{xx}-|A_1|^2A_1 -2\mu|A_2|^2A_1-i\alpha|A_1|^2A_1-2i\eta|A_2|^2A_1, \quad (1)$$

$$(A_2)_t-c_2(A_2)_x=\gamma A_2+D_2(A_2)_{xx}+i\beta D_2(A_2)_{xx}-|A_2|^2A_2 -2\mu|A_1|^2A_2-i\alpha|A_2|^2A_2-2i\eta|A_1|^2A_2, \quad (2)$$

where A_1 and A_2 are complex envelope functions of the two interacting waves (the left- and right-traveling waves in the 1D geometry, or the waves colliding at the DW in 2D; in the latter case, x is the coordinate perpendicular to the DW [3,5]). The coefficients $\pm c_{1,2}$, γ, D, and $\beta D_{1,2}$ account for linear properties of the medium, i.e., respectively, the mean group velocities, overcriticality, dissipation, and spatial dispersion. In the 1D case, there is a full symmetry between the two waves, hence the subscripts 1 and 2 attached to the coefficients c and D can be dropped. In the 2D case, the DW may be asymmetric, [5], i.e., the normal to it may have unequal angles θ_1 and θ_2 with the carrier wave vectors of the two waves, so that

$$c_n=c\cos\theta_n\,,\; D_n=D\cos^2\theta_n\,. \quad (3)$$

As will be demonstrated below, an important difference of the DW in the wave system from that in the nonoscillatory system is the fact that the asymmetric DW will be *moving* at a certain velocity u. In order to find a steady solution, it is then natural to consider Eqs. (1) and (2) in the reference frame moving with the velocity u, modifying Eqs. (3) as follows:

$$c_{1,2}=c\cos\theta_n\mp u\,. \quad (4)$$

The signs in front of the term u in Eq. (4) are opposite for $n=1$ and $n=2$ to compensate the opposite signs in front of c_1 and c_2 in Eqs. (1) and (2). The meaning of different nonlinear terms in Eqs. (1) and (2) is obvious. In particular, the real coefficients μ, α, and η account, respectively, for the nonlinear cross-dissipation, self-phase modulation, and cross-phase modulation.

The main issue to be addressed in this work is obtaining DW solutions of Eqs. (1) and (2) in an approximate analytical form. Guided by the experience gained in solving other problems for the generalized GL equations (e.g., finding a solution for a stable solitary pulse in the quintic GL equation [19]), one should attack the problem assuming that all the dispersive parameters β, α , and η are small. In the zeroth approximation, it is necessary to find a DW solution to Eqs. (1) and (2) for the case when these parameters are absent but the group velocities are present. In this approximation, one may look for a real time-independent solution that satisfies the equations

$$c_1(A_1)_x=\gamma A_1+D_1(A_1)_{xx}-A_1^3-2\mu A_2^2A_1\,, \quad (5)$$

$$-c_2(A_2)_x=\gamma A_2+D_2(A_2)_{xx}-A_2^3-2\mu A_1^2A_2\,. \quad (6)$$

Equations (5) and (6) cannot be solved analytically (except for the case when one may completely neglect the second derivatives [14]; however, this case seems oversimplified). Moreover, the analysis developed in Ref. [5] has shown that, even in the case $c_{1,2}=0$, exact solutions can be found only in exceptional cases. The most important tractable case was that when the nonlinear coupling coefficient 2μ was close to 1,

$$2\mu=1+\epsilon,\quad \epsilon\ll 1\,. \quad (7)$$

The inequality (7) will be adopted in this work too. Simultaneously, it will be assumed that the group velocities $c_{1,2}$ are also small. Note that it makes sense to consider only positive ϵ, as otherwise the solutions with constant $A_1\neq 0, A_2=0$ or $A_2\neq 0, A_1=0$, which correspond to the uniform phases separated by the DW, are unstable [5].

In the case when ϵ and $c_{1,2}$ are small, it is natural, following Ref. [5], to introduce the polar variables:

$$A_1\equiv R\cos\chi,\, A_2\equiv R\sin\chi\,. \quad (8)$$

It is easy to understand that, in the lowest approximation with respect to the small parameters, one may assume the amplitude R to be constant, reducing Eqs. (5) and (6) to a single equation for the angle χ. The simplest way to derive this equation is to notice that Eqs. (5) and (6) with $c_{1,2}=0$ can be deduced from the Hamiltonian

$$H=\frac{1}{2}\left[D_1\left(\frac{dA_1}{dx}\right)^2+D_2\left(\frac{dA_2}{dx}\right)^2\right]+\frac{1}{2}\gamma(A_1^2+A_2^2) -\frac{1}{4}(A_1^2+A_2^2)^2-\frac{1}{2}\epsilon A_1^2A_2^2\,. \quad (9)$$

The terms in Eqs. (5) and (6) proportional to $c_{1,2}$ break the conservation of the Hamiltonian according to the following obvious equation:

$$\frac{dH}{dx}=c_1\left(\frac{dA_1}{dx}\right)^2-c_2\left(\frac{dA_2}{dx}\right)^2\,. \quad (10)$$

Next, one may make use of the well-known balance equations technique (see, e.g., Ref. [20]), inserting Eqs. (8) into Eq. (10), assuming in accordance with what was said above that $R=R_0\equiv$ const, and calculating the left- and right-hand sides of the equation. After a simple algebra, this procedure gives rise to the following equation for $\chi(x)$:

$$(D_1\sin^2\chi+D_2\cos^2\chi)\chi''+\frac{1}{2}(D_1-D_2)\sin(2\chi)(\chi')^2 -\frac{1}{4}\epsilon R_0^2\sin(4\chi)=(c_1\sin^2\chi-c_2\cos^2\chi)\chi'\,, \quad (11)$$

the prime standing for $\frac{d}{dx}$. In the symmetric case ($D_1=D_2\equiv D$, $c_1=c_2\equiv c$), Eq. (11) simplifies to

$$\chi''=\frac{1}{4}\epsilon D^{-1}R_0^2\sin(4\chi)-cD^{-1}\cos(2\chi)\chi'\,. \quad (12)$$

Note that at $c_{1,2}=\epsilon=0$ the Hamiltonian (9) conserves the angular momentum $M=A_1A_2'-A_2A_1'\equiv R^2\chi'$. It is straight-

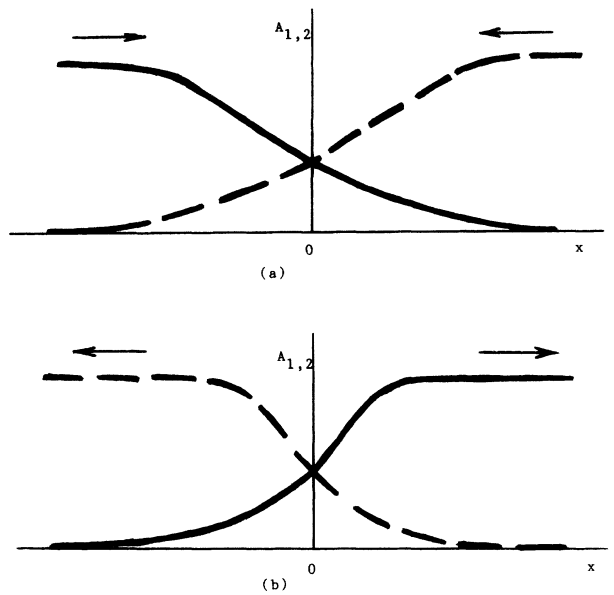

FIG. 1. The schematic structure of the sink (a) and source (b) solutions as given by Eqs. (8) and (13). The continuous and dashed lines depict, respectively, the amplitudes $A_1(x)$ and $A_2(x)$. The arrows show the sign of the group velocity associated with each amplitude.

forward to check that application of the balance-equations to the quantity M when c and ϵ are different from zero leads exactly to the same Eq. (12).

The DW solution to Eq. (12) can be found in an exact form:

$$\chi(x)=\tan^{-1}[\exp(\kappa x)], \quad \kappa=\frac{1}{2}D^{-1}(-c\pm\sqrt{c^2+4\epsilon D R_0^2}). \tag{13}$$

To construe this solution, one should insert it back into Eqs. (8). Then, setting, for the definiteness, $c>0$, one sees that the solutions with the upper and lower signs in Eq. (13) take the form shown, respectively, in Figs. 1(a) and 1(b). In both cases, one has asymptotically, at $x\rightarrow\pm\infty$, one of the amplitudes $A_{1,2}$ equal to R_0 and another equal to zero. Thus, this solution may indeed be regarded as a boundary separating domains filled with two different phases. In Fig. 1, the arrows show the sign of the group velocity of the waves in each phase according to the choice $c>0$. One immediately notices that the DW shown in Fig. 1(a) is a sink, while the one in Fig. 1(b) is a source. As follows from Eq. (13), the sink and source have different widths $L\sim|\kappa|^{-1}$:

$$L_{\mathrm{sink}}/L_{\mathrm{source}}=\frac{\sqrt{4\epsilon D R_0^2+c^2}+c}{\sqrt{4\epsilon D R_0^2+c^2}-c}. \tag{14}$$

So, Eq. (14) predicts that the sink should be broader than the source. Plots representing a detailed structure of the sinks and sources observed in the traveling-wave convection can be found in Ref. [15]. It seems that, according to those plots (see, in particular, Fig. 8a), the experimentally observed sinks are indeed broader than the sources. However, the difference in their widths is not very conspicuous, which can also be explained in the framework of Eq. (14). Indeed, it is known from the empirical data for the traveling-wave convection (see Ref. [21] and references therein) that the effective group velocity of the traveling waves is rather small. At the same time, an actual value of the parameter ϵ defined by Eq. (6) that corresponds to the real traveling-wave convection should not be really small. Then, Eq. (17) with c small and ϵ nonsmall yields the ratio of the two widths, which is only slightly larger than 1.

The analysis developed above applies only to the symmetric case. It will be shown now that in the asymmetric case ($c_1\neq c_2, D_1\neq D_2$) the DW will be moving, which is a serious difference from the nonwave systems [5]. To tackle the asymmetric situation, one may consider the case when $c_{1,2}$ are much smaller than ϵ. In the zeroth approximation, $c_{1,2}=0$, the asymmetric DW does exist, as was said above, and in this approximation the corresponding function $\chi(x)$ is determined by the equation [5]

$$\chi'=\frac{1}{2}\sqrt{\epsilon}R\sin(2\chi)(D_1\sin^2\chi+D_2\cos^2\chi)^{-1/2}. \tag{15}$$

Next, substitution of Eqs. (8) with $R_0=\mathrm{const}$ and $c_1\neq c_2$ into the energy-balance equation (10) yields

$$\frac{dH}{dx}=R_0^2(c_1\sin^2\chi-c_2\cos^2\chi)(\chi')^2. \tag{16}$$

To proceed further, it is necessary to insert Eq. (15) into Eq. (16) and calculate the net change ΔH of the Hamiltonian along the solution from $x=-\infty$ to $x=+\infty$:

$$\Delta H\equiv\int_{-\infty}^{+\infty}\frac{dH}{dx}\,dx. \tag{17}$$

Calculation of the integral yields

$$\Delta H=\pm\frac{1}{3}(\sqrt{D_1}+\sqrt{D_2})^{-2}\sqrt{\epsilon}R_0^3(2c_1\sqrt{D_2}-2c_2\sqrt{D_1}+c_1\sqrt{D_1}-c_2\sqrt{D_2}), \tag{18}$$

where + and − correspond, respectively, to the source and sink. On the other hand, the DW boundary conditions according to which $A_1=0$, $A_2^2=R_0$ at one infinity, and $A_2=0$, $A_1^2=R_0$ at the opposite infinity correspond to equal values of the Hamiltonian (9), which demands $\Delta H=0$. Now, inserting into Eq. (18) the coefficients $D_{1,2}$ from Eq. (3) and $c_{1,2}$ from Eq. (4), one immediately finds that ΔH vanishes at

$$u=\frac{1}{3}c(\cos\theta_1-\cos\theta_2). \tag{19}$$

Thus, the asymmetric DW moves at the velocity given by Eq. (19). It is relevant to emphasize that this expression has a dynamical origin, and it should not be interpreted as a kinematic formula for the velocity of a point of intersection of two moving lines.

Thus far, the dispersive terms in Eqs. (1) and (2) have been ignored. They will now be taken into account as small perturbations, and it will be demonstrated that they lend each wave A_1 and A_2 a uniquely determined wave number. Only the symmetric case will be considered. The solution is sought as

$$A_n = R_n(x)\exp[i\phi_n(x) - i\omega t], \tag{20}$$

where $n=1,2$, and the amplitude functions $R_{1,2}(x)$ are taken as given by the right-hand sides of Eqs. (8), with $R \equiv R_0$ and $\chi(x)$ as per Eq. (13). Substitution of Eq. (20) into Eqs. (1) and (2) leads to equations for the phases which can be conveniently written as follows:

$$D(R_n^2 q_n)' = -\omega R_n^2 - \beta R_n R_n'' + \alpha R_n^4 + 2\eta R_{3-n} R_n^2, \tag{21}$$

where $q_n \equiv \phi'$. At $x = \pm\infty$, Eqs. (21) yield an expression for the frequency: $\omega = \alpha R_0^2$. The next step is to find the asymptotic wave numbers $Q_n \equiv q_n(x = \pm\infty)$. Integrating both parts of Eq. (21) over x from $-\infty$ to $+\infty$, one notices that the integral of the left-hand side, which is a full derivative, is $\pm D R_0^2 Q_n$, while the integral of the right-hand side can be calculated explicitly if one inserts the expressions (8) for R_n, and then the expression (13) for $\chi(x)$. Eventually, it is easy to find

$$Q_n = (-1)^n (D\kappa)^{-1} R_0^2 \left[\left(\frac{1}{2}\alpha - \eta \right) R_0^2 - \frac{1}{2}\beta\kappa^2 \right]. \tag{22}$$

Equations (22) demonstrate how the DW performs the *wave number selection* for both domains, which is a well-known general problem of the pattern formation theory [10,11].

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[1] M.C. Cross, Phys. Rev. A **25**, 1065 (1982).
[2] P. Manneville and Y. Pomeau, Philos. Mag. A **48**, 607 (1983).
[3] Y. Pomeau, Physica D **23**, 3 (1986).
[4] M.C. Cross, G. Tesauro, and H.S. Greenside, Physica D **23**, 12 (1986).
[5] B.A. Malomed, A.A. Nepomnyashchy, and M.I. Tribelsky, Phys. Rev. A **42**, 7244 (1990).
[6] E.Y. Kuo and M.C. Cross, Phys. Rev. E **47**, R2245 (1993).
[7] G. Ahlers, D.S. Cannell, and V. Steinberg, Phys. Rev. Lett. **54**, 1373 (1985); V. Steinberg, G. Ahlers, and D.S. Cannell, Phys. Scr. **32**, 534 (1985); E. Bodenschatz, J.R. de Bruyn, G. Ahlers, and D.S. Cannell, Phys. Rev. Lett. **67**, 3078 (1991).
[8] W.S. Edwards and S. Fauve, J. Fluid Mech. (to be published).
[9] L. Kramer, E. Ben-Jacob, H. Brandt, and M.C. Cross, Phys. Rev. Lett. **49**, 1891 (1982); Y. Pomeau and S. Zaleski, J. Phys. (Paris) Lett. **44**, L135 (1983).
[10] H. Riecke and H.-G. Paap, Phys. Rev. A **33**, 547 (1986); Phys. Rev. Lett. **59**, 2570 (1987).
[11] W. Schöpf and W. Zimmermann, Phys. Rev. A **41**, 1145 (1990).
[12] E. Moses, J. Fineberg, and V. Steinberg, Phys. Rev. A **35**, 2757 (1987); P. Kolodner, D. Bensimon, and C.M. Surko, Phys. Rev. Lett. **60**, 1723 (1988).
[13] E. Knobloch, Phys. Rev. A **34**, 1538 (1986); A.E. Dean, E. Knobloch, and J. Toomre, Phys. Rev. A **37**, 1817 (1988).
[14] P. Coullet, T. Frisch, and F. Plaza, Physica D **62**, 75 (1993).
[15] P. Kolodner, Phys. Rev. A **46**, 6431 (1992).
[16] D.N. Christodoulides, Opt. Commun. **86**, 431 (1991).
[17] M.C. Cross, Phys. Rev. Lett. **57**, 2935 (1986).
[18] B.A. Malomed, Z. Phys. B **55**, 241 (1984); **55**, 249 (1984).
[19] W. van Saarloos and P.C. Hohenberg, Phys. Rev. Lett. **64**, 749 (1990); V. Hakim, P. Jakobsen, and Y. Pomeau, Europhys. Lett. **11**, 19 (1990); B.A. Malomed and A.A. Nepomnyashchy, Phys. Rev. A **42**, 6009 (1990).
[20] P.S. Lomdahl, O.H. Olsen, and M.R. Samuelsen, Phys. Rev. A **29**, 350 (1984).
[21] P. Kolodner, J.A. Glazier, and H.L. Williams, Phys. Rev. Lett. **65**, 1579 (1991).