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Infinitely divisible distributions in turbulence

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The imbedding of the scale similarity of random fields into the theory of infinitely divisible probability distributions is considered. The general probability distribution for the breakdown coefficients of turbulent energy dissipation is obtained along with corresponding similarity exponents. Related issues of self-similarity and asymptotic behavior of statistical characteristics are also considered.

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The concept of scale similarity of random fields was developed more than a quarter century ago [1–3] (see also a more recent account of this theory [4] with some additional details, including the multifractal representation). The original purpose of this concept was to describe the phenomenon of self-similar intermittency of turbulent flows which was observed experimentally [5]. However, the concept is quite general and its applications (and rediscoveries) have emerged in many areas of science, as diverse as biology and astrophysics. The main purpose of this Rapid Communication is to imbed the concept of scale similarity into the theory of infinitely divisible probability distributions. This will give us access to the well developed mathematical apparatus, which can be used not only for the description of experimental data but also for derivation of scale similarity from basic principles (from the Navier-Stokes equations in the case of turbulence).

We can distinguish between discrete and continuous self-similarity. In the discrete case there is a preferable scale factor leading to the logarithmically periodic modulations [1,4] (this phenomenon sometimes is called lacunarity [6,7]). In this paper we concentrate on the continuous self-similarity without a preferable scale factor.

For simplicity consider a one-dimensional section of a non-negative scalar field (e.g., dissipation rate). This is in accord with experimental reading in time (with the aid of the "frozen-flow" hypothesis). Thus ϵ_r is the dissipation (or similar quantity, see Ref. [3]) averaged over the segment r. For a turbulent flow which is locally isotropic in scales less than a certain external scale L, the two- and three-dimensional statistical characteristics have the same form (see obvious exceptions in Ref. [4]).

Consider inertial range of scales $l_* \ll r \ll L$, where l_* is an inertial scale, which can differ from the Kolmogorov internal scale because of the intermittency correction [1,8,9]. In the inertial range we shall single out three segments inserted in one another with length $r<\rho<l$ and introduce corresponding breakdown coefficients (BDC's):

$$q_{r,l} \equiv \epsilon_r/\epsilon_l \leq l/r, \tag{1}$$

$$q_{r,l} = q_{r,\rho} q_{\rho,l}. \tag{2}$$

In (1) we utilized the fact that the dissipation rate is non-negative. The scale similarity is determined by the following conditions: (i) probability distribution for BDC's depends only on the ratio of the corresponding scales and (ii) $q_{r,\rho}$ and $q_{\rho,l}$ are statistically independent [instead of (ii) we can use a less restrictive condition [4]]. More general conditions, which take into account exact relative positions of segments, are considered in Ref. [3] (see also discussion below). For the moments of BDC's from conditions of scale similarity and (2), we have

$$\langle q_{r,l}^p \rangle = a_p(l/r) = a_p(\rho/r) a_p(l/\rho), \tag{3}$$

$$a_p(l/r) = (l/r)^{\mu(p)}, \quad \mu(0)=0. \tag{4}$$

Here $\langle \ \rangle$ means statistical averaging and we used arbitrariness of ρ and normalization of probability. In Refs. [2,3] it was shown that

$$\mu(1)=0, \quad 0<\mu(2)\equiv\mu<1, \tag{5}$$

$$\mu(p+\delta)-\mu(p) \leq \delta \quad (\delta \geq 0) \tag{6}$$

$$\mu(p) \leq \mu + p - 2 \quad (p \geq 2). \tag{7}$$

Inequality (6) follows from (1), and (7) follows from (6) and (5). The probability density $W(q,l/r)$ for $q_{r,l}$ is uniquely defined by the set of $\mu(p)$ with integer p $(p=0,1,2,\dots)$, because (7) ensures the fulfillment of the Carleman condition [10]:

$$\sum_{p=1}^{\infty} (a_{2p})^{-1/2p} = \infty. \tag{8}$$

If function $\mu(p)$ has analytical continuation into the complex domain, then the characteristic function of $\ln(q_{r,l})$ has the form

$$\psi(s,l/r) = \langle \exp(is \ln q_{r,l}) \rangle = \int_0^{l/r} q^{is} W(q,l/r)\, dq = (l/r)^{\mu(is)}. \tag{9}$$

The last equality in (9) can be inverted, giving

$$W(q,l/r)=\frac{1}{2\pi q}\int_{-\infty}^{\infty}\exp[-is\ \ln q+\mu(is)\ln(l/r)]ds. \quad (10)$$

The standard way to construct a model is to make an assumption, say, about $W(q,2)$. From (4) we obtain

$$\mu(p)=\log_2\left[\int_0^2 q^p W(q,2)dq\right]. \quad (11)$$

Then, from (10) [or from a lengthier formula [3,4] when $\mu(p)$ is not analytical] we may calculate $W(q,l/r)$. In order for the model to have physical and mathematical sense, we must ensure that *for arbitrary* l/r the probability density W is non-negative and properly normalized by integration over the finite interval $[0,l/r]$. Generally, it is not easy to verify these conditions analytically or even numerically. Thus, many models, proposed in the literature, remain questionable.

The solution to this problem comes from the observation that for arbitrary l/r and arbitrary integer n, Eq. (9) can be written in the form

$$\psi(s,l/r)=\psi^n(s,(l/r)^{1/n}). \quad (12)$$

Thus, $\ln q_{r,l}$ has the infinitely divisible probability distribution [10]. Now we can use the Lévy-Baxter-Shapiro (LBS) theorem [10], which gives the general form of the characteristic function for the infinitely divisible distribution, concentrated on interval $[0,\infty)$:

$$\chi(s)=\exp\left\{ibs-\int_0^{\infty}\frac{1-e^{isx}}{x}P(dx)\right\},\quad b\geqslant 0. \quad (13)$$

Here P is a measure on the open interval $(0,\infty)$ such that $(1+x)^{-1}$ is integrable with respect to P. From (9) and (13), by using variable $z_{r,l}=-\ln[(r/l)q_{r,l}]$ $(0\leqslant z_{r,l}<\infty)$, we get

$$\mu(p)=\kappa p-\int_0^{\infty}\frac{1-e^{-px}}{x}F(dx), \quad (14)$$

$$\kappa=1-\frac{b}{\ln(l/r)}\leqslant 1,\quad P(dx)=\ln(l/r)F(dx). \quad (15)$$

Here measure F has the same (stated above) properties as P. From the first equality in (5) we have an additional condition on the measure:

$$\int_0^{\infty}\frac{1-e^{-x}}{x}F(dx)=\kappa\leqslant 1. \quad (16)$$

All other properties of $\mu(p)$ and $W(q,l/r)$ are ensured by the theorem (13). Thus we get an alternative approach for modeling, namely, by choosing measure F.

Consider, for example, measure with the density

$$F'(x)=Ax^{\alpha-1}\exp(-x/\sigma), \quad (17)$$

where A, α, and σ are positive constants. From (14) and (16) we get

$$\mu(p)=\kappa\left[p-\frac{(p\sigma+1)^{1-\alpha}-1}{(\sigma+1)^{1-\alpha}-1}\right],\quad \alpha\neq 1 \quad (18)$$

$$\mu(p)=\kappa\left[p-\frac{\ln(p\sigma+1)}{\ln(\sigma+1)}\right],\quad \alpha=1. \quad (19)$$

A particular case of (19) with $\kappa=\sigma=1$ was considered in Ref. [3] and corresponds to the constant density $W(q,2)$. The density $W(q,l/r)$ for arbitrary l/r was calculated analytically in Ref. [3] and it has all the necessary properties. Another case of (19) with $\kappa=1$, $\sigma\approx 0.283$ was considered recently [11] and compared with experimental data on $\mu(p)$. Formula (19) with $\kappa=1$ corresponds to the Γ distribution [10,11] for $z_{r,l}$. A particular case of (18) with $\kappa=1$, $\alpha=\frac{1}{2}$ was also considered recently [12] and corresponds to the distribution of $z_{r,l}$ presented in Ref. [10], Problem 14, Chap. 13. As far as I know, the general formula (18) and corresponding probability distribution has not been considered before, at least, in the literature on turbulence.

Experimental data on $\mu(p)$ (see, for example, Ref. [13]) corresponds to a smooth curve, which can be easily fitted by choosing parameters in (18) or (19). A more sensitive criterion will be comparison with detailed experimental data on probability density $W(q,l/r)$ for a variety of scale factor l/r. Some measurements of $W(q,l/r)$ have been presented in the literature [14,15]. However, in order to find an optional model, we need a more systematic study. A particular interest is in the asymptotic behavior of $W(q,l/r)$ when q approaches its maximum value l/r. Physically it corresponds to local events when all dissipation in the interval l is concentrated in the subinterval $r<l$.

Let us note that, having in mind the scale similarity conditions, the optimal position of subinterval r is when its center coincides with the center of interval l. To demonstrate this, consider the opposite situation when one of the boundaries of subinterval r coincides with a boundary of interval l. In this case, the complementary subinterval $l-r$ can be considered on the same footing as subinterval r, taking into account the local isotropy of the turbulent dissipation field. From definition (1) we get a relation between corresponding BDC's:

$$q_{r,l}=\lambda-(\lambda-1)q_{l-r,l},\quad \lambda=l/r>1. \quad (20)$$

Here λ is the scale factor for $q_{r,l}$, and the corresponding scale factor for $q_{l-r,l}$ is $\lambda/(\lambda-1)$. By squaring both sides of (20), statistical averaging and using (5), we have

$$\langle q_{r,l}^2\rangle=\lambda(2-\lambda)+(\lambda-1)^2\langle q_{l-r,l}^2\rangle. \quad (21)$$

Here the statistical averaging is over the whole dissipation field or over the joint probability distributions of two BDC's. Equation (4), with the definition of μ in (5), gives

$$\lambda^{\mu}=\lambda(2-\lambda)+(\lambda-1)^2\left(\frac{\lambda}{\lambda-1}\right)^{\mu}. \quad (22)$$

We see that, no matter the value of μ (experimentally $\mu\approx 0.2$), Eq. (22) cannot be satisfied for arbitrary λ. An exception is $\lambda=2$. For the central position of the subinterval we do not have this contradiction with scale similarity.

Returning to the asymptotic of $W(q,\lambda)$ where q approaches its maximum value λ, consider the case of a gap:

$$\int_{\lambda_1}^{\lambda} W(q,\lambda)dq=0, \quad \lambda_1<\lambda. \tag{23}$$

From (4) and (23) we get

$$\mu(p)=\log_\lambda\left\{\int_0^{\lambda} q^p W(q,\lambda)dq\right\}\leqslant p\ \log_\lambda(\lambda_1). \tag{24}$$

Thus

$$h=\lim_{p\to\infty}\frac{\mu(p)}{p}\leqslant\log_\lambda(\lambda_1)<1. \tag{25}$$

The value of h is important for the description of turbulent velocity increments [16,17]. If there is no gap, then

$$1\geqslant h\geqslant \lim_{p\to\infty}\frac{1}{p}\log_\lambda\left\{\lambda_i^p\int_{\lambda_i}^{\lambda} W(q,\lambda)dq\right\}=\log_\lambda(\lambda_i) \tag{26}$$

for any $\lambda_i<\lambda$. Thus $h=1$. We hope that the presented results, including the connection between self-similarity and infinitely divisible distributions, can serve as a basis for more detailed experimental, analytical and numerical studies of turbulence and other phenomena with scale similarity.

After this paper was submitted for publication, we became aware of two works which are relevant to the subject [18,19]. The approach in Refs. [18,19] is more heuristic. Let us note, the Lévy-Khinchin representation in Ref. [18] does not take into account that the energy dissipation rate ϵ is non-negative. The non-negativity leads to important constraints (5)–(7) and to the Carleman condition (8) (see details in Refs. [3,4]). The LBS theorem (13) does the precise job, reflecting all these facts. The concrete model, studied in Refs. [18,19], corresponds to $h=\frac{2}{3}$ and, thus, to a substantial gap (23) in the probability density function. Such a gap contradicts the existing experimental data [14,15]. Let us stress that more detailed experimental studies of $W(q,\lambda)$ are needed in order to find an optimal model.

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[1] E. A. Novikov, Dokl. Akad. Nauk SSSR **168**, 1279 (1966) [Sov. Phys. Dokl. **11**, 497 (1966)].
[2] E. A. Novikov, Dokl. Akad. Nauk SSSR **184**, 1072 (1969) [Sov. Phys. Dokl. **14**, 104 (1969)].
[3] E. A. Novikov, Prikl. Mat. Mekh. **35**, 266 (1971) [Appl. Math. Mech. **35**, 231 (1971)].
[4] E. A. Novikov, Phys. Fluids A **2**, 819 (1990).
[5] E. A. Novikov and R. W. Stewart, Izv. Ser. Geofiz. No. 3, 408 (1964) [Izv. Akad. Nauk SSSR Geophys. Ser. No. 3, 245 (1964)].
[6] B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983).
[7] L. A. Smith, J. D. Fournier, and E. A. Spiegel, Phys. Lett. **114A**, 465 (1986).
[8] E. A. Novikov, Appl. Math. Mech. **27**, 1445 (1963).
[9] G. Paladin and A. Vulpiani, Phys. Rev. A **35**, 1971 (1987).
[10] W. Feller, *An Introduction to Probability Theory and its Applications* (Wiley, New York, 1991) (reprinted from 1966 edition), Vol. 2.
[11] Y. Saito, J. Phys. Soc. Jpn. **61**, 403 (1992).
[12] P. L. Vanyan (unpublished).
[13] C. Meneveau and K. R. Sreenivasan, Nucl. Phys. B, Proc. Suppl. **2**, 49 (1987).
[14] C. W. Van Atta and T. T. Yeh, J. Fluid Mech. **71**, 417 (1975).
[15] A. B. Chhabra and K. R. Sreenivasan, Phys. Rev. Lett. **68**, 2762 (1992).
[16] E. A. Novikov, Phys. Rev. A **46**, R6147 (1992).
[17] G. Pedrizzetti and E. A. Novikov, J. Fluid Mech. (to be published).
[18] Z. S. She and E. Waymire (unpublished).
[19] B. Dubrulle, Phys. Rev. Lett. **73**, 959 (1994).