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Oscillatory instability of crack propagations in quasistatic fracture

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Quasistatic crack propagations in a thin plate are studied theoretically. The Griffith theory is applied to determine a crack extension condition and the motion of crack tips in straight propagation. A linear stability problem for the straight propagation is formulated, based on the assumption that the crack tip moves in such a way that a singular shear stress is made to vanish. It is shown that straight propagations become unstable under certain conditions and that an oscillatory propagation appears. The critical conditions are calculated quantitatively, and the results are compared with the corresponding experiments.

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The study of fracture has developed greatly since Griffith wrote a breakthrough paper in 1920 [1]. This progress, however, has been mainly in the fields of engineering science. Phenomena at the most fundamental level of the physical processes involved, such as crack speed, crack branching, dynamical instability, etc., have not yet been fully understood [2,3]. One of the obstacles to this understanding is that well-controlled experiments are difficult to perform. Recently, however, Yuse and Sano have carried out a nice experiment by which they were able to make reproducible crack patterns in a thin glass plate [4]. In their experiment, a heated thin glass plate with a notch is dipped into cold water at a constant velocity v. It was observed that various forms of crack patterns developed depending on the velocity v and the temperature difference δT between the heater and the water. Nothing happens when δT and v are small enough, but as these parameters are increased, a straight crack starts to extend along the center line of the strip of glass at a certain point. If the parameters are increased further, a transition from the straight crack to a wavy crack occurs. Being motivated by the experiment, numerical simulations based on simple models have been performed [5,6], and the transitions have been simulated in these systems. The experiment has also been analyzed theoretically [7], and the velocity dependence of the fracture energy has been estimated from

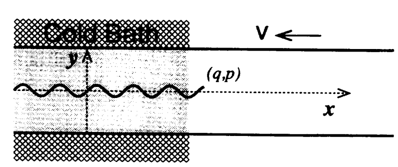

FIG. 1. Schematic figure of an experimental configuration. The glass plate is submerged in the cold bath represented by the dotted region.

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the experimental data for transition points to the wavy crack. In this paper, we will present a theoretical analysis for the experiment and show that the transition to an oscillatory propagation can be understood as a Hopf bifurcation.

The experimental configuration we will analyze is illustrated in Fig. 1. The coordinate system (x,y) is fixed on the infinitely long strip of thin glass whose boundaries are located at $y=\pm W/2$. The portion satisfying $x<s(t)$ of the glass plate is in a cold bath, where t denotes the time. As the interface at $x=s(t)$ moves upward (right in the figure) at $ds/dt=v$, the crack tip also advances at an average speed v. This fact implies that the crack tip moves adiabatically with respect to the change of the external conditions. We thus parametrize the time evolution of the position of the crack tip by s as $(q(s),p(s))$. Such a fracture is called a quasistatic fracture, because the time t comes into the problem only through $s(t)$.

The problem we address is to derive an equation of motion for the crack tip. In order to solve it, we focus on infinitely narrow cracks for mathematical simplicity. Then, in the linear elastic theory, stresses σ_{ij} diverge at the crack tip. Using a polar coordinate system (r,θ), we can expand σ_{ij} in $\sqrt{r}$ around the crack tip, and the singular part of the stress is known to be characterized by stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ related with basic "modes" of crack-surface displacement called mode I (opening mode) and mode II (sliding mode), respectively as

$$\sigma_{ij}^{(\rm sing)}=\frac{K_{\rm I}}{\sqrt{2\pi r}}\Sigma_{ij}^{(1)}(\theta)+\frac{K_{\rm II}}{\sqrt{2\pi r}}\Sigma_{ij}^{(2)}(\theta), \tag{1}$$

where $\Sigma_{ij}^{(1)}(\theta)$ and $\Sigma_{ij}^{(2)}(\theta)$ are universal functions [8]. We will express the equation of motion for the crack tip using these stress intensity factors in the next paragraph.

According to the Griffith theory [1], when the strain-energy release rate G exceeds the energy needed to create a new surface Γ, the crack begins to extend at a speed which is on the order of the propagation of sound. If a temperature profile were fixed, however, the crack would be arrested immediately at a position where G equals Γ, because elastic forces arising from nonuniform thermal expansions are localized around a region where the temperature gradient is large. Then, if a temperature profile were translated infinitesimally along the plate, the crack tip would move, because $G>\Gamma$, and then stop again at $G=\Gamma$ after infinitesimal extension. Since the translation velocity v is much smaller than the sound velocity in glass, the crack propagates so as to maintain the equality $G=\Gamma$. Another condition, which is necessary to determine the crack tip position $(q(s),p(s))$, is given from observations that a crack in brittle material extends toward the direction where the shear stress vanishes [9]. This leads to the condition that the equality

$$K_{\rm II}=0 \tag{2}$$

is maintained while the crack extends smoothly [9]. In this case, the energy release rate G is related to the stress intensity factor $K_{\rm I}$ as $G=K_{\rm I}^2/E$, where E is a Young modulus [10]. By introducing a critical stress intensity factor $K_{\rm I}^c$ such that $K_{\rm I}^c=\sqrt{\Gamma E}$, the equation $G=\Gamma$ is reduced to

$$K_{\rm I}=K_{\rm I}^c. \tag{3}$$

(2) and (3) are our basic equations describing crack propagations in quasistatic fracture. Note that the equations define a non-Markov dynamical system because $K_{\rm I}$ and $K_{\rm II}$ depend on the crack pattern which is identical to the history of the crack tip. We will discuss motion of the crack tip based on Eqs. (2) and (3).

First, we describe an outline of our analysis. Suppose that a crack is positioned on the center line of a glass plate where the shear stress vanishes. Then, the equality $K_{\rm II}=0$ holds for the straight crack. Since $K_{\rm I}$ is a function of $q(s)-s$ due to translational symmetry in the x direction, we obtain

$$q(s)=s+q_0, \tag{4}$$

where q_0 is a constant given by (3). Note that there are two solutions, if they exist, but only the larger one is physically relevant because it corresponds to the position where $K_{\rm I}$ decreases for infinitesimal crack extension [11]. When there is no solution of (4), the crack cannot extend. Therefore the critical condition for the straight crack extension is given by an existence condition for solutions of Eq. (3). We next investigate the linear stability of the straight propagation. We can define an eigenvalue problem for the equation obtained by linearizing (2) in p, and the long time behavior of $p(s)$ is characterized by the eigenvalue with the largest real part, denoted by z_*, such that $p(s)\sim\exp(sz_*)$ for $s\to\infty$. When the real part of z_*, $\mathrm{Re}(z_*)$, is positive, the straight propagation is unstable, and the crack tip deviates against the center line. Then, if $\mathrm{Im}(z_*)\neq 0$, the crack tip oscillates with a wavelength $2\pi/\mathrm{Im}(z_*)$. Therefore the oscillatory instability observed in experiments can be explained by checking the change of the sign of $\mathrm{Re}(z_*)$ and finiteness of $\mathrm{Im}(z_*)$. The transition point to the oscillatory propagation is given by the condition $\mathrm{Re}(z_*)=0$.

Now, we will express the stress intensity factors $K_{\rm I}$ and $K_{\rm II}$ in terms of the stresses σ_{ij}^0 on glass plates without cracks. Since the existence of a crack alters only the boundary conditions, the stresses on the glass plate with a crack are given by $\sigma_{ij}^0+\sigma_{ij}^*$, where σ_{ij}^* are the stresses without the temperature gradient but with fictitious external forces introduced so as to satisfy the boundary conditions along the crack. Then, a singularity appears in the stresses σ_{ij}^*, but it is not easy to calculate σ_{ij}^* under the proper boundary conditions for the plate with a finite width W. For a plate with infinite width, however, the expression for the stress intensity factors has been given within the linear approximation in p [9], and in the present configuration they are

$$K_{\rm I}=\sqrt{\frac{2}{\pi}}\int_0^\infty du\,\frac{\sigma_{yy}^0(q_u,0)}{\sqrt{u}}, \tag{5}$$

$$K_{\rm II}=\sqrt{\frac{2}{\pi}}\int_0^\infty du\,\frac{1}{\sqrt{u}}\left\{\frac{\partial\sigma^0_{xx}(q_u,0)}{\partial u}\,p(s)+\frac{1}{2}\,\sigma^0_{yy}(q_u,0)\,\frac{dp(s)}{ds}\right\}+\frac{1}{\sqrt{2\pi}}\int_0^\infty du\,\frac{\sigma^0_{xx}(q_u,0)-\sigma^0_{yy}(q_u,0)}{u^{3/2}}[p(s_u)-p(s)], \tag{6}$$

where we employ the abbreviated notations $q_u \equiv s+q_0-u$ and $s_u \equiv s-u$. Note that $K_{\rm I}$ has no linear term in p and the apparent s dependence of $K_{\rm I}$ is deceptive because $\sigma^0_{ij}(q_u,0)$ depends only on $q_u-s=q_0-u$. The validity of this "infinite plate approximation" will be discussed later. Here, the expressions for the stresses $\sigma^0_{xx}(x,0)$ and $\sigma^0_{yy}(x,0)$ are easily derived in the form [12]

$$\sigma^0_{xx}(x,0)=-E\alpha\int_{-\infty}^{\infty}dke^{ikx}\hat{T}(k)f_-(k), \tag{7}$$

and

$$\sigma^0_{yy}(x,0)=-E\alpha\int_{-\infty}^{\infty}dke^{ikx}\hat{T}(k)[1-f_+(k)], \tag{8}$$

with

$$f_\pm(k)=\frac{kW\cosh(kW/2)\pm 2\sinh(kW/2)}{kW+\sinh(kW)}, \tag{9}$$

where α is a thermal expansion coefficient, and $\hat{T}(k)$ is a Fourier mode of the temperature field $T(x)$ relative to the cold bath. We assume that $T(x)$ is given by a steady moving solution at a velocity v of the diffusion equation $\partial_t T=D\Delta T$, where D is a thermal diffusion constant. That is,

$$T(x,t)=\theta(x-s)\,\delta T\{1-\exp[-(x-s)/d_0]\}, \tag{10}$$

where $\theta(x)$ stands for the step function and we have introduced a thermal diffusive length: $d_0=D/v$.

Using expressions (5) and (7)–(10), we can numerically calculate q_0 from (3). Then, substituting $p(s)=\exp(sz)$ into $K_{\rm II}=0$ with the expression (6), we obtain the eigenvalue equation:

$$Q(z)\equiv Q_0+Q_1 z+\delta Q(z)=0, \tag{11}$$

where

$$Q_0=-\sqrt{\frac{2}{\pi}}\int_0^\infty du\,\frac{\partial_x\sigma^0_{xx}(q_u,0)}{\sqrt{u}},\quad Q_1=\frac{1}{2}K_{\rm I} \tag{12}$$

and

$$\delta Q(z)=\sqrt{\frac{1}{2\pi}}\int_0^\infty du\,\frac{\sigma^0_{xx}(q_u,0)-\sigma^0_{yy}(q_u,0)}{u^{3/2}}\times(e^{-uz}-1). \tag{13}$$

Note that $Q(z)$ does not depend on s for the same reason as above. Solving numerically the eigenvalue equation (11) with (12),(13) and (7)–(10), we can find the eigenvalue with the largest real part, z_*.

We now introduce two dimensionless control parameters defined by $R=E\alpha\delta TW^{1/2}/K^c_{\rm I}$ and $\mu=W/d_0$, which are proportional to the temperature difference and dipping speed, respectively. We further define ξ as the ratio of the crack length out of the cold bath to the system width W: $\xi=q_0/W$. Then, $K_{\rm I}$ and $Q(z)$ are expressed as $K_{\rm I}=K^c_{\rm I}RF_{\rm I}(\mu,\xi)$, $Q(z)=K^c_{\rm I}RF_{\rm II}(\mu,\xi,\tilde{z})/W$, where $F_{\rm I}$ and $F_{\rm II}$ are dimensionless functions calculated numerically, and $\tilde{z}=zW$. We first fix the parameter value $\mu=100$. For sufficiently small R, there is no solution ξ satisfying $RF_{\rm I}(\mu,\xi)=1$, which is equivalent to $K_{\rm I}=K^c_{\rm I}$. In this case, the crack cannot extend. For $R>R^{(1)}_c$, $RF_{\rm I}(\mu,\xi)=1$ has a solution ξ, and then $\tilde{z}_*$ is determined from the equation $F_{\rm II}(\mu,\xi,\tilde{z}_*)=0$. When the temperature difference is increased, the ξ increases, and $\mathrm{Re}[\tilde{z}_*]$ becomes positive beyond $R^{(2)}_c$, as shown in Fig. 2. That is, the straight propagation is stable for $R^{(1)}_c<R<R^{(2)}_c$, while unstable for $R>R^{(2)}_c$. We have also found that $\mathrm{Im}[\tilde{z}_*]\neq 0$, which means that a Hopf bifurcation occurs at $R^{(2)}_c$ with a critical wavelength $\lambda_c=2\pi W/\mathrm{Im}[\tilde{z}_*]$. Here, $R^{(1)}_c$, $R^{(2)}_c$, and λ_c were computed as $R^{(1)}_c=3.7$, $R^{(2)}_c=11.8$, and $\lambda_c=0.06W$. Similarly, by calculating $R^{(1)}_c$ and $R^{(2)}_c$ for several values of μ, we can draw a phase diagram in a μ-R space. Our phase diagram shown in Fig. 3 is qualitatively the same as that obtained by experiments [4].

In order to see the correspondence with experiments more quantitatively, we use the following values of material constants typical for glass plates: $\alpha=7.7\times10^{-6}$ [K^{-1}], $E=7.1\times10^{10}$ [Pa], $\Gamma=8$ [J m^{-2}], and $D=4.7\times10^{-3}$ [cm^2/s]. Then, for glass plates with $W=2.4$ [cm], $\mu=100$ corresponds to $v=2.0$ [mm/s], and the critical tem-

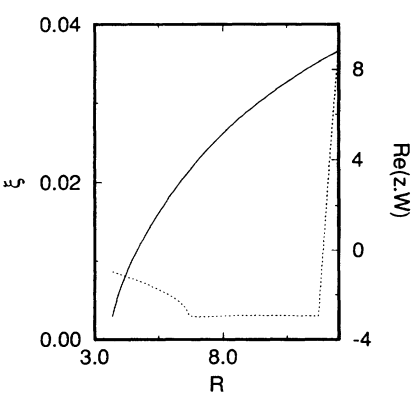

FIG. 2. ξ versus R (solid line) and $\mathrm{Re}(z_*W)$ versus R (dotted line). Vertical axes of solid and dotted lines are graduated on the left and right, respectively. These graphs start from $R^{(1)}_c$, and $\mathrm{Re}(z_*W)$ is positive for $R\geqslant R^{(2)}_c$. Here, $R^{(1)}_c=3.7$, and $R^{(2)}_c=11.8$. The dotted graph is kinked at $R=6.7$ and $R=11.7$, where the eigenvalues with the largest and second largest real part are interchanged.

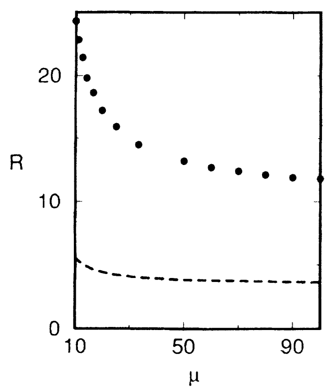

FIG. 3. Phase diagram in a μ-R space. A crack can extend in the region above the broken line. The transition to oscillatory propagation occurs at the curve defined by the dots. Below this curve, straight propagation is stable.

peratures $\delta T_c^{(1)}$ and $\delta T_c^{(2)}$ are computed as $\delta T_c^{(1)}=33$ [K], and $\delta T_c^{(2)}=105$ [K]. In the experiment [4], $T_c^{(1)}$ is in good agreement with our result, while $T_c^{(2)}$ is somewhat less than ours. (For example, under the same condition as ours, $\delta T_c^{(1)}$ and $\delta T_c^{(2)}$ were measured as about 30 [K] and 80 [K], respectively.) Further, in our theory, the critical wavelength λ_c turns out to be proportional to W for the limiting case $\mu \to \infty$. This scaling relation was found experimentally [4] and also confirmed by the numerical simulation [6]. Unfortunately, however, our value of λ_c/W, which is calculated as 0.05, is substantially smaller than the corresponding experimental value 0.3.

The only approximate treatment in our theoretical framework is the "infinite plate approximation" for the stress intensity factors in expressions (5) and (6). The approximation works better for larger W/l, where l is a decay length of the stresses $\sigma_{ij}^0(x,0)$. In the present problem, however, since l is proportional to W for the case $\mu \gg 1$, our approximation does not give a precise value for any asymptotic cases. This should be the reason why the discrepancy in wavelength arises. It is important to improve this approximation.

Recently, Marder attempted to explain the oscillatory instability [7] using the Cotterell and Rice criterion (CR criterion) for the instability of straight cracks [9], which is different from the one we have used here. The CR criterion states that straight crack propagations along the x direction are unstable when the nonsingular part of σ_{xx} at the crack tip is positive. We can show that the present formulation leads to the CR criterion if we ignore the x dependence of $\sigma_{ij}^0(x,0)$ and replace $\sigma_{ij}^0(x,0)$ by the values at the crack tip in (6)–(13). We believe that our formulation determines the transition more precisely than the CR criterion. Also, it should be pointed out that the CR criterion itself cannot be used to derive the oscillatory propagation. More detailed explanation of the correspondence between the two theories will be presented in a separate paper.

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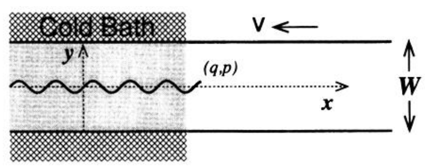

FIG. 1. Schematic figure of an experimental configuration. The glass plate is submerged in the cold bath represented by the dotted region.