Generalized Markov coarse graining and spectral decompositions of chaotic piecewise linear maps

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Spectral decompositions of the evolution operator for probability densities are obtained for the most general one-dimensional piecewise linear Markov maps and a large class of repellers. The eigenvalues obtained with respect to the space of functions piecewise analytic over the minimal Markov partition equal the reciprocals of the zeros of the Ruelle zeta functions. The logarithms of the zeros correspond to the decay rates of time correlation functions of analytic observables when the system is mixing. The space can also be extended to include piecewise analytic observables permitted to have discontinuities at the elements of any given periodic orbit(s), so that local behavior of observables can be considered. The new spectra associated with the extension are surprisingly simple and are related to the relative stability factors of the given orbit(s). Finally, arbitrarily slowly decaying periodic and aperiodic nonanalytic eigenmodes are constructed.

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I. INTRODUCTION

Sensitivity to initial conditions, together with the fact that experimental or numerical determination of the state of a physical system cannot be determined with infinite accuracy, entails that a statistical description is a natural approach to systems undergoing chaotic dynamics. At the basis of the statistical description is a linear equation governing the evolution of probability densities, which for discrete time systems such as one-dimensional recurrences is known as the Frobenius-Perron equation. The central question, then, is to determine the eigenvalues and associated eigenstates of the associated linear evolution operator U—the Frobenius-Perron operator [1]. This depends, in turn, both on the underlying dynamics and the choice of space of functions F on which this operator is acting. Once this information is available, a variety of properties, such as time correlation functions, can be evaluated for observables compatible with the choice of F. Moreover, the dynamical evolution of almost pointlike initial conditions can be simply described, both before and after the Lyapounov time, if F includes initial densities having as support small regions of phase space.

The spectral properties of the Frobenius-Perron operator have received considerable attention in the past [2–4], especially for the particular class of maps known as Markov analytic maps [5]. While the spectra cannot lie beyond the unit circle in the complex plane, their nature can vary, depending on the choice of F, from being continuous within the unit circle when F is a Hilbert space [6] to being discrete when F is a space of functions piecewise analytic over a particular discrete partition of phase space known as the minimal Markov partition [5]. The choice of a given functional space is thus far from academic. For instance, as is well known, the spectra are related to characteristic times of the system. For example, the logarithm of the absolute value of a point eigenvalue defines the decay rates of corresponding time correlation functions. Furthermore, F is connected with the scales of the phenomena that can be probed in phase space through the probabilistic description.

In general, U acts on an infinite dimensional space. Describing its properties for arbitrary densities is impossible, even for dynamical systems as simple at first sight as piecewise linear maps. In the present paper the conditions for reducing this infinite dimensional problem to a finite one are analyzed by identifying finite dimensional function spaces that are invariant under U. Each invariant space F is spanned by a basis of functions and, if U acts on a basis function, the result is a linear combination of the basis. The action of U on any basis function is thus equivalent to that of a finite dimensional matrix W acting on a vector. Therefore, any function lying in F can be represented in terms of the basis as a vector and evolved with W. Moreover, the eigenvalues and eigenvectors of W define the eigenvalues and eigenfunctions of U residing in F. The choice of space F depends both on the nature of the initial densities or measurable functions of interest and the invariance requirement.

One particular way to reduce the study of U to a finite dimensional problem is to perform a coarse graining, wherein distributions $\rho(x)$ at each time step are locally averaged in the cells of a finite partition of the system's phase space. For an M cell partition $\mathcal{P}$, a linear operator E performing the coarse graining can be defined

$$E\rho(x) = \sum_{i=1}^{M} \left(\frac{1}{\Delta_i} \int_{C_i} \rho(x)dx \right) \chi_{C_i}(x), \quad (1)$$

where $\chi_{C_i}(x)$ equals one if x belongs to the ith cell C_i and is zero otherwise, and Δ_i is the size of C_i.

When the relation

$$(EUE)^n = EU^nE \quad (2)$$

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is valid, the coarse grained description of the evolution is generated by a time independent finite dimensional matrix $W = EUE$ and a Markov coarse graining (MCG) is said to be possible [7]. A sufficient (but not necessary) condition for this is that the coarse graining and evolution operators commute when acting on coarse grained functions

$$[E,U]E = (EU - UE)E = EUE - UE = 0. \quad (3)$$

Concrete examples of MCG's have been constructed both for conservative and dissipative dynamical systems. One class of such examples, which is the chief concern of this paper, are piecewise linear maps on the unit interval [7,8] whose nondifferentiable points fall in a finite number of iterations onto a periodic orbit(s). Such maps are called piecewise linear Markov maps. The set of iterates of the nondifferentiable points forms a minimal Markov [5] partition and with respect to this partition, a MCG exists. Indeed, in this case, $EUE - UE = 0$. That is, initial densities that are piecewise constant over the partition remain so for all time. Thus the space F of functions piecewise constant over the cells of the partition is invariant under U, and the eigenvalues and right eigenvectors of W define eigenvalues and eigenfunctions of U restricted to F. Hence invariant densities are easily calculated [9]. The logarithms of the absolute values of the spectra equal the decay rates observed in time correlation functions of measurable functions lying in the same space F. Finer scale descriptions can be easily made by adding to the points defining the minimal Markov partition either a finite number of successive inverse images of one or more of these points or all the points of a finite number of periodic orbits of the iterative law $f(x)$. Both methods of partition refinement can be done simultaneously. With respect to the new partition, a MCG also exists. We notice, in this respect, that fine scale behavior attracts increasing interest in statistical mechanics in connection, for instance, with transport properties such as diffusion coefficients and escape rates [10] and low dimensional systems [11]. A simple generalization of spaces of piecewise constant functions are spaces of piecewise polynomials, that is, spaces of functions that are polynomial within each cell of $\mathcal{P}$ but are permitted to have discontinuities at each cell boundary. As it turns out, piecewise polynomial spaces of any order are also invariant under U. Therefore eigenfunctions and eigenvalues are again easily calculated. Moreover, any function $g(x)$ that is piecewise analytic (whose derivatives for all orders are bounded) over $\mathcal{P}$ can be expressed piecewise over each partition cell in terms of a uniformly convergent Taylor expansion. Alternatively, $g(x)$ can be expanded in terms of the basis of the corresponding eigenspaces of U. Thus spectral decompositions of U can be obtained. The extension of classical coarse graining to this more general and representative function space is the principal goal of the present work.

The outline of the rest of the paper and principal results are as follows. In Sec. II we define piecewise linear Markov maps and repellers and generalize the notion of Markov coarse graining to obtain a spectral decomposition of U. In this case, U is restricted to the space of functions that can be expressed, exactly or up to a good approximation, over each cell of $\mathcal{P}$ as a finite Taylor expansion. The spectra are arranged in groups corresponding to the order of the piecewise monomial space.

In Sec. III we consider the case where $\mathcal{P}$ is the minimal Markov partition. In this case, the logarithms of the absolute values of the spectra equal the decay rates of the time correlation functions of all analytic measurable functions. We prove this by showing that they equal the reciprocals of the zeros of the corresponding zeta functions [12].

In Sec. IV we consider the case of refined partitions suitable for local statistical descriptions. In particular, we derive a simple formula for their spectra. Each eigenvalue corresponds to a given periodic orbit and the associated time correlation functions exhibit a periodic decay. The results of Secs. III and IV are illustrated in Sec. V with simple examples of maps from the tent family.

In Sec. VI the eigenvalues associated with measurable functions that are generalized piecewise polynomials (powers of x raised to a fractional or to a complex exponent) are considered. It turns out that exponents can be chosen so that the corresponding spectra lie anywhere in the unit circle. Arbitrarily slowly decaying eigenstates can thus be constructed which are highly localized about the periodic orbits.

There exists extensive literature about piecewise linear Markov maps. The main subjects of concern have been the behavior of time correlation functions and the calculation of invariant measures rather than a full analysis of the Frobenius-Perron operator. In the context of the present paper, the works of Mori *et al.* [8] and Györgi and Szepfalusy [13] are particularly relevant. They investigated the properties of time correlation functions of polynomial observables by considering the corresponding eigenvalue problem of U. In a related work [14] evidence was provided that certain piecewise linear maps that do not possess finite minimal Markov partitions can be well approximated as Markov maps, which is important because the Markov property is rarely satisfied exactly. However, this problem has not yet been resolved conclusively [15,16]. Roeperstorff [17] and later others [18,19] obtained a spectral decomposition of U for r-adic maps (e.g., Bernouili shift map). An additional motivation for studying piecewise linear maps is that many of the results obtained for such systems can be generalized to everywhere expanding maps (with curvature) [16] and certain higher dimensional conservative hyperbolic maps [20].

The present paper extends this earlier work to all piecewise linear Markov maps and repellers and to a wider class of function spaces such as spaces of piecewise analytic functions. Further, it reveals the existence of a wider class of eigenvalues related to the unstable periodic orbits. Finally, it provides a systematic algorithm for constructing the transition matrix governing the evolution of probability vectors, which are the representation of the densities in each function space.

II. GENERALIZED MARKOV COARSE GRAINING IN A PIECEWISE MONOMIAL BASIS

A piecewise linear or analytic map f: $\mathbf{I} \to \mathbf{I}$, where $\mathbf{I} = [0,1]$, is Markovian if it has the following properties [5].

(i) The number of nondifferentiable points $d_1, d_2, \ldots, d_N$ is finite.

(ii) Their iterates form a finite set of points $\mathcal{P}_{\mathcal{M}}$ $\{a_0, a_1, \ldots, a_{N'}\}$, where $a_k < a_{k+1}, a_0 = 0$ and $a_{N'} = 1$.

(iii) There exists an integer k such that $|\frac{d}{dx} f^k(x)| > 1$ for some $k \geq 1$ and $\forall x \in \mathbf{I}$.

The set of points $\mathcal{P}_{\mathcal{M}}$ can be used to partition the unit interval into cells $C_i = [a_{i-1}, a_i]$. The resulting partition is referred to as the minimal Markov partition. We remark that in some cases the minimal Markov partition may consist of only one cell, the unit interval itself.

The probability density $\rho_n(x)$ at time n, descriptive of the statistical state of the system, assumed to be an L^1 function, evolves according to the Frobenius-Perron operator

$$\rho_{n+1}(x) = U\rho_n(x) = \int_0^1 \delta(x - f(y))\rho_n(y)dy = \sum_\alpha \rho_n(f_\alpha^{-1}(x)) \left| \frac{d}{dx} f_\alpha^{-1}(x) \right|, \quad (4)$$

where $f_\alpha^{-1}(x)$ denote the inverse branches of $f(x)$. It is understood that if an inverse branch $f_\alpha^{-1}(x)$ is not defined for a point x, then the corresponding term in the sum Eq. (4) is set equal to zero at this point. Throughout the rest of this paper derivatives will often be indicated by the following suffices: $\frac{d}{dx} f(x) = f'(x)$ and $\frac{d^n}{dx^n} f(x) = f^{(n)}(x)$.

We shall also be interested in the properties of a class of piecewise linear repellers. Let $f(x)$ denote the map of any system belonging to this class. $f(x)$ has the following properties.

(i) The unit interval $\mathbf{I}$ is mapped onto a union of itself and another set of points D, where $\mathbf{I}$ and D are mutually disjoint

$$f(\mathbf{I}) = \mathbf{I} \bigcup D. \quad (5)$$

(ii) Points that leave $\mathbf{I}$ never return

$$f^n(D) \bigcap \mathbf{I} = \emptyset \quad \forall n. \quad (6)$$

(iii) The iterates of nondifferentiable points of $f(x)$ that are in $\mathbf{I}$ are finite in number and define $\mathcal{P}_{\mathcal{M}}$. This implies that the iterates of nondifferentiable points that remain in $\mathbf{I}$ are mapped onto one or more periodic orbits.

(iv) There exists a k such that $|\frac{d}{dx} f^k(x)| > 1$ for some $k \geq 1$ and $\forall x \in \mathbf{I}$.

We shall restrict the Frobenius-Perron operator U so that it acts only on functions whose support is not outside the unit interval, thus ignoring what happens to densities after they have left $\mathbf{I}$. The set of points $\mathcal{P}_{\mathcal{M}}$ for repellers will effectively form, for our purposes, a minimal Markov partition of $\mathbf{I}$. From here on, $f(x)$ denotes a map belonging to either of the above classes.

We now need to define the type of function basis and its dual which shall be used in the representations of U. As shall be clear shortly, each possible basis is associated with any M-cell Markov partition $\mathcal{P}$ of the following class: the minimal Markov partition and refinements formed by adding to the points defining $\mathcal{P}_{\mathcal{M}}$ either (a) a finite number of successive inverse images of one or more of these points or (b) all the points of a finite number of periodic orbits of $f(x)$. Both methods of partition refinement can be done simultaneously.

We define the monomial family whose support is the cell $C_i = [a_{i-1}, a_i]$ as

$$\phi_i(n,x) = \begin{cases} (x - a_{i-1})^n & \text{if } x \in C_i \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

where $n = 0, 1, 2, \ldots$. Let Φ_N be the function space spanned by the set of monomial families of order N associated with the partition $\mathcal{P}$, $\{\phi_i(n,x)$: $i = 1, 2, \ldots, M$; $n = 0, 1, \ldots, N\}$. The dual functional to a monomial of order n, having support in the cell $C_i = [a_{i-1}, a_i]$, is

$$\phi_i^\dagger(n,x) = \lim_{\epsilon \to 0^+} \frac{(-1)^n}{n!} \frac{d^n}{dx^n} \delta(x + \epsilon - a_{i-1}). \quad (8)$$

The above functionals define the functional space $\Phi_N^\dagger$, which is the dual space to Φ_N. We notice that the set of $\phi_i^\dagger$'s simply gives the expansion coefficients of the Taylor series of a function $g(x)$ evaluated at the left boundary a_{i-1} of the cell C_i

$$\int_0^1 \phi_i^\dagger(n,x) g(x) dx = \lim_{\epsilon \to 0^+} \frac{1}{n!} g^{(n)}(a_{i-1} + \epsilon). \quad (9)$$

Note that to avoid ambiguities at the cell boundaries only upper derivatives are considered, and Dirac δ-function distributions are defined as

$$\int g(x)\delta(x - a) = \lim_{\epsilon \to 0^+} g(a + \epsilon). \quad (10)$$

It is straightforward to show that $\phi_i(n,x)$ and the corresponding dual form a biorthonormal set

$$\int_0^1 \phi_i^\dagger(n,x)\phi_j(m,x) dx = \delta(i,j)\delta(n,m). \quad (11)$$

We now need to identify the sort of function $g(x)$ that can be represented, finitely with respect to our basis, either exactly or up to a good approximation. This will certainly be the case if the expansion of $g(x)$ in each cell C_i is uniformly convergent. A sufficient condition for this is obtained by simply using Taylor's mean value theorem with Lagrange's form of remainder and demanding that the remainder be very small. If the nth derivative of $g(x)$ exists for all $x \in C_i$, then

$$g(a_{i-1}+x) = S_n(x) + R_n(x), \tag{12}$$

where

$$S_n(x) = \sum_{r=0}^{n-1} \frac{x^r}{r!}\, g^{(r)}(a_{i-1}),$$
$$R_n(x) = \frac{x^n}{n!} g^{(n)}(a_{i-1}+\theta x), \tag{13}$$

and $0 \le \theta \le 1$. In fact, it is easy to show that, if $\mathcal{P}$ consists of more than one cell, the Taylor expansions of functions piecewise analytic (whose infinite order derivatives are finite) over $\mathcal{P}$ are uniformly convergent. It suffices to show that an upper bound $\hat{R}_n$ on $|\hat{R}_n(x)|$ exists which is uniformly decreasing with increasing n. Specifically,

$$|R_n(x)| \le \frac{\Delta^n}{n!} \max[|g^{(n)}(a_{i-1}+\theta x)|] = \hat{R}_n, \tag{14}$$

where $\Delta = a_i - a_{i-1} < 1$. As $|g^{(n)}(a_{i-1}+\theta x)|$ is finite, uniform convergence of $\hat{R}_n$ to zero is guaranteed. Similarly, it follows that the space of piecewise monomials of the unit interval is dense in the space Ξ of functions which are piecewise analytic over $\mathcal{P}$ and whose infinite order derivatives are bounded. Thus a finite piecewise Taylor expansion of any function of Ξ can be made to any degree of accuracy.

We shall now compile a number of identities needed for the following sections which are valid for all piecewise linear Markov maps and repellers and the aforementioned partitions.

(i) Any inverse branch $f_\alpha^{-1}(x)$ of a map $f(x)$ has the properties

$$f_\alpha^{-1}(C_i) \bigcap C_j \ne \emptyset \Rightarrow f_\alpha^{-1}(C_i) \subset C_j \quad \forall C_i,\, C_j. \tag{15}$$

A corollary of the first identity is that for each cell C_i, $f(C_i)$ is a union of cells.

(ii) The derivatives of the inverse branches are piecewise constant over each cell,

$$\frac{d}{dx} f_\alpha^{-1}(x) = c \quad \forall x \in C_i, \tag{16}$$

where c is a constant.

(iii) Consider a monomial of order n whose support is over the union of k adjacent cells of a partition,

$$\psi(x) = \begin{cases} (x-a_{i-1})^n & \text{if } x \in C_i \bigcup C_{i+1} \bigcup \cdots \bigcup C_{i+k-1} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

One can easily show that

$$\psi(x) = \sum_{j=i}^{i+k-1} \phi_j(n,x) + \sum_{j=i}^{i+k-1}\sum_{l=0}^{n-1} a_{jl}\phi_j(l,x), \tag{18}$$

except perhaps at the points $a_i, a_{i+1}, \ldots, a_{k-2}$, where a_{jl} are expansion coefficients. As these points are finite in number, their exclusion has no measurable effect after integration. Note that the coefficients of $\phi_j(n,x)$ in Eq. (18) equal one.

(iv) Using identities (15)–(18) one can easily show that the functional space Φ_n is invariant under U. That is,

$$[E_n, U]E_n = E_n U E_n - U E_n = 0, \tag{19}$$

where the projector E_n acting on a function $g(x)$,

$$E_n g(x) = \sum_{l=0}^{n}\sum_{j=1}^{M} \left(\int_0^1 \phi_j^\dagger(l,y) g(y) dy \right) \phi_j(l,x), \tag{20}$$

can be viewed as a generalized coarse graining operator. Therefore, $U\phi_i(l,x)$ is a linear combination of the basis elements, which implies that U can be represented as a time independent transition matrix when acting on these states. More specifically,

$$U\phi_i(n,x) = \sum_\alpha \phi_i(n, f_\alpha^{-1}(x)) |f_\alpha^{-1\prime}(x)| = \sum_{l=0}^{n}\sum_{j=1}^{M} W_{lj\,ni}\phi_j(l,x). \tag{21}$$

For expository convenience we have written the expansion coefficients $W_{lj\,ni}$ as a fourth-order tensor, although they really form a matrix. A crucial property of this basis is that monomial families of order N can undergo transitions to linear combinations of monomial families of order no higher than N. As a result, if the basis elements are ordered according to

$$\begin{gathered} \phi_1(0,x), \phi_2(0,x), \ldots, \phi_M(0,x), \\ \phi_1(1,x), \phi_2(1,x), \ldots, \phi_M(1,x), \ldots, \\ \phi_1(n,x), \phi_2(n,x), \ldots, \phi_M(n,x), \ldots, \end{gathered} \tag{22}$$

the matrix W has the block form

$$W = \begin{pmatrix} w(0) & \cdot & \cdot & \cdot & \cdot \\ 0 & w(1) & \cdot & \cdot & \cdot \\ 0 & 0 & w(2) & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot \\ 0 & 0 & 0 & 0 & w(4) \end{pmatrix}, \tag{23}$$

where the $w(i)$ and the dots denote $M \times M$ matrices and M is the number of cells in the relevant partition. We can immediately use standard results of linear algebra to explore the properties of U through its matrix representation W when acting on the above states. In particular, the fact that the basis functions and their corresponding dual distributions form a biorthonormal set [Eq. (11)] leads to the following results.

(i) The right (left) basis elements for which W is in its Jordan form give the expansion coefficients for the basis of functions (distributions) for which U restricted to $\Phi_n^\dagger \bigotimes \Phi_n$ is in its Jordan form.

(ii) This canonical basis of functions $\{v_i(x)\}$ and its dual $\{v_i^\dagger(x)\}$ form a biorthonormal set, that is,

$$\int v^{\dagger}{}_i(x) v_j(x) dx \;=\; \delta_{i,j}. \tag{24}$$

Thus the eigenfunctions or eigendistributions of U restricted to $\Phi_n^{\dagger} \bigotimes \Phi_n$ can be calculated in a finite number of algebraic manipulations.

III. EIGENVALUE PROBLEM OF THE TRANSITION MATRIX W AND CONNECTION WITH THE ZETA FUNCTION FORMALISM

In Sec. II a simple method for obtaining spectral decompositions for piecewise analytic spaces was presented. Initial densities residing in this space evolve with a time independent matrix and the equilibrium density or measure (when unique) can be straightforwardly obtained. Thus all the statistical moments associated with long time averages for observables (for example, position x) in this functional space can be easily calculated, including time correlation functions and their corresponding decay properties. It is therefore natural to inquire about the connection between this method and one of the most elegant methods yet developed for investigating the statistical properties of a hyperbolic dynamical system, namely, the zeta function formalism developed by Ruelle and others [4]. The formalism has been successfully applied by several authors [21,22,12]. The purpose of the present section is to establish the nature of this connection.

The zeta function or thermodynamic formalism provides a means to calculate all statistical moments associated with long time averages of analytical observables, such as the Kolmogorov-Sinai entropy, Lyapounov exponents, Hausdorf dimensions, etc., provided the equilibrium measure is known. Independent of a knowledge of the equilibrium measure, it also gives the decay rates of time correlation functions of such observables. There are a variety of zeta functions. The most commonly used one is known as the Selberg-Smale zeta function $Z(z)$, which for Markov analytic maps can be easily derived from the Fredholm determinant of the Frobenius-Perron operator U [5]. That is,

$$Z(z) = \det(1 - zU) = \prod_{k=0}^{\infty} \frac{1}{\zeta_k(z)}, \tag{25}$$

where

$$\zeta_k(z) = \prod_p \left(1 - \frac{z^{n_p}}{|\Lambda_p|\Lambda_p^{k-1}}\right) \tag{26}$$

and $\prod_p, n_p$, and Λ_p denote the product over all periodic orbits, the period of the pth periodic orbit, and its relative stability or instability factor $f'(x_1)f'(x_2)\cdots f'(x_p)$, respectively. Equation (25) can be derived using the formula

$$\begin{aligned}\det(1 - zT) &= \exp\{\mathrm{Tr}\,[\ln(1 - zT)]\} \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{Tr}\,(T^n)\right),\end{aligned} \tag{27}$$

valid for the Fredholm determinant of nuclear operators T of order zero and nonsingular matrices [5]. The functions $\zeta_k(z)$ are known as Ruelle zeta functions. Mayer has proved for analytical Markov maps that $\frac{1}{Z(z)}$ is a meromorphic function on the entire complex plane. The poles of $\frac{1}{Z(z)}$ are reciprocals of the eigenvalues of U acting on the space Θ of piecewise analytical functions associated with $\mathcal{P}_{\mathcal{M}}$. The space Φ of piecewise monomial functions associated with $\mathcal{P}_{\mathcal{M}}$ is a subspace of the Θ. Therefore, the eigenvalues of U restricted to Φ should be elements of the spectrum of U restricted to Θ. We shall now prove that the spectra are identical, by deriving the Smale-Selberg zeta function from the matrix representation W of U restricted to Φ_n. The proof consists of showing that the eigenvalues of W are the reciprocals of the poles of $\frac{1}{Z(z)}$. The secular equation of W, for a piecewise monomial space of order N, reduces to

$$\det(W - \lambda I) = \prod_{k=0}^{N} \det[w(k) - \lambda I]. \tag{28}$$

Let us now apply formula (27) to $w(k)$:

$$\begin{aligned}\det[1 - zw(k)] &= \exp\left(\mathrm{Tr}\,\{\ln[1 - zw(k)]\}\right) \\ &= \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{Tr}\,[w(k)^n]\right).\end{aligned} \tag{29}$$

To calculate the trace of $w(k), \sum_i w_{ii}(k)$, where $w_{ii}(k)$ is the coefficient of transition of $\phi_i(k,x)$ to itself, we notice that the relevant terms in Eq. (21) satisfy

$$f_\alpha^{-1}(C_i) \bigcap C_i \neq \emptyset. \tag{30}$$

It follows that

$$\begin{aligned}w_{ii}(k) = &\sum_{f_\alpha^{-1}(C_i)\bigcap C_i \neq \emptyset} \int \phi_i^{\dagger}(k,x)\phi_i(k, f_\alpha^{-1}(x)) \\ &\times |f_\alpha^{-1\prime}(x)| dx.\end{aligned} \tag{31}$$

On the other hand, we notice that Eq. (30) implies that there is a fixed point of $f(x)$ contained in $f_\alpha^{-1}(C_i)$. Let us denote it by x_*. Then from Eqs. (15) and (16)

$$f_\alpha^{-1}(x) = \frac{x}{f'(x_*)} + c \;\; \forall x \in C_i, \tag{32}$$

where c is a constant. Therefore,

$$\phi_i(k, f_\alpha^{-1}(x))|f_\alpha^{-1\prime}(x)| = \frac{(\frac{x}{f'(x_*)} + c)^k}{|f'(x_*)|} \tag{33}$$

and

$$w_{ii}(k) = \sum_{x_* \in C_i} \frac{1}{f'^k(x_*)|f'(x_*)|}. \tag{34}$$

Thus

$$\mathrm{Tr}\,[w(k)] = \sum_{x_*} \frac{1}{f'^k(x_*)|f'(x_*)|}. \tag{35}$$

One can show in a similar fashion that

$$\mathrm{Tr}\,[w(k)^n] = \sum_{f^n(x_*)=x_*} \frac{1}{f^{n\prime k}(x_*)|f^{n\prime}(x_*)|}, \tag{36}$$

where the sum is now over fixed points of $f^n(x)$.

Let us now come back to Eq. (29). $w_{ii}^n(k)$ is nonzero only if the cell C_i contains an element(s) of a periodic orbit p, of period N_p, such that N_p is a divisor of n. We shall denote the set of all periodic orbits whose period N_p is a divisor of n by N_p/n. We shall adopt the argument and notation of Artuso *et al.* [12]. In general, the nth order trace picks up contributions from N_p/n. Therefore

$$\mathrm{Tr}\,[w(k)^n] = \sum_{N_p/n} N_p t_p^{\frac{n}{N_p}}, \tag{37}$$

where

$$t_p = \frac{1}{f^{N_p\prime}(x_*)^k |f^{N_p\prime}(x_*)|} \tag{38}$$

and x_* is a fixed point of $f^{N_p}(x_*)$. Let $r = \frac{n}{N_p}$. We can express Eq. (29) as

$$\det[1 - zw(k)] = \exp\left(-\sum_p \sum_{r=1}^{\infty} \frac{(z^{N_p}t_p)^r}{r}\right)$$
$$= \exp\left(\prod_p \ln(1 - z^{N_p}t_p)\right) = \prod_p (1 - z^{N_p}t_p) = \prod_p \left(1 - \frac{z^{N_p}}{f^{N_p\prime}(x_*)^k |f^{N_p\prime}(x_*)|}\right) = \frac{1}{\zeta_k(z)}. \tag{39}$$

We conclude that all the zeros of $\zeta_k(z)$, and therefore the poles of $Z(z)$, can be obtained from Eq. (28). A more informative conclusion is that we can identify which $\zeta_k(z)$ are relevant for a given set of measurable functions. That is, if a set of measurable functions can be expressed (exactly or up to a good approximation) as linear combinations of piecewise monomials (associated with $\mathcal{P_M}$) of order no higher than N, only the $\zeta_k(z)$ for $k \leq N$ are relevant for the corresponding time correlation functions. Thus the nature of the link between our method and the zeta function formalism is clearly established.

One advantage of our method compared to the zeta function formalism for piecewise linear Markov maps is that our method is far easier to implement. Furthermore, we can evaluate explicitly all time correlation functions whose corresponding observables are piecewise analytic (whose infinite order derivatives are bounded), whereas with the zeta formalism only the corresponding decay rates are calculated in practice.

IV. EFFECT OF REFINEMENT OF THE MINIMAL MARKOV PARTITION WITH PERIODIC ORBITS

The investigation of the properties of statistical systems at different length scales has attracted much interest recently [10]. It is therefore desirable to consider this problem for simple models amenable to analytical calculations. This is the main theme of the present section.

The study of local statistical behavior involves the calculation of statistical moments associated with observables having support over local regions. This is not possible with analytic observables whose support is over the entire interval, because the calculation of their statistical moments entails an integration over all of phase space. Consequently, they can only describe essentially global aspects of the system. Such "global" aspects were the subject of Sec. III, where we characterized the spectra of spectral decompositions associated with $\mathcal{P_M}$. To describe local statistical behavior, we shall now use the results of Sec. II to characterize the eigenvalues of spectral decompositions associated with refined partitions. The partitions are formed by adding to the points defining $\mathcal{P_M}$ all the points of a finite number of periodic orbits. The periodic orbits are chosen so that the points defining the new partition $\mathcal{P}$ lie on or close to the boundaries of the local regions of interest. We shall establish the following properties.

(i) There exists a countable infinity of eigenvalues of U associated with each periodic orbit $\{x_1, x_2, \ldots, x_p\}$.

(ii) Corresponding to each eigenvalue there is a unique eigenfunction which is contained in the space of monomial families associated with the partition $\mathcal{P}$.

(iii) The eigenvalues are

$$\{\lambda\} = \frac{1}{\{[f^{p\prime}(x_l)]^{k+1}\}^{1/p}}$$
$$= \left|\frac{1}{\{[f^{p\prime}(x_l)]^{k+1}\}^{1/p}}\right| \exp(i\gamma\pi j/p), \tag{40}$$

where

$$\gamma = \begin{cases} 1 \text{ if } [f^{p\prime}(x_l)]^{k+1} < 0 \\ 2 \text{ otherwise,} \end{cases} \tag{41}$$

where $k = 0, 1, 2, 3, \ldots$; $j = 1, 2, 3, \ldots, p$; and x_l belongs to a periodic orbit of period p.

They describe a decaying oscillatory mode with period p if the argument of the pth root is positive and $2p$ if it is negative. They play no role for time correlation functions of observables that are piecewise analytic with respect to $\mathcal{P}_{\mathcal{M}}$. However, they are relevant for the statistical moments of measurable functions, where each function has the following properties. (i) It can be well approximated over each cell of $\mathcal{P}$ as a finite Taylor series. (ii) It has at least one discontinuity (or discontinuity in a derivative of lower order than the order of truncation in the Taylor expansion) at or close to an element of one of the periodic orbits used to refine $\mathcal{P}_{\mathcal{M}}$.

We now proceed to the proof of the above statements. Let $\Phi_n = \{\phi_i(n,x)\}$ denote the functional space spanned by the basis of monomial families of order N associated with a $\mathcal{P}_{\mathcal{M}}$ having M cells. For each point of the periodic orbit x_i define the set of monomials having support $[x_i, a_{i*}]$ as

$$\psi_i(k,x) = \begin{cases} (x-x_i)^k & \text{if } x \in [x_i, a_{i*}] \\ 0 & \text{otherwise,} \end{cases} \tag{42}$$

where C_{i_*} denotes a cell of $\mathcal{P}_{\mathcal{M}}$ such that $x_i \in C_{i*} = [a_{i*-1}, a_{i*}]$ and $k = 0,1,2,\ldots,N$. We extend the space Φ_n to a space $\tilde{\Phi}_n$ by adding the monomials [Eq. (42)] as new basis functions to the set spanning Φ_n. The new additional dual basis distributions are

$$\psi_i^\dagger(n,x) = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}[\delta(x-x_i) - \delta(x-a_{i*-1})]. \tag{43}$$

One can readily verify that the extended basis and its dual form a biorthonormal set. By Eqs. (15)–(18) $\tilde{\Phi}_n$ is an invariant functional space of U. Therefore, denoting for convenience the new basis by $\{\tilde{\phi}_i(n,x)\}$ we have, in analogy to Eq. (21),

$$\begin{aligned} U\tilde{\phi}_i(n,x) &= \sum_\alpha \tilde{\phi}_i(n, f_\alpha^{-1}(x))|f^{-1'}{}_\alpha(x)| \\ &= \sum_{l=0}^{n}\sum_{j=1}^{M+1} \breve{W}_{lj\,ni}\tilde{\phi}_j(l,x), \end{aligned} \tag{44}$$

where $\breve{W}_{lj\,ni}$ is the matrix representation of U in this basis. A monomial $\psi_i(n,x)$ of order n having a discontinuity at x_i, will have a discontinuity at x_{i+1} after one iteration, but not at any other points of the periodic orbit. Furthermore, $U\psi_i(n,x)$ is a linear combination of monomials, including $\psi_{i+1}(k,x)$, where $k = 0,1,2,\ldots,n$. Discontinuities at x_{i+1} in the sum Eq. (44) only occur for the term whose corresponding inverse branch $f_\alpha^{-1}(x)$ maps x_{i+1} to x_i. Using this fact together with Eqs. (15)–(18) one can easily show that

$$\psi_i(n, f^{-1}{}_\alpha(x))|f^{-1'}{}_\alpha(x)| = \frac{\psi_{i+1}(n,x)}{f'(x_i)^{n+1}} + \Upsilon, \tag{45}$$

where Υ consists of terms which do not contribute to the transition to $\psi_{i+1}(n,x)$. Indeed,

$$\begin{aligned} \int \psi_j^\dagger(n,x)U\psi_i(n,x)dx \\ &= \int \psi_j^\dagger(n,x)\psi_i(n, f_\alpha^{-1}(x))|f_\alpha^{-1}(x)|dx \\ &= \int \psi_j^\dagger(n,x)\frac{\psi_{i+1}(n,x)}{f'(x_i)^{n+1}}dx \\ &= \frac{\delta(j,i+1)}{f'(x_i)^{n+1}}. \end{aligned} \tag{46}$$

Consider the following arrangement of the basis functions for $k = 0,1,2,\ldots,n$,

$$\begin{aligned} &\phi_1(k,x), \phi_2(k,x), \ldots, \phi_m(k,x), \\ &\psi_1(k,x), \psi_2(k,x), \ldots, \psi_p(k,x), \end{aligned} \tag{47}$$

that is, for each value of k, the basis functions associated with $\mathcal{P}_{\mathcal{M}}$ occur first, followed by the basis functions associated with the periodic orbit. Then the nth order block of the matrix representation $\breve{W}$ of U for this basis has the schematic form

$$\breve{W} = \begin{pmatrix} A(n) & B(n) \\ 0 & C(n) \end{pmatrix}. \tag{48}$$

$A(n)$ is a $M \times M$ matrix giving the transition coefficients of elements of $\{\phi_i(k,x)\}$ going to elements of $\{\phi_j(k,x)\}$. $B(n)$ is a $p \times M$ matrix giving the transition of elements of $\{\psi_i(k,x)\}$ going to elements of $\{\phi_j(k,x)\}$. $C(n)$ is a $p \times p$ matrix giving the transition coefficients of elements of $\{\psi_i(k,x)\}$ going to elements of $\{\psi_j(k,x)\}$. If the periodic orbit is a fixed point x_* (i.e., $p = 1$), then $C(n)$ is a 1×1 matrix, so that $C(n) = f'^{-(n+1)}(x_*)$. For any orbit of higher period, $C(n)$ has the form

$$C(n) = \begin{pmatrix} 0 & 0 & 0 & 0 & \kappa(x_p) \\ \kappa(x_1) & 0 & 0 & 0 & 0 \\ 0 & \kappa(x_2) & 0 & 0 & 0 \\ \vdots & \vdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa(x_{p-1}) & 0 \end{pmatrix}, \tag{49}$$

where $\kappa(x) = f'^{-(n+1)}(x)$ and, as before, monomial families of order n cannot undergo transitions to higher orders. Therefore, the secular equation reduces to

$$\begin{aligned} |\breve{W} - \lambda I| &= \prod_{k=0}^{k=N} |w(k) - \lambda I| \\ &= \prod_{k=0}^{k=N} |A(k) - \lambda I|\,|C(k) - \lambda I|, \end{aligned} \tag{50}$$

so the eigenvalues arising from the refinements are the eigenvalues of $C(k)$. These are precisely the ones listed in Eq. (40). In fact, there are strong analytical reasons to believe that the results of this section can be generalized to everywhere expanding piecewise analytic maps which do not have to possess a minimal Markov partition [16] and certain higher dimensional hyperbolic conservative maps [23].

V. EXAMPLES

We shall now illustrate the results of the preceding sections on simple chaotic maps. Let us first consider the tent map

$$f(x) = \begin{cases} mx & \text{if } 0 \le x \le 0.5 \\ m(1-x) & \text{otherwise} \end{cases} \tag{51}$$

for $m \ge 2$. If $m > 2$, then the map is a repeller, so that the support of the invariant measure is a cantor set [24]. However, the only effect in the evolution of initially smooth densities [25] is that at each iteration their norm can decrease if part of the density leaves the interval $\mathbf{I} = [0,1]$. That is, within the interval the evolving density never has a fractal support. The Frobenius-Perron operator takes the form

$$\rho_{n+1}(x) = \frac{1}{m}\left[\rho_n\left(\frac{x}{m}\right) + \rho_n\left(1 - \frac{x}{m}\right)\right]. \tag{52}$$

We are only concerned with the evolution of densities (or portions thereof) while they remain in the interval $[0,1]$. The effective minimal Markov partition coincides with the unit interval itself. Following Sec. II, a biorthonormal set is provided by the monomials with support the unit interval, i.e,

$$1,\ x,\ x^2,\ x^3, \ldots x^n, \ldots, \tag{53}$$

and their corresponding duals

$$\delta(x),\ -\frac{d}{dx}\delta(x),\ (-1)^2\frac{d^2}{dx^2}\delta(x),$$
$$(-1)^3\frac{d^3}{dx^3}\delta(x), \ldots, (-1)^n\frac{d^n}{dx^n}\delta(x), \ldots. \tag{54}$$

For this basis it is easy to see that the transition matrix is upper triangular, with diagonal elements

$$\frac{2}{m},\ 0,\ \frac{1}{m^3},\ 0,\ \frac{1}{m^5}, \ldots. \tag{55}$$

The diagonal terms are just the eigenvalues and their reciprocals equal the poles of the zeta function $Z(z)$ defined in Sec. III. When $m > 2$, the eigenvalue $\frac{2}{m}$ gives the escape rate out of the unit interval for an initially uniform distribution. The zero eigenvalues show that monomials of odd order cannot undergo transitions to themselves in one iteration, which is due to the symmetry of the map about $\frac{1}{2}$. Let us now refine the minimal partition with the addition of the fixed point $x_* = \frac{m}{1+m}$ and extend the basis with monomials having as support $[x_*, 1]$, i.e.,

$$\chi_{[x_*,1]}(x),\ (x - x_*)\chi_{[x_*,1]}(x), \ldots, (x - x_*)^n\chi_{[x_*,1]}(x) \ldots, \tag{56}$$

the new dual basis elements being

$$\tilde{\delta}(x),\ -\frac{d}{dx}\tilde{\delta}(x), \ldots, (-1)^n\frac{d^n}{dx^n}\tilde{\delta}(x), \ldots, \tag{57}$$

where

$$\tilde{\delta}(x) = \delta(x - x_*) - \delta(x). \tag{58}$$

One can readily show that the extended basis and its dual form a biorthonormal set. Equation (40) states that for each periodic orbit of period p new eigenvalues are obtained,

$$\{\lambda\} = \frac{1}{\{[f^{p\prime}(x_*)]^{k+1}\}^{1/p}} = \frac{1}{m}\exp(i\pi\gamma j/p), \tag{59}$$

where $k = 0, 1, 2, 3, \ldots$; $j = 1, 2, 3, \ldots, p$; and

$$\gamma = \begin{cases} 1 & \text{if } [f^{p\prime}(x_*)]^{k+1} < 0 \\ 2 & \text{otherwise.} \end{cases} \tag{60}$$

The first few eigenvalues generated by the above refinement are then $\frac{-1}{m}, \frac{1}{m^2}, \frac{-1}{m^3}, \ldots$ and the overall transition matrix has the form of Eq. (23),

$$W_{ki} = \begin{pmatrix} 2m^{-1} & m^{-1} & m^{-1} & m^{-1} & m^{-1} & m^{-1} & \cdots \\ 0 & -m^{-1} & 0 & -m^{-1} & 0 & -m^{-1} & \cdots \\ 0 & 0 & 0 & -m^{-2} & -2m^{-2} & -2m^{-2} & \cdots \\ 0 & 0 & 0 & m^{-2} & 0 & 2m^{-2} & \cdots \\ 0 & 0 & 0 & 0 & 2m^{-3} & m^{-3} & \cdots \\ 0 & 0 & 0 & 0 & 0 & m^{-3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \cdots \end{pmatrix}, \tag{61}$$

while each successive 2×2 block along the diagonal has the form of Eq. (48). Actually for this map W is diagonalizable so that the right (left) eigenvectors of W define eigenfunctions (dual eigendistributions) of the Frobenius-Perron operator U, which can be normalized to form a biorthonormal set. More generally, W with respect to its canonical basis reduces to its Jordan form. The eigenfunctions corresponding to the first four eigenvalues $(\frac{2}{m}, \frac{-1}{m}, 0, \frac{1}{m^2})$ are

$$\begin{aligned} v_1(x) &= \chi_{[0,1]}(x), \\ v_2(x) &= \frac{-\chi_{[0,1]}(x)}{3} + \chi_{[x_*,1]}(x), \\ v_3(x) &= \chi_{[0,1]}(x) - \frac{x\chi_{[0,1]}(x)}{2}, \\ v_4(x) &= \frac{m^2\chi_{[0,1]}(x)}{(2m-1)(1+m)} - \frac{m\chi_{[x_*,1]}}{1+m} \\ &\quad -x\chi_{[0,1]}(x) + (x - x_*)\chi_{[x_*,1]}(x). \end{aligned} \tag{62}$$

The dual eigendistributions can be expanded in terms of the dual basis. For the eigendistributions corresponding to the eigenfunctions above, the first four terms in the expansion are

$$v_1^\dagger(x) = \delta(x) + \frac{1}{3}\tilde{\delta}(x) - \frac{1}{2}\frac{d}{dx}\delta(x) - \frac{2(3-m)}{3(2m-1)}\frac{d}{dx}\tilde{\delta}(x) + \cdots,$$
$$v_2^\dagger(x) = \tilde{\delta}(x) + \frac{m}{m+1}\frac{d}{dx}\tilde{\delta}(x) + \cdots,$$
$$v_3^\dagger(x) = -2\frac{d}{dx}\delta(x) - 2\frac{d}{dx}\tilde{\delta}(x) + \cdots,$$
$$v_4^\dagger(x) = -\frac{d}{dx}\tilde{\delta}(x) + \cdots, \tag{63}$$

where $\tilde{\delta}(x)$ is the same as before. Let $g(x)$ be any function which can be well approximated with the first two terms of a Taylor series [see Eq. (12)] over each cell of the partition. $g(x)$ can be expanded in terms of the eigenfunctions of Eq. (62),

$$g(x) \cong \sum_{i=0}^{4}\left(\int_0^1 v_i^\dagger(y)g(y)dy\right)v_i(x). \tag{64}$$

It is straightforward to extend this treatment to higher orders and to other periodic orbits.

For values of m less than 2 the tent map no longer sends the unit interval onto itself. However, for $m > \sqrt{2}$ the interval $[f(f(\frac{1}{2})), f(\frac{1}{2})] = [a,b]$ is mapped onto itself, so that our results will be directly applicable if the critical point is mapped onto a periodic orbit or fixed point. A slope value for which the analysis is particularly simple is $m = \frac{1+\sqrt{5}}{2}$. The minimal Markov partition is $\{a, \frac{1}{2}, b\}$. Again we can refine the partition with the fixed point x_*, and in addition consider the dynamics in the interval $[0,a]$, so that the new Markov partition is $\{0, a, \frac{1}{2}, x_*, b, \}$. The analysis proceeds as in the previous example. The eigenvalues associated with the decay out of the region $[0,a]$ are simply

$$\{m^{-1}, m^{-2}, m^{-3}, \ldots\}. \tag{65}$$

The eigenvalues associated with the fixed point are

$$\frac{-1}{m}, \frac{1}{m^2}, \frac{-1}{m^3}, \ldots. \tag{66}$$

The eigenvalues corresponding to the minimal Markov partition, and thus the zeros of the zeta function [Eq. (29)] are

$$1, \frac{-1}{m^3}, \frac{-1}{m^3}e^{1.0944i\pi}, \frac{-1}{m^3}e^{-1.0944i\pi}, \ldots. \tag{67}$$

VI. EXTENSIONS TO NONANALYTIC BASIS FUNCTIONS

So far we have described the properties of U acting cn spaces of piecewise analytic functions associated with the minimal Markov partitions and its refinements. We have seen explicitly that the choice of partition and corresponding function space determines the possible spectra and thus characteristic times that can be observed in time correlation functions. Since there is no reason that the initial density or observable should be analytic, it is natural to inquire about the spectral properties of U when acting on nonpiecewise analytic functions. Consider first a simple generalization of a monomial, x raised to a noninteger power α, $\psi(x) = x^\alpha$, and the tent map with $m \geq 2$. We have then

$$U\psi(x) = \frac{(\frac{x}{m})^\alpha}{m} + \frac{(1-\frac{x}{m})^\alpha}{m}. \tag{68}$$

The first term is proportional to $\psi(x)$, while the second is expandable as an infinite power series. No exact representation of U in terms of a finite matrix can now be expected. However, since $m > 1$ and $x \in [0,1]$, we can still apply the method of Sec. II since the expansion converges uniformly and calculate the new eigenfunction to any desired accuracy. The new eigenvalue is clearly $m^{\alpha+1}$, which has the same form as the eigenvalues associated with monomials having the unit interval as support. For $-1 < \alpha < 0$, $\psi(x)$ is divergent at zero but nevertheless integrable. We can choose α so that the corresponding eigenvalue lies anywhere between zero and one. Indeed, if α is complex with its real part greater than -1, $\psi(x)$ remains integrable [with integrable singularities at zero if $\mathrm{Re}(\alpha) < 0$] and eigenvalues can be obtained anywhere in the unit circle, implying that the corresponding characteristic times can be arbitrarily long. We can extend the basis to include the function $(1-x)^\alpha$, finding that its corresponding eigenvalue is zero. Notice a general similarity with the analysis of Feigenbaum *et al.* [24] of scaling properties of multifractal repellers. They are able to map this problem into an eigenvalue problem of an operator reminiscent of, but not identical to, the Frobenius-Perron operator. In our analysis we are concerned exclusively with the eigenvalue problem of the Frobenius-Perron operator.

Let us now consider the problem of general piecewise linear Markov maps. We need to define functions that are a generalization of noninteger powers of x and then show how a finite invariant functions space containing them can be constructed. Therefore, let

$$\tilde{\psi}_i(\alpha,\eta,c,x) = \begin{cases} (x - a_{i-1} + c)^\alpha & \text{if } \eta = \mathrm{L},\ x \in C_i \\ (a_i - x + c)^\alpha & \text{if } \eta = R,\ x \in C_i \\ 0 & \text{otherwise,} \end{cases} \tag{69}$$

where $c \geq 0$ and the index η has two possible values L and R to denote each of the nonzero functions of Eq. (69). In the following we shall refer to these functions as generalized piecewise monomials (GPM's) if $c = 0$ and generalized piecewise polynomials (GPP's) if $c \neq 0$. We shall show that no matter how many times U is applied to a GPM with exponent α, the resulting density can be expanded in terms of a finite number of GPM's, GPP's, and piecewise monomials. The key element of the proof is that if U is applied to a GPP, the result is a linear combination of GPP's having the same exponent α, but a larger c. Let $m_{\beta,j}$ denote the value of $|f'(f_\beta^{-1}(x))|$, $x \in C_j$. If the functions in Eq. (40) are evolved one time step one finds

$$\tilde{\psi}_j(\alpha,\eta,c,f_\beta^{-1}(x))|f_\beta^{-1\prime}(x)| = \frac{\tilde{\psi}_k(\alpha,\eta',\hat{c},x,)}{m_{\beta,k}^{\alpha+1}} \tag{70}$$

for

$$x \in C_k, \quad f_\beta^{-1}(C_k) \bigcap C_j \neq \emptyset,$$

where

$$\hat{c} = c\, m_{\beta,k} \text{ if } a_{j-1} \in f_\beta^{-1}(C_k) \text{ or } a_j \in f_\beta^{-1}(C_k), \qquad (71)$$
$$\hat{c} > c\, m_{\beta,k} \text{ otherwise,}$$

as easily proved using relations (15) and (16). If $\hat{c} > a_k - a_{k-1}$, $\tilde{\psi}_j(\alpha, \hat{c}, x)$ can be expressed as a uniformly convergent power series in x, for all $x \in C_j$. Moreover, if $c \neq 0$, $U^n \tilde{\psi}_j(\alpha, c, x)$ can be expanded in a similar fashion, after a finite number of iterations since $\hat{c}$ is growing exponentially quickly. In fact, after a maximum of

$$N_{\max} = \frac{\ln(\frac{\Delta_1 m_1}{\Delta_2})}{\ln(m_2)}, \qquad (72)$$

iteration $\hat{c}$ is either zero or greater than the width of the largest cell. Here Δ_1, Δ_2, m_1, and m_2 denote the width of the largest and smallest cells, the maximum slope, and the minimum slope, respectively, and $N_{\max}$ is an upper bound. Therefore if an initial density is a given GPM, in the subsequent evolution it can be expressed as a linear combination of piecewise monomials, a finite number GPM's and GPP's (where the number of different values of c is finite). Thus an invariant functional space can be constructed having the above functions as a basis. Only certain transitions between the basis functions are possible, as indicated in the following table.

	GPM	GPP	PM
GPM	yes (a)	no	no
GPP	yes (b)	yes (d)	no
PM	yes (c)	yes (e)	yes (f)

(73)

where PM denotes piecewise monomials. Thus the matrix W generating the evolution of the states has the schematic form

$$W = \begin{pmatrix} A & 0 & 0 \\ B & D & 0 \\ C & E & F \end{pmatrix}, \qquad (74)$$

where the matrix blocks A, B, C, D, E, and F correspond to transitions (a)–(f) listed in table (73) and the eigenvalues of W are the eigenvalues of A, D, and F, respectively. The eigenvalues of F are just the eigenvalues of U acting on the space of piecewise monomial functions. All the eigenvalues of D are zero, because a GPP as it evolves can never undergo a transition to itself [see Eq. (71)]. The eigenvalues of A when $\mathcal{P}$ is the minimal Markov partition are difficult to characterize for general piecewise linear Markov maps. However, if $\mathcal{P}$ is a refined partition of the type considered in Sec. IV, the eigenvalues associated with periodic orbits can be easily calculated. The derivation is very similar to that of Sec. IV, except that for every exponent α there are two GPM's,

$$(x_l - x)^\alpha \chi_{[a_{i*-1}, x_l]}(x)$$

and

$$(x - x_l)^\alpha \chi_{[x_l, a_{i*}]}(x),$$

corresponding to each point of the periodic orbit, whereas in Sec. IV there is only one piecewise monomial for every integer exponent k. The definition of a_{i*} was given directly after Eq. (42). $C(n)$ of Eq. (49) is now replaced with a $2p \times 2p$ matrix $C'(n)$. $C'(n)$ can be written as a $p \times p$ matrix having essentially the same form as $C(n)$, except that each element is a 2×2 matrix. It follows immediately that the eigenvalues are roots of the equations

$$\lambda^{2p} + \frac{1}{|f^{p\prime}(x_l)|^{2(\alpha+1)}} = 0 \text{ if } f^{p\prime}(x_l) < 0,$$
$$\lambda^{2p} - \frac{1}{|f^{p\prime}(x_l)|^{2(\alpha+1)}} = 0 \text{ if } f^{p\prime}(x_l) > 0, \qquad (75)$$

where x_l is any element of the periodic orbit. Therefore the eigenvalues are

$$\frac{1}{|f^{p\prime}(x_l)|^{\frac{\alpha+1}{2p}}} \exp\left(\frac{\pi j}{p}\right) \text{ if } f^{p\prime}(x_l) > 0,$$
$$j = 0, 1, 2, \ldots, p, p+1, \ldots, 2p-1, \qquad (76)$$

$$\frac{1}{|f^{p\prime}(x_l)|^{\frac{\alpha+1}{2p}}} \exp\left(\frac{\pi j}{2p}\right) \text{ if } f^{p\prime}(x_l) < 0,$$
$$j = 0, 1, 2, \ldots, p-1, p, p+1, \ldots, 2p-1. \qquad (77)$$

The eigenmodes describe a periodic oscillatory decay if the imaginary part of α is given by

$$\alpha_I = \frac{g\; 2\pi}{\ln[|f^{p\prime}(x_l)|]}, \qquad (78)$$

where g is any fraction, and an aperiodic oscillatory decay otherwise. Again, if the real part of α, α_R, is close to -1, then the eigenvalues decay arbitrarily slowly. Of course, as observables are normally real valued, when α is complex the basis [see Eq. (69) ($c = 0$)] needs to be extended for its complex conjugate α^*. In this case the observables or initial densities of interest are typically of the form

$$\tilde{\psi}_i(\alpha, L, 0, x) + \tilde{\psi}_i(\alpha*, L, 0, x)$$
$$= 2(x - a_{i-1})^{\alpha_R} \cos[\alpha_I \ln(x - a_{i-1})] \chi_i(x). \qquad (79)$$

A striking property of this type of function is that it oscillates infinitely quickly as x approaches a_{i-1} if α_I is nonzero.

Let us illustrate the latter results with the tent map, choosing the partition $\{0, x_*, 1\}$, so

$$a_0 = 0,\; a_1 = x_* = \frac{m}{m+1},\; a_2 = 1 \qquad (80)$$

and

$$\begin{aligned}\psi_1(\alpha,L,0,x) &= x^\alpha \chi_{[0,x_*]}(x),\\ \psi_1(\alpha,R,0,x) &= (x_*-x)^\alpha \chi_{[0,x_*]}(x),\\ \psi_2(\alpha,L,0,x) &= (x-x_*)^\alpha \chi_{[x_*,1]}(x),\\ \psi_2(\alpha,R,0,x) &= (1-x)^\alpha \chi_{[x_*,1]}(x).\end{aligned} \tag{81}$$

Evolving each of these functions once we obtain

$$\begin{aligned}U\psi_1(\alpha,L,0,x) &= \frac{(\frac{x}{m})^\alpha}{m} + \frac{(1-\frac{x}{m})^\alpha}{m},\\ U\psi_1(\alpha,R,0,x) &= \frac{(x_*-\frac{x}{m})^\alpha}{m} + \frac{(\frac{x-x_*}{m})^\alpha}{m}\chi_{[x_*,1]}(x),\\ U\psi_2(\alpha,L,0,x) &= \frac{(\frac{x_*-x}{m})^\alpha}{m}\chi_{[0,x_*]}(x),\\ U\psi_2(\alpha,R,0,x) &= \frac{(\frac{x}{m})^\alpha}{m}\chi_{[0,x_*]}(x).\end{aligned} \tag{82}$$

The right-hand sides of each of the last four equations can be written as linear combinations of the functions Eq. (81) and the monomials

$$x^k\chi_{[0,x_*]}(x),\ (x-x_*)^k\chi_{[x_*,1]}(x), \tag{83}$$

where $k = 0, 1, 2, \ldots$. Again we can find a value of k such that the monomial expansions have converged sufficiently so that truncation produces a negligible error. Thus we can find a finite dimensional functional space including the generalized piecewise monomials [Eq. (81)] that up to an arbitrary approximation is invariant under U. The new eigenvalues corresponding to the fixed point are

$$\lambda_\pm = \pm\frac{1}{|f'(x_*)|^{\alpha+1}}. \tag{84}$$

VII. CONCLUSION

Spectral decompositions of the Frobenius-Perron operator have been obtained for all piecewise linear Markov maps and associated repellerlike maps. This was achieved by restricting the operator to an invariant space spanned by a piecewise monomial basis associated with any Markov partition $\mathcal{P}$ of the following class: the minimal Markov partition, its self-refinements, and refinements made with the addition of the elements of a finite number of periodic orbits. This representation is relevant for all initial densities or measurable functions, which, over each cell of a $\mathcal{P}$, can be expressed (or suitably approximated) as a finite Taylor series, and includes all functions piecewise analytic over $\mathcal{P}$. The main results are the following.

(i) The spectra and right eigenfunctions for the zeroth order monomial projection correspond to those obtained in earlier work using a Markov coarse graining projection [7,26].

(ii) The eigenvalues can be grouped into sets [Eqs. (23) and (39)], each corresponding to a zeta function and a piecewise monomial space of a given order.

(iii) Right and left eigenfunctions or eigendistributions associated with periodic orbits have been constructed and a simple general formula for their spectra [Eq. (40)] has been given. This construction enables local behavior to be predicted, in particular the characteristic times of time correlation functions, which generally exhibit a periodic oscillatory decay.

(iv) The decay rates associated with observables that are generalized piecewise polynomials (powers of x raised to a fractional or to a complex exponent) have been considered [Eq. (69)]. Eigenvalues [Eqs. (76) and (77)] and methods for obtaining eigenfunctions have been given. Exponents can be chosen so that the corresponding spectra lie anywhere in the unit circle, thus arbitrarily slowly decaying eigenstates can be constructed which are highly localized about the periodic orbits.

Our results confirm further the conclusion of recent work [2,19] that the spectra of the Frobenius-Perron operator, and thus the possible measurable characteristic times for chaotic maps, depend critically on the functional space under consideration. More importantly they show how invariant functional spaces can be constructed in a systematic and efficient manner according to the set of observables that are of immediate interest. Numerical investigations and analytic work suggest that the analysis can be extended using an approximation technique to chaotic piecewise linear non-Markov maps and, to some extent, everywhere expanding maps. This problem will be the subject of a future publication [16], where higher dimensional systems will also be explored (see also [27] for related references).

Note added. After submission of this paper, G. Vattay informed us that he had also related the spectra of U acting on functions piecewise analytic over the minimal Markov partition to the Ruelle zeta function [28]. He also drew our attention to Ref. [20], which we recommend to the reader.

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