Symmetry in phase space for a system with a singular higher-order Lagrangian

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A generalized first Noether theorem for singular Nth-order Lagrangians and generalized Noether identities for a variant system in phase space is derived. The strong and weak conservation laws in a canonical formalism are also obtained. Based on the canonical action, the generalized Poincaré-Cartan integral invariant (GPCII) for singular Nth-order Lagrangians is deduced. The GPCII connected with canonical equations of a constrained system is discussed. A counterexample to a conjecture of Dirac for a system with a singular higher-order Lagrangian is given. Applying the theories to the Yang-Mills field theory we find a new PBRST charge. Some additional information about the Lagrangian multipliers connected with first-class constraints is given.

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I. INTRODUCTION

Dynamical systems described in terms of higher-order Lagrangians were first given by Ostrogradsky [1,2] (see Ref. [2] and references cited therein). Recently, higher-order derivative Lagrangians obtained by many authors have exhibited a lot of interesting aspects in connection with the gauge theories [3], gravity [4,5], modified Korteweg–de Vries (KdV) equations [6], supersymmetry [7,8], string models [9,10], and other problems [11–15]. Higher-order Lagrangians have emerged as effective Lagrangians in gauge theories [16]. Baker *et al.* argued that at large distances (strong coupling), the Yang-Mills theory could be approximated by an effective Lagrangian containing the second derivative of field-strength tensors [17]. In recent years, there has been a certain amount of interest in applying the methods of molecular dynamics to systems containing complex molecules discussed by de Leeuw, Perram, and Peterson [18]. This has led researchers in this field to make practical use of constrained dynamics with the holonomic constraints which arise in classical statistical mechanics. In addition, in some models of field theories the field variables are subjected to holonomic constraints (for example, the nonlinear σ model [19], etc.). These holonomic constrained systems can be treated as constrained Hamiltonian systems by introducing Lagrangian multipliers. The approach presented here is a more general case: studying the symmetry properties in canonical formalism for singular Nth-order Lagrangians (N refers to the highest time derivative in the Lagrangian) with the aid of the Ostrogradsky transformation and the Dirac theory of a constrained system.

As is well known, the discussion of the symmetry properties of a system is usually based on examination of the Lagrangian in configuration space. Classical Noether theorems were formulated in terms of Lagrange's variables. For a system with a regular Lagrangian and a finite number of degrees of freedom, the invariance under a finite continuous group in terms of the canonical variables was discussed by Djukic [20]. A different version of the Noether theorem for constrained Lagrangian systems has been proved by Ferrario and Passerini [21]. The extended second Noether theorem has also been discussed by Lusanna [22]. The system with a singular Lagrangian is subject to some inherent phase-space constraints, and is called a constrained Hamiltonian system [23]. The symmetry properties of this system in canonical formalism for ordinary singular and second-order Lagrangians have been given in our previous work [24–26]. The generalization of those results to the singular Nth-order Lagrangians is straightforward. In this paper, the generalized first Noether theorem (GFNT) and generalized Noether identity (GNI) in canonical formalism for singular Nth-order Lagrangians have been derived and some new applications are given. The GPCII for singular Nth-order Lagrangians has also been deduced. The invalidity of Dirac's conjecture is further discussed.

The paper is organized as follows. In the beginning of Sec. II, we discuss some aspects of the Dirac theory for singular Nth-order Lagrangians relevant to the following discussion. Then, we derive the GFNT and GNI in canonical formalism and point out that in some variant systems there may also be a Dirac constraint. Another form of GNI is also formulated, combined with the concept of a weak quasi-invariant system, with which one can further analyze the canonical constraints. The strong and weak conservation laws are formulated in a more general form. Based on the symmetry properties of the constrained Hamiltonian system, we present in Sec. III a counterexample without linearization of constraints for a system with a singular higher-order Lagrangian. In this example Dirac's conjecture fails. In Sec. IV, we present our approach for obtaining the GPCII for singular Nth-order Lagrangians in which the constraints depend on time explicitly. The advantages of this derivation of the GPCII are that one can easily discuss the connection between the GPCII and canonical equations for a constrained Hamiltonian system. In Sec. V, the applications of the theory to the Yang-Mills theory with singular higher-order Lagrangians are given and a new partial

Becchi-Rouet-Stora-Tyutin (PBRST) charge and some additional information about the Lagrange multipliers connecting with the first-class constraints are obtained. Section VI is devoted to the conclusions.

II. GFNT AND GNI FOR SINGULAR *N*th-ORDER LAGRANGIANS

A. Preliminaries

We start this section by reviewing very briefly the transition from Lagrangian to Hamiltonian formalism for singular higher-order Lagrangians given in a paper [13], by Saito, Sugano, Ohta, and Kimura. Consider a system whose Lagrangian is given by

$$L=L(t,q_{(0)},q_{(1)},\ldots,q_{(N)}),\quad q=[q^1,q^2,\ldots,q^n],$$
$$q_{(0)}=q,\quad q_{(s)}=D^s q,\quad D=\frac{d}{dt}\ .$$

The Euler-Lagrange equations of this system are given by

$$\sum_{s=0}^{N}(-1)^s D^s\left[\frac{\partial L}{\partial \dot{q}_{(s)}}\right]=0\ . \tag{2.1}$$

The Ostrogradsky transformation introduces canonical momenta

$$p_i^{(N-1)}=\frac{\partial L}{\partial q^i_{(N)}}\ , \tag{2.2a}$$

$$p_i^{(s-1)}=\frac{\partial L}{\partial q^i_{(s)}}-\dot{p}_i^{(s)}\quad (s=1,2,\ldots,N-1)\ , \tag{2.2b}$$

and using these relations one can go over from the Lagrangian description to the Hamiltonian description. The canonical Hamiltonian is defined by [13] (the summation is taken over repeated indices)

$$\begin{aligned}H_c&=p_i^{(s)}q^i_{(s+1)}-L\\&=\sum_{i=1}^{n}\sum_{s=0}^{N-1}p_i^{(s)}q^i_{(s+1)}-L(t,q_{(0)},q_{(1)},\ldots,q_{(N)})\ .\end{aligned} \tag{2.3}$$

We suppose the extended Hessian matrix

$$H_{ij}=\frac{\partial^2 L}{\partial q^i_{(N)}\partial q^j_{(N)}}\quad (i,j=1,2,\ldots,n) \tag{2.4}$$

to be singular and its rank to be $(n-R)$. Thus, one cannot solve for all $q^i_{(N)}$ from (2.2a) because $\det|H_{ij}|=0$; this implies the existence of constraints [27]

$$\phi_a^0(t,q^i_{(s)},p_i^{(s)})\approx 0\quad (a=1,2,\ldots,R)\ , \tag{2.5}$$

where the sign $\approx$ (weak equality) means equality on the constrained hypersurface. Equations (2.5) are called the primary constraints. The total Hamiltonian is given by

$$H_T=H_c+\lambda^a\phi_a^0\ , \tag{2.6}$$

where $\lambda^a(t)$ $(a=1,2,\ldots,R)$ are Lagrange multipliers. The canonical equations of the singular *N*th-order Lagrangians are given by [28]

$$\dot{q}^i_{(s)}\approx\{q^i_{(s)},H_T\},\quad \dot{p}_i^{(s)}\approx\{p_i^{(s)},H_T\}\ , \tag{2.7}$$

where { , } denotes the generalized Poisson bracket

$$\{u,v\}=\frac{\partial u}{\partial q^i_{(s)}}\frac{\partial v}{\partial p_i^{(s)}}-\frac{\partial u}{\partial p_i^{(s)}}\frac{\partial v}{\partial q^i_{(s)}}\ . \tag{2.8}$$

The stationary conditions of the primary constraints ϕ_a^0 enable one to define successively the secondary constraints

$$\phi_a^1=\hat{X}\phi_a^0=\frac{\partial\phi_a^0}{\partial t}+\{\phi_a^0,H_T\}\approx 0\ , \tag{2.9a}$$

$$\phi_a^k=\hat{X}\phi_a^{k-1}=\frac{\partial\phi_a^{k-1}}{\partial t}+\{\phi_a^{k-1},H_T\}\approx 0\ . \tag{2.9b}$$

This algorithm is continued until

$$\phi_a^{m+1}=\hat{X}\phi_a^m=C^b_{ak}\phi_b^k\quad (k\le m) \tag{2.9c}$$

is satisfied. All the constraints ϕ_s are classified into two classes. A ϕ_a is defined to be first class if $\{\phi_a,\phi_b\}=0$ (mod ϕ_c) for all ϕ_b; otherwise it is second class.

B. GFNT

Let us consider the transformation properties of the system under the continuous group with the infinitesimal transformation given by

$$\begin{aligned}&t'=t+\Delta t,\\&q'^i_{(s)}(t')=q^i_{(s)}(t)+\Delta q^i_{(s)}(t)\ ,\\&p_i^{(s)\prime}(t')=p_i^{(s)}(t)+\Delta p_i^{(s)}(t)\end{aligned} \tag{2.10}$$

where $\Delta q^i_{(s)},\Delta p_i^{(s)}$ are total variations which can be expressed in terms of the simultaneous variations $\delta q^i_{(s)},\delta p_i^{(s)}$,

$$\Delta q^i_{(s)}=\delta q^i_{(s)}+\dot{q}^i_{(s)}\Delta t,\quad \Delta p_i^{(s)}=\delta p_i^{(s)}+\dot{p}_i^{(s)}\Delta t\ . \tag{2.11}$$

Let it be supposed that the variation of canonical action integral

$$I_p=\int_{t_1}^{t_2}L_p dt=\int_{t_1}^{t_2}(p_i^{(s)}q^i_{(s+1)}-H_c)dt \tag{2.12}$$

under the transformation (2.10) is given by

$$\Delta I_p=\int_{t_1}^{t_2}(D\Lambda+V)dt\ , \tag{2.13}$$

where Λ and V are functions of t, $q^i_{(s)}$ and $p_i^{(s)}$. Under the transformation (2.10), from (2.12) and (2.13) one has

$$\delta p_i^{(s)}\frac{\delta I_p}{\delta p_i^{(s)}}+\delta q^i_{(s)}\frac{\delta I_p}{\delta q^i_{(s)}}+D[p_i^{(s)}\delta q^i_{(s)}+(p_i^{(s)}q^i_{(s+1)}-H_c)\Delta t]=D\Lambda+V\ , \tag{2.14}$$

where

$$\frac{\delta I_p}{\delta p_i^{(s)}}=\dot{q}^i_{(s)}-\frac{\partial H_c}{\partial p_i^{(s)}},\quad \frac{\delta I_p}{\delta q^i_{(s)}}=-\dot{p}_i^{(s)}-\frac{\partial H_c}{\partial q^i_{(s)}}\ . \tag{2.15}$$

Let us first consider the finite continuous group G_r and let

$$\Delta t=\varepsilon_\sigma \tau^\sigma(t,q^i_{(s)},p_i^{(s)}),$$
$$\Delta q^i_{(s)}=\varepsilon_\sigma \xi^{i\sigma}_{(s)}(t,q^i_{(s)},p_i^{(s)}), \tag{2.10'}$$
$$\Delta p_i^{(s)}=\varepsilon_\sigma \eta_i^{(s)\sigma}(t,q^i_{(s)},p_i^{(s)}) ,$$

where ε_σ ($\sigma=1,2,\ldots,r$) are parameters. For a weakly quasi-invariant system [22] $V \triangleq 0$, where $\triangleq$ means "evaluated on the trajectory of motion." Let it be supposed that the constraint conditions (2.5) are satisfied,

$$\delta\phi_a^0=\frac{\partial\phi_a^0}{\partial q^i_{(s)}}\delta q^i_{(s)}+\frac{\partial\phi_a^0}{\partial p_i^{(s)}}\delta p_i^{(s)}\approx 0 , \tag{2.16}$$

under the transformation (2.10′). These conditions imply that the constraints $\phi_a^0\approx 0$ are invariant under the simultaneous variations determined by (2.10). Introducing a set of Lagrange multipliers $\lambda^a(t)$, combining the expressions (2.14) and (2.16), and using the canonical equations (2.7), one obtains

$$D[p_i^{(s)}(\xi^{i\sigma}_{(s)}-\dot q^i_{(s)}\tau^\sigma)+(p_i^{(s)}q^i_{(s+1)}-H_c)\tau^\sigma]=D\Lambda^\sigma \quad (\sigma=1,2,\ldots,r) , \tag{2.17}$$

where Λ^σ satisfy $\Lambda=\varepsilon_\sigma\Lambda^\sigma$. Therefore, we have the following generalized first Neother theorem (GNFT) in canonical formalism for a singular Nth-order Lagrangian. If, under the transformation (2.10′), the canonical action (2.12) is a weak quasi-invariant and the constraint conditions (2.5) are invariant under the simultaneous variations $\delta q^i_{(s)}$ and $\delta p_i^{(s)}$ induced by (2.10′), then the r expressions

$$p_i^{(s)}\xi^{i\sigma}_{(s)}-H_c\tau^\sigma-\Lambda^\sigma=\text{const} \quad (\sigma=1,2,\ldots,r) \tag{2.18}$$

are constants of motion for the singular Nth-order Lagrangian. This theorem is a generalization of our previous results [24–26].

For example, if the Lagrangian L and constraints ϕ_a^0 do not depend on time explicitly, for the translation Δt of time t, one has $\Delta L_p=0$ and $\Delta\phi_a^0\approx 0$; hence

$$\delta\phi_a^0=\frac{\partial\phi_a^0}{\partial q^i_{(s)}}\delta q^i_{(s)}+\frac{\partial\phi_a^0}{\partial p_i^{(s)}}\delta p_i^{(s)}$$
$$=-\left[\frac{\partial\phi_a^0}{\partial q^i_{(s)}}\dot q^i_{(s)}+\frac{\partial\phi_a^0}{\partial p_i^{(s)}}\dot p_i^{(s)}\right]\Delta t=-\frac{d\phi_a^0}{dt}\Delta t . \tag{2.19}$$

According to the stationary conditions of constraint, the conditions (2.16) are satisfied. In this case the GFNT gives us the conservation law of generalized energy. Conversely, if one requires the conservation of generalized energy in agreement with the Lagrange description, then the constraint must be preserved in time. This is just Dirac's consistency condition for the constraint.

The GFNT in canonical formalism can be easily extended to the case when the system is subjected to extra holonomic constraints $f_b(t,q^1_{(0)},\ldots q^n_{(0)})=0$. In this case one needs to require further that the extra holonomic constraints are invariant under simultaneous variations $\delta q^i_{(0)}$ induced by (2.10′).

C. GNI

As is well known in the massive Yang-Mills theory, the Lagrangian is not invariant under gauge transformation; gauge-invariant interaction of massive Fermi fields with gauge fields is not invariant under the chirality transformation of the Fermi fields; the effective Lagrangian with Faddeev-Popov ghost fields is not invariant under the gauge transformation; the invariance is restored only under the BRS transformation. Therefore, for the Lagrangian of a system which is not invariant under an infinite continuous group, the discussion of the transformation properties is necessary. The discussion of this problem in configuration space was given in our previous work [29]. Now we discuss the transformation properties for those systems under an infinite continuous group $G_{\infty r}$ in extended phase space. This will lead to the GNI in canonical formalism. In quantum theory, the GNI corresponds to the Ward-Takahashi identity.

Let us consider an infinite continuous group $G_{\infty r}$ generated by the transformation of time and canonical variables given by

$$t'=t+\Delta t=t+\sum_{j=0}^{J}A_j^\sigma D^j\varepsilon_\sigma(t) ,$$
$$q'^i_{(s)}(t')=q^i_{(s)}(t)+\Delta q^i_{(s)}(t)$$
$$=q^i_{(s)}(t)+\sum_{k=0}^{K}B^{i\sigma}_{(s)k}D^k\varepsilon_\sigma(t) , \tag{2.20}$$
$$p_i^{(s)\prime}(t')=p_i^{(s)}(t)+\Delta p_i^{(s)}(t)$$
$$=p_i^{(s)}(t)+\sum_{m=0}^{M}C_{im}^{(s)\sigma}D^m\varepsilon_\sigma(t) ,$$

where $\varepsilon_\sigma(t)$ ($\sigma=1,2,\ldots,r$) are arbitrary infinitesimal functions and A, B, C's are functions of t, $q^i_{(s)}(t)$, and $p_i^{(s)}(t)$. Under the transformation (2.20), from (2.14) one has

$$\int_{t_1}^{t_2}\left[\delta p_i^{(s)}\frac{\delta I_p}{\delta p_i^{(s)}}+\delta q^i_{(s)}\frac{\delta I_p}{\delta q^i_{(s)}}-V^\sigma\varepsilon_\sigma\right]dt$$
$$=\int_{t_1}^{t_2}D[\Lambda^\sigma\varepsilon_\sigma-p_i^{(s)}\delta q^i_{(s)}-(p_i^{(s)}q^i_{(s+1)}-H_c)\Delta t]dt , \tag{2.21}$$

where

$$\Lambda^\sigma=\sum_{i=0}^{I}u_i^\sigma D^i, \quad V^\sigma=\sum_{n=0}^{N_0}v_n^\sigma D^n , \tag{2.22}$$

and u,v's are functions of t, $q^i_{(s)}$, and $p_i^{(s)}$. Since $\varepsilon_\sigma(t)$ are arbitrary, one can choose $\varepsilon_\sigma(t)$ and their derivatives up to a required order to vanish on the end points of the interval $[t_1,t_2]$; then the integral of the right-hand side of (2.21) is reduced to zero. We repeat the integral by parts of the remaining terms on the left-hand side of the identity (2.21); appealing to the arbitrariness of the $\varepsilon_\sigma(t)$ one can force the end-point terms to vanish, after which one can apply the fundamental lemma of calculus of variations to conclude that

$$\sum_{m=0}^{M}(-1)^m D^m\left[C_{im}^{(s)\sigma}\frac{\delta I_p}{\delta p_i^{(s)}}\right]+\sum_{j=0}^{J}(-1)^{j+1}D^j\left[A_j^\sigma \dot{p}_i^{(s)}\frac{\delta I_p}{\delta p_i^{(s)}}\right]+\sum_{k=0}^{K}(-1)^k D^k\left[B_{(s)k}^{i\sigma}\frac{\delta I_p}{\delta q_{(s)}^i}\right]$$
$$+\sum_{j=0}^{J}(-1)^{j+1}D^j\left[A_j^\sigma \dot{q}_{(s)}^i\frac{\delta I_p}{\delta q_{(s)}^i}\right]=\sum_{n=0}^{N_0}(-1)^n D^n v_n^\sigma . \quad (2.23)$$

Thus, we have the following generalized second Noether theorem in canonical formalism. If the variation of canonical action is given by (2.13) under the transformation (2.20), then there are r identities (2.23) between the functional derivative $\delta I_p/\delta q_{(s)}^i, \delta I_p/\delta p_i^{(s)}$ and their derivatives up to some fixed order. These identities (2.23) are called generalized Noether identities (GNI's) in canonical formalism. If $V\neq 0$, these systems are called variant systems. In the case of invariance ($V=0$), the right-hand side of (2.23) equals zero. Thus, we have identity relations between the functional derivatives and their derivatives and this leads to a reduction in the number of linearly independent functional derivatives $\delta I_p/\delta q_{(s)}^i$ and $\delta I_p/\delta p_i^{(s)}$.

Let us now derive the alternative form of the GNI in phase space. If we consider a set of local transformations for fixed s ($s=c$),

$$\begin{aligned}\Delta t&=\alpha\psi^\sigma(t)\omega_\sigma(t) ,\\ \Delta q_{(c)}^i&=\alpha\psi_{(c)}^{i\sigma}(t)\omega_\sigma(t) ,\\ \Delta p_i^{(c)}&=\alpha\psi_i^{(c)\sigma}(t)\omega_\sigma(t) ,\end{aligned} \quad (2.24)$$

where $\omega_\sigma(t)$ ($\sigma=1,2,\ldots,r$) are arbitrary functions of time and α is an infinitesimal parameter, under which it is supposed that the canonical action is weakly quasi-invariant, i.e.,

$$\Delta I_p=\int_{t_1}^{t_2}\alpha\{D[\Lambda^\sigma(t,q_{(s)}^i,p_i^{(s)})\omega_\sigma(t)] + V(t,q_{(s)}^i,p_i^{(s)})\omega_\sigma(t)\}dt , \quad (2.25)$$

where $V\doteq 0$ and Λ^σ and V^σ are functions of t and the canonical variables. Then, one can get

$$\int_{t_1}^{t_2}\left[\frac{\delta I_p}{\delta p_i^{(c)}}(\psi_i^{(c)\sigma}-\dot{p}_i^{(c)}\psi^\sigma)\omega_\sigma+\frac{\delta I_p}{\delta q_{(c)}^i}(\psi_{(c)}^{i\sigma}-\dot{q}_{(c)}^i\psi^\sigma)\omega_\sigma-V^\sigma\omega_\sigma\right]dt=(\Lambda^\sigma-p_i^{(c)}\psi_{(c)}^{i\sigma}-H_c\psi^\sigma)\omega_\sigma\Big|_{t_i}^{t_2} . \quad (2.26)$$

As $\omega_\sigma(t)$ are arbitrary functions, the integrands of volume and surface terms have to vanish identically and we get the two sets of Noether identities in phase space,

$$\frac{\delta I_p}{\delta p_i^{(c)}}(\psi_i^{(c)\sigma}-\dot{p}_i^{(c)}\psi^\sigma)+\frac{\delta I_p}{\delta q_{(c)}^i}(\psi_{(c)}^{i\sigma}-\dot{q}_{(c)}^i\psi^\sigma)-V^\sigma=0 , \quad (2.27)$$

$$\Lambda^\sigma-p_i^{(c)}\psi_{(c)}^{i\sigma}-H^c\psi^\sigma=0 . \quad (2.28)$$

Using the equations of motion, one can obtain some constraints in phase space from these Noether identities (2.27) and (2.28).

D. Dirac constraint

Let us now give a preliminary application of the GNI to the Dirac theory of constrained systems. Let us put $\Delta t=0$ in (2.20), as is usually done in the discussion of gauge transformation. Let it be assumed that the variation of canonical action satisfies (2.21) with $\Delta t=0$ on the right-hand side. For the sake of convenience, it is supposed that $M<K, N_0<K$ and, in this case, the GNI becomes

$$\sum_{m=0}^{M}(-1)^m D^m\left[C_{im}^{(s)\sigma}\left[\dot{q}_{(s)}^i-\frac{\partial H_c}{\partial p_i^{(s)}}\right]\right]+\sum_{k=0}^{K}(-1)^k D^k\left[B_{(s)k}^{i\sigma}\left[-\dot{p}_i^{(s)}-\frac{\partial H_c}{\partial q_{(s)}^i}\right]\right]=\sum_{n=0}^{N_0}(-1)^n D^n v_n^\sigma . \quad (2.29)$$

Using the Ostrogradsky transformation (2.2), one finds

$$p_i^{(0)}=\sum_{s=1}^{N}(-1)^{s-1}D^{(s-1)}\left[\frac{\partial L}{\partial q_{(s)}^i}\right] \quad (2.30)$$

and

$$D\left[\frac{\partial L}{\partial q_{(N)}^i}\right]=\sum_{s=0}^{N}\left[\frac{\partial^2 L}{\partial q_{(N)}^i\partial q_{(s)}^j}\right]q_{(s+1)}^j . \quad (2.31)$$

Substituting the expressions (2.30) and (2.31) into the identities (2.29), one finds that in these identities the highest derivatives of q^i must occur in the terms $D^K(B^{i\sigma}_{(0)K}\dot{p}_i^{(0)})$ which contain the $(2N+K)$th-order derivatives of q^i, and these must cancel each other irrespective of other terms as Bergmann did for the singular first-order Lagrangian [30]

$$B^{i\sigma}_{(0)K}\frac{\partial^2 L}{\partial q^i_{(N)}\partial q^j_{(N)}}q^i_{(2N+K)}=0 . \tag{2.32}$$

These conditions are to be fulfilled for any $(2N+K)$th-order derivatives of q^i; thus, one obtains

$$B^{i\sigma}_{(0)K}\frac{\partial^2 L}{\partial q^i_{(N)}\partial q^j_{(N)}}=0 . \tag{2.33}$$

Because $B^{i\sigma}_{(0)K}$ are not all identically zero, this implies that

$$\det\left|\frac{\partial^2 L}{\partial q^i_{(N)}\partial q^j_{(N)}}\right|=0 . \tag{2.34}$$

Then, the extended Hessian matrix is degenerate and therefore we conclude that this variant system also has a Dirac constraint. For example, the massive Yang-Mills field theories belong to this category.

E. Strong and weak conservation laws

According to the GNI (2.23), one can obtain strong conservation laws or exact differential identities for higher-order Lagrangians in certain cases. The strong conservation laws are valid whether the equations of motion are satisfied or not. Along the dynamical trajectory of motion one finds weak conservation laws.

Let it be supposed that in transformation (2.20) one uses

$$\begin{aligned}\Delta t&=a_0^\sigma \varepsilon_\sigma(t) ,\\ \Delta q^i_{(s)}&=(b^{i\sigma}_{0(s)}+b^{i\sigma}_{1(s)}D)\varepsilon_\sigma(t) ,\\ \Delta p_i^{(s)}&=(c^{(s)\sigma}_{0i}+c^{(s)\sigma}_{1i}D)\varepsilon_\sigma(t) ,\end{aligned} \tag{2.35}$$

where a, b, c's are functions of $t,q^i_{(s)}(t),p_i^{(s)}(t)$ and it be supposed that the variation of canonical action satisfies (2.21). Multiplying the GNI (2.23) by $\varepsilon_\sigma(t)$ and subtracting the result from the basic identity (2.21), one obtains the exact differential identity

$$D\left\{\left[b^{i\sigma}_{1(s)}\frac{\delta I_p}{\delta q^i_{(s)}}+c^{(s)\sigma}_{1i}\frac{\delta I_p}{\delta p_i^{(s)}} +p_i^{(s)}(b^{i\sigma}_{0(s)}+b^{i\sigma}_{1(s)}D)-H_c a_0^\sigma-\Lambda^\sigma -\sum_{n=1}^{N_0}\sum_{j=0}^{n-1}(-1)^j(D^j v_n^\sigma)D^{n-j-1}\right]\varepsilon_\sigma(t)\right\}=0 , \tag{2.36}$$

which implies the existence of strong conservation laws.

If the transformation group has a subgroup and $\varepsilon_\sigma=\varepsilon_\rho^0\xi_\sigma^\rho(t)$, where ε_ρ^0 are numerical parameters of the Lie group, one can get the weak conservation laws along the dynamical trajectory of motion

$$\left[b^{i\sigma}_{1(s)}\frac{\partial H_1}{\partial q^i_{(s)}}+c^{(s)\sigma}_{1i}\frac{\partial H_1}{\partial p_i^{(s)}}+p_i^{(s)}(b^{i\sigma}_{0(s)}+b^{i\sigma}_{1(s)}D)-H_c a_0^\sigma-\Lambda^\sigma-\sum_{n=1}^{N_0}\sum_{j=0}^{n-1}(-1)^j(D^j v_n^\sigma)D^{n-j-1}\right]\xi_\sigma^\rho=\text{const}\quad(\rho=1,2,\ldots,r) . \tag{2.37}$$

Thus, we have seen that the GNI may be converted into weak conservation laws in certain cases even if the canonical action of the system is not invariant under the specific local transformation. This algorithm deriving conservation laws differs from the first Noether theorem, where the invariance under a finite continuous group implies the existence of conservation laws. To illustrate this result, we shall discuss it further in Sec. V.

III. DIRAC'S CONJECTURE FOR A SYSTEM WITH A SINGULAR HIGHER-ORDER LAGRANGIAN

At present, Dirac's theory of constrained systems plays an important role in theoretical physics, especially in modern quantum field theory. With its help, many of the central problems which appeared in the development of the quantization procedures of gauge and gravitational fields have been solved [28]. However, in spite of these general achievements, some basic problems in the theory are still being discussed widely in the literature, one of them being Dirac's conjecture [23]. Dirac in his work on generalized canonical formalism conjectured that all first-class constraints are independent generators of the gauge transformation which generate equivalent transformations among physical states. It had been pointed by Henneaux, Teiteboim, and Zanell [31], Costa, Girotti, and Simões [32], and Cabo [33] that this problem is closely connected with the problem of whether Dirac's procedure in terms of the extended Hamiltonian H_E and the Lagrangian description are equivalent. From time to time there have been objections to Dirac's conjecture [34–38]. Costa, Girotti, and Simões pointed out that all these objections are based on the straightforward observation that the equations of motion derived from H_E are not strictly equivalent to the corresponding Lagrange equations [32]. Several examples given by Allcock [39], Cawley [40], and Frenkle [41] indicate that Dirac's conjecture is invalid when the constraints are written in linearized form. Recently, Qi has argued that Dirac's conjecture is valid [42]. A number of counterexamples are reexamined in which the constraints are not linearized as Cawley and others have done. Owing to the lack of a rigorous proof of Dirac's conjecture [22,43] (or even a proof that it is not correct), we have provided two examples of a system with a singular first-order Lagrangian to show that Dirac's conjecture fails [24,26]. Our examples differ from others in that we do not write the constraint in linearized form. Some points in our previous counterexamples need further discussion. First, if one writes the constraint in linearized form as is done in the literature [24,26], the full set of secondary constraints becomes second class. Second, because the secondary constraint p_x enters into the total Hamiltonian in these examples, the total Hamiltonian H_T is not differentiable on the constraint submanifold. We have presented another example of how to avoid these ambiguities [44].

We did not know of any system with a singular higher-order Lagrangian in which Dirac's conjecture was invalid. Now we will further discuss this problem. Dirac's conjecture for a singular Nth-order Lagrangian states that all first-class constraints in those systems are independent generators of gauge transformation. If this conjecture holds true, then the dynamics of a system possessing primary $\{\phi_a^0\}$ and secondary $\{\phi_a^k\}$ first-class constraints should be correctly described by the equations of motion arising from the extended Hamiltonian

$$H_E=H_c+H'=H_c+\lambda^a\phi_a^0+\mu_k^a\phi_a^k=H_T+\mu_k^a\phi_a^k , \tag{3.1}$$

where $\mu_k^a(t)$ are also Lagrange multipliers.

The present paper discusses the validity of Dirac's conjecture for a system with a singular higher-order Lagrangian from the viewpoint of the generalized Noether theorems in phase space. Let us consider whether conservation laws derived from H_E via GFNT in canonical formalism are equivalent to conservation laws arising from Lagrange's formalism via the classical first Noether theorem [45]. In this way one can judge the validity of Dirac's conjectures as we did in our previous discussion for a singular first-order Lagrangian [24,26]. Using the GNI (2.23) one can also examine this problem. If Dirac's conjecture holds true in a problem, the canonical equations of constrained Hamiltonian systems are derived from the extended Hamiltonian H_E. Along the trajectory of motion, from (2.23) one has

$$\sum_{m=0}^{M}(-1)^m D^m\left[C_{im}^{(s)\sigma}\frac{\partial H'}{\partial p_i^{(s)}}\right]+\sum_{j=0}^{M}(-1)^{j+1}D^j\left[A_j^\sigma \dot{p}_i^{(s)}\frac{\partial H'}{\partial p_i^{(s)}}\right]+\sum_{k=0}^{K}(-1)^k D^k\left[B_{(s)k}^{i\sigma}\frac{\partial H'}{\partial q_{(s)}^i}\right]+\sum_{j=0}^{J}(-1)^{j+1}D^j\left[A_j^\sigma \dot{q}_{(s)}^i\frac{\partial H'}{\partial q_{(s)}^i}\right]=\sum_{n=0}^{N_0}(-1)^n D^n v_n^\sigma . \tag{3.2}$$

The expressions (3.2) may become either a trivial equality or may give more relationships for the Lagrange multipliers connected with the first-class constraints [26]. If these expressions (3.2) give us inconsistent results for an admissible Lagrangian, then Dirac's conjecture regarding secondary first-class constraints may be invalid in this circumstance [26].

Let us consider a model with Lagrangian [46]

$$L=x_{(2)}z_{(2)}+x_{(1)}z_{(1)}+x_{(0)}z_{(0)}-y_{(1)}z_{(2)}-y_{(0)}z_{(1)} . \tag{3.3}$$

The Lagrangian (3.3) is invariant under a "scale" transformation

$$\begin{aligned} x'_{(s)}&=\rho^{-1}x_{(s)} ,\\ y'_{(s)}&=\rho^{-1}y_{(s)} ,\\ z'_{(s)}&=\rho z_{(s)} \quad (s=0,1,2) , \end{aligned} \tag{3.4}$$

where ρ is a numerical parameter. This leads to the conservation law via the classical first Noether theorem given by Anderson [45].

The canonical momenta $p_x^{(1)},p_y^{(1)},p_z^{(1)}$ and $p_x^{(0)},p_y^{(0)},p_z^{(0)}$ that conjugate to $x_{(1)},y_{(1)},z_{(1)}$ and $x_{(0)},y_{(0)},z_{(0)}$ are

$$\begin{aligned} &p_x^{(1)}=z_{(2)},\ \ p_y^{(1)}=0,\ \ p_z^{(1)}=x_{(2)}-y_{(1)} ,\\ &p_x^{(0)}=z_{(1)}-\dot{p}_x^{(1)},\ \ p_y^{(0)}=-z_{(2)},\\ &p_z^{(0)}=x_{(1)}-y_{(0)}-\dot{p}_z^{(1)} , \end{aligned} \tag{3.5}$$

respectively. The canonical Hamiltonian is given by

$$\begin{aligned} H_c=&p_x^{(1)}p_z^{(1)}+p_x^{(0)}x_{(1)}+p_x^{(0)}y_{(1)}+p_z^{(0)}z_{(1)}\\ &+p_x^{(1)}y_{(1)}-x_{(1)}z_{(1)}-x_{(0)}z_{(0)}+y_{(0)}z_{(1)} \end{aligned} \tag{3.6}$$

with the primary constraint $\phi^0=p_y^{(1)}\approx 0$. The total Hamiltonian is given by $H_T=H_c+\lambda\phi^0$, where $\lambda(t)$ is a Lagrange multiplier. The stationary condition of the constraint $\{\phi^k,H_T\}\approx 0$ yields the following secondary constraints:

$$\phi^1=\{\phi^0,H_T\}=-p_x^{(1)}-p_y^{(0)}\approx 0 , \tag{3.7}$$

$$\phi^2=\{\phi^1,H_T\}=p_x^{(0)}\approx 0 , \tag{3.8}$$

$$\phi^3=\{\phi^2,H_T\}=z_{(0)}\approx 0 . \tag{3.9}$$

All constraints $\{\phi^k\}$ $(k=0,1,2,3)$ are first class. The phase space Lagrangian and the primary constraint ϕ^0 are invariant under the following transformation:

$$\begin{aligned} &x'_{(s)}=\rho^{-1}x_{(s)},\ \ y'_{(s)}=\rho^{-1}y_{(s)},\ \ z'_{(s)}=\rho z_{(s)} ,\\ &p_x^{(1)\prime}=\rho p_x^{(1)},\ \ p_z^{(0)\prime}=\rho^{-1}p_z^{(0)} \quad (s=0,1) . \end{aligned} \tag{3.10}$$

Using the results (2.18) of the GFNT one obtains the conservation law

$$p_z^{(s)}z_{(s)}-p_x^{(s)}x_{(s)}-p_y^{(s)}y_{(s)}=\text{const} , \tag{3.11}$$

which can also be obtained by using Lagrange's variables via the classical first Noether theorem [45].

If Dirac's conjecture holds true, then the dynamics of this system should be described by the equations of motion arising from the extended Hamiltonian

$H_E=H_T+\mu_1\phi^1+\mu_2\phi^2+\mu_3\phi^3$. All secondary first-class constraints in this Hamiltonian are taken into account as Costa, Girotti, and Simões did for a singular first-order Lagrangian [32]. According to the GFNT, for the existence of the conservation law (3.11) one must further require that all secondary constraints ϕ^k $(k=1,2,3)$ satisfy the conditions (2.16), but these secondary first-class constraints cannot satisfy the conditions (2.16) under the transformation (3.10). Hence one cannot obtain the conservation law (3.11) from the extended Hamiltonian H_E.

Now let us give a brief discussion about the generator of gauge transformation of a constrained Hamiltonian system. Let it be supposed that the set of all independent constraints is first class and let these constraints be divided into primary ϕ_a^0 and secondary χ_b ones. According to Dirac's prescription, the generator G of the gauge transformation for a system can be written as

$$G=\theta^a(t)\phi_a^0+\omega^b(t)\chi_b \ . \tag{3.12}$$

It has been pointed out by Saito *et al.* [13] that the generator G of the gauge transformation must be conservative. So we have

$$\frac{d\theta^a}{dt}\phi_a^0+\theta^a\{\phi_a^0,H_c\}+\frac{d\omega^b}{dt}\chi_b+\omega^b\{\chi_b,H_c\}=0 \quad (\mathrm{mod}\,\phi_a^0) \ . \tag{3.13}$$

The quantities $\{\phi_a^0 H_c\}$ and $\{\chi_b,H_c\}$ can be expressed as

$$\{\chi_b,H_c\}=\alpha_{be}\chi_e \quad (\mathrm{mod}\,\phi_a^0) \ , \tag{3.14a}$$

$$\{\phi_a^0,H_c\}=\beta_{af}\chi_f \quad (\mathrm{mod}\,\phi_a^0) \ . \tag{3.14b}$$

Substituting (3.14) into (3.13) and taking into account the linear independence of the constraints, one obtains the following differential equations relating the coefficients $\theta^a(t)$ and $\omega^b(t)$ in generator (3.12):

$$\frac{d\omega^b}{dt}+\alpha_{be}\omega_e+\beta_{ba}\theta^a=0 \quad (\mathrm{mod}\,\phi_a^0) \ . \tag{3.15}$$

This result has been given by Galvão and Boechat for an ordinary singular Lagrangian [43]. But they have discarded the last term on the left-hand side of Eq. (3.8) in their paper. Here we have given a simple treatment to obtain the result, clarifying some confusion in their paper.

Applying this result to the above example, Eq. (3.12) is given by

$$G=\theta(t)\phi^0+\omega_1(t)\phi^1+\omega_2(t)\phi^2+\omega_3(t)\phi^3 \ . \tag{3.16}$$

From (3.14) and (3.15), one finds $\dot\omega(t)=-\theta(t)$, $\dot\omega_2(t)=-\omega_1(t)$, $\dot\omega_3(t)=-\omega_2(t)$. Let $\theta(t)=\dddot\varepsilon(t)$; then the generator (3.16) assumes the form

$$G=-\varepsilon(t)z_{(0)}+\dot\varepsilon(t)p_x^{(0)}+\ddot\varepsilon(t)(p_x^{(0)}+p_y^{(0)})+\dddot\varepsilon(t)p_y^{(1)} \ . \tag{3.17}$$

This result can be obtained by using the method that has been given in our previous paper [26].

The generator (3.17) produces the following transformation:

$$\begin{aligned}&\delta x_{(s)}=\{x_{(s)},G\}=\varepsilon^{(s+1)}(t) \ ,\\ &\delta y_{(s)}=\varepsilon^{(s+2)}(t),\\ &\delta z_{(s)}=0, \quad \delta p_z^{(0)}=\varepsilon(t) \ ,\\ &\delta p_z^{(1)}=\delta p_x^{(s)}=\delta p_y^{(s)}=0 \quad (s=0,1) \ ,\end{aligned} \tag{3.18}$$

where $\varepsilon^{(n)}=D^n\varepsilon(t)$. Under the transformation (3.18) the conserved quantity (3.11) is gauge invariant, and therefore this constant of motion is a physical, observable quantity in the Dirac sense. The total Hamiltonian H_T and the extended Hamiltonian H_E may generate different equations of motion for gauge-dependent variables. As pointed out by Henneaux, Teiteboim, and Zanell [31] and Costa , Girotti, and Simões [32], although one can change the equations of motion, one cannot change the gauge-invariant quantities of the system. The gauge-invariant constant of motion (3.11) cannot be obtained from the extended Hamiltonian H_E; this means that Dirac's conjecture fails in this example in which there is no linearization of the constraint. This example shall be investigated with the aid of GPCII later.

IV. GPCII FOR A SINGULAR HIGHER-ORDER LAGRANGIAN

A. GPCII

The Poincaré-Cartan integral invariant plays an important role in classical mechanics and field theories. The generalization of this invariant to the case with an ordinary singular Lagrangian has been given by Benavent and Gomis [47] and Dominici and Gomis [48], and some applications have also been given by Dominici and Gomis [49] and Sugano [50]. Now a more general case will be discussed. Let us consider a system whose Nth-order Lagrangian is singular and depends on time explicitly; the GPCII for this system has been deduced. Our starting point differs from traditional ones, which are based on the analysis in configuration space. Here, the treatment is based on canonical action and transformation properties of the system in extended phase space. We deduce the GPCII for a system with a singular Nth-order Lagrangian in which the constraint conditions are invariant under the simultaneous variation of canonical variables. It is easy to show the connection between the GPCII and the canonical equations of a constrained Hamiltonian system from this point of view.

Let us consider a system whose Lagrangian $L(t,q_{(0)},\ldots,q_{(N)})$ is singular and depends on time t explicitly. This system is subjected to some inherent phase-space constraints. Let ϕ_α $(\alpha=1,2,\ldots,A)$ denote the primary first-class constraints and θ_m $(m=1,2,\ldots,M)$ denote all the second-class constraints. The canonical equations for this system can be written as [51]

$$\dot q^i_{(s)}=\frac{\partial H_c}{\partial p_i^{(s)}}+\lambda^\alpha\frac{\partial\phi_\alpha}{\partial p_i^{(s)}} -\frac{\partial\theta_m}{\partial p_i^{(s)}}\Delta^{-1}_{mm'}[\{\theta_{m'},H_c\}+\partial\theta_{m'}/\partial t] \ , \tag{4.1a}$$

$$\dot{p}_i^{(s)} = -\frac{\partial H_c}{\partial q^i_{(s)}} - \lambda^\alpha \frac{\partial \phi_\alpha}{\partial q^i_{(s)}} + \frac{\partial \theta_m}{\partial q^i_{(s)}} \Delta^{-1}_{mm'}[\{\theta_{m'}, H_c + \partial\theta_{m'}/\partial t\}] \,, \tag{4.1b}$$

where $\lambda^\alpha(t)$ are Lagrange multipliers and, $\Delta_{mm'} = \{\theta_m, \theta_{m'}\}$. In general, the constraints ϕ_α and θ_m are also explicitly dependent on time.

Let us consider the following transformation in extended phase space:

$$t \to t' = t + \Delta t(\omega) \,,$$
$$q^i_{(s)}(t) \to q'^i_{(s)}(t') = q^i_{(s)}(t) + \Delta q^i_{(s)}(t,\omega) \,, \tag{4.2a}$$
$$p_i^{(s)}(t) \to p_i^{(s)\prime}(t') = p_i^{(s)}(t) + \Delta p_i^{(s)}(t,\omega) \,,$$

where ω is a parameter which satisfies

$$q'^i_{(s)}(t,0) = q^i_{(s)}(t), \quad p_i^{(s)\prime}(t,0) = p_i^{(s)}(t) \,. \tag{4.2b}$$

Under the transformation (4.2), the variation of canonical action (2.12) is given by

$$\Delta I_p = I'_p(\omega)\Delta\omega = \int_{t_1}^{t_2} \left\{ \frac{\delta I_p}{\delta p_i^{(s)}} \delta p_i^{(s)} + \frac{\delta I_p}{\delta q^i_{(s)}} \delta q^i_{(s)} + D[p_i^{(s)} \delta q^i_{(s)} + (p_i^{(s)} q^i_{(s+1)} - H_c)\Delta t] \right\} dt \,. \tag{4.3}$$

Let it be supposed that the simultaneous variation $\delta q^i_{(s)}$ and $\delta p_i^{(s)}$ determined by (4.2a) satisfy the following conditions:

$$\frac{\partial \phi_\alpha}{\partial p_i^{(s)}} \delta p_i^{(s)} + \frac{\partial \phi_\alpha}{\partial q^i_{(s)}} \delta q^i_{(s)} \approx 0 \,, \tag{4.4}$$

$$\frac{\partial \theta_m}{\partial p_i^{(s)}} \delta p_i^{(s)} + \frac{\partial \theta_m}{\partial q^i_{(s)}} \delta q^i_{(s)} \approx 0 \,. \tag{4.5}$$

Using the Lagrange multipliers $\lambda^\alpha(t)$ and $\lambda^m(t)$ and combining the expressions (4.3), (4.4), and (4.5), one obtains

$$\Delta I_p = I'_p(\omega)\Delta\omega = \int_{t_1}^{t_2} \left[\left[\frac{\delta I_p}{\delta p_i^{(s)}} + \lambda^\alpha \frac{\partial \phi_\alpha}{\partial p_i^{(s)}} + \lambda^m \frac{\partial \theta_m}{\partial p_i^{(s)}} \right] \delta p_i^{(s)} + \left[\frac{\delta I_p}{\delta q^i_{(s)}} + \lambda^\alpha \frac{\partial \phi_\alpha}{\partial q^i_{(s)}} + \lambda^m \frac{\partial \theta_m}{\partial q^i_{(s)}} \right] \delta q^i_{(s)} + D(p_i^{(s)} \Delta q^i_{(s)} - H_c \Delta t) \right] dt \,. \tag{4.6}$$

The stationary conditions of constraint yield the following results [51]:

$$\lambda_m \approx -\Delta^{-1}_{mm'}[\{\theta_{m'}, H_c\} + \partial\theta_{m'}/\partial t] \,. \tag{4.7}$$

Substituting Eq. (4.7) into Eq. (4.6) and using the canonical equations of the constrained Hamiltonian system (4.1), one gets

$$\Delta I_p = I'_p(\omega)\Delta\omega = [p_i^{(s)} \Delta q^i_{(s)} - H_c \Delta t]_1^2 \,. \tag{4.8}$$

In the extended phase space spanned by the variables $q^i_{(s)}$, $p_i^{(s)}$ and t, one can choose a closed curve which satisfies the constraint conditions $\phi^\alpha \approx 0$, $\theta_m \approx 0$. This closed curve C_1 lies in some subspace Γ owing to these constraints. Let it be supposed that the equation of this closed curve C_1 is given by

$$t = t_1(\omega), \quad q^i_{(s)} = q^i_{(s)1}(\omega), \quad p_i^{(s)} = p_{i1}^{(s)}(\omega) \,, \tag{4.9}$$

where $\omega = 0$ and $\omega = l$ are some points on C_1. Through any point on C_1 there is a dynamical trajectory of the motion. The dynamical trajectories through every point on C_1 form a tube of trajectories. Choose another closed curve C_2 on this tube that encircles this tube and intersects the generatrix of the tube only once. Suppose the equation of C_2 is given by

$$t = t_2(\omega), \quad q^i_{(s)} = q^i_{(s)2}(\omega), \quad p_i^{(s)} = p_{i2}^{(s)}(\omega) \,. \tag{4.10}$$

Taking the integral for the expression (4.8) in the interval $[0, l]$ along the curve C_1 and C_2, one obtains

$$W = \oint_{C_R} [p_i^{(s)} \Delta q^i_{(s)} - H_c \Delta t] = \mathcal{J} \quad (k = 1,2) \tag{4.11}$$

(where $\mathcal{J}$ denotes invariant). Consequently, for any simple closed curve C lying in the subspace Γ of the extended phase space defined by the constraints, the integral (4.11) is invariant with respect to an arbitrary displacement (with deformation) of the contour C along any tube of dynamical trajectories. W is called the generalized Poincaré-Cartan integral invariant (GPCII) for a system with a singular Nth-order Lagrangian.

It is worthwhile to point out that owing to the stationary condition of constraint the conditions (4.4) and (4.5) imply that the constraints ϕ_α and θ_m are also invariant under the total variation of canonical variables including time:

$$\frac{\partial \phi_\alpha}{\partial t} \Delta t + \frac{\partial \phi_\alpha}{\partial q^i_{(s)}} \Delta q^i_{(s)} + \frac{\partial \phi_\alpha}{\partial p_i^{(s)}} \Delta p_i^{(s)} \approx 0 \,, \tag{4.12}$$

$$\frac{\partial \theta_m}{\partial t} \Delta t + \frac{\partial \theta_m}{\partial q^i_{(s)}} \Delta q^i_{(s)} + \frac{\partial \theta_m}{\partial p_i^{(s)}} \Delta p_i^{(s)} \approx 0 \,. \tag{4.13}$$

In the literature [47,48], the system with a singular first-order Lagrangian was discussed in which the constraint does not depend on time explicitly, which is a special case of the more general consideration given above.

B. GPCII and canonical equations

Let us consider a dynamical system described by an Nth-order Lagrangian. Due to the singularity of the Lagrangian, the motion of the system is restricted to a hypersurface of the phase space, determined by a set of constraints in which the first-class constraints are

$$\Lambda_a(t,q^i_{(s)},p_i^{(s)})\approx 0 \quad (a=1,2,\ldots,A) , \tag{4.14}$$

and the second-class constraints are

$$\theta_m(t,q^i_{(s)},p_i^{(s)})\approx 0 \quad (m=1,2,\ldots,M) . \tag{4.15}$$

Let it be supposed that the dynamical trajectories of the system satisfy a set of differential equations involving functions $\lambda^a(t)$ $(a=1,2,\ldots,A)$,

$$\begin{aligned}\dot q^i_{(s)} &\approx f^i_{(s)}(t,q^i_{(s)},p_i^{(s)},\lambda^a) ,\\ \dot p_i^{(s)} &\approx g_i^{(s)}(t,q^i_{(s)},p_i^{(s)},\lambda^a) .\end{aligned} \tag{4.16}$$

Let H_c be a function with the property

$$\frac{\partial \Lambda_a}{\partial t}+\{\Lambda_a,H_c\}\approx 0 . \tag{4.17}$$

Then, Eqs. (4.16) are the canonical equations if the generalized Poincaré-Cartan integral (4.11) is invariant.

In fact, following Refs. [48,52], introducing an auxiliary variable, and using GPCII (4.11), one can obtain

$$\left[g_i^{(s)}+\frac{\partial H_c}{\partial q^i_{(s)}}\right]\Delta q^i_{(s)}+\left[-f^i_{(s)}+\frac{\partial H_c}{\partial p_i^{(s)}}\right]\Delta p_i^{(s)} +\left[-\frac{dH_c}{dt}+\frac{\partial H_c}{\partial t}\right]\Delta t\approx 0 . \tag{4.18}$$

Due to constraints, $\Delta q^i_{(s)}$ and $\Delta p_i^{(s)}$ are not independent and satisfy

$$\frac{\partial \Lambda_a}{\partial q^i_{(s)}}\delta q^i_{(s)}+\frac{\partial \Lambda_a}{\partial p_i^{(s)}}\delta p_i^{(s)}\approx 0 , \tag{4.19}$$

$$\frac{\partial \theta_m}{\partial q^i_{(s)}}\delta q^i_{(s)}+\frac{\partial \theta_m}{\partial p_i^{(s)}}\delta p_i^{(s)}\approx 0 . \tag{4.20}$$

Introducing the Lagrange multipliers $\lambda^a(t)$ and $\lambda^m(t)$, from (4.17–(4.20) one obtains

$$\dot q^i_{(s)}\approx f^i_{(s)}\approx\frac{\partial H_c}{\partial p_i^{(s)}}+\lambda^a\frac{\partial \Lambda_a}{\partial p_i^{(s)}}+\lambda^m\frac{\partial \theta_m}{\partial p_i^{(s)}} , \tag{4.21a}$$

$$\dot p_i^{(s)}\approx g_{i_{(s)}}\approx-\frac{\partial H_c}{\partial q^i_{(s)}}-\lambda^a\frac{\partial \Lambda_a}{\partial q^i_{(s)}}-\lambda^m\frac{\partial \theta_m}{\partial q^i_{(s)}} , \tag{4.21b}$$

$$\frac{dH_c}{dt}\approx\frac{\partial H_c}{\partial t}+\lambda^a\frac{\partial \Lambda_a}{\partial t}+\lambda^m\frac{\partial \theta_m}{\partial t} . \tag{4.21c}$$

Substituting Eqs. (4.7) into Eqs. (4.21a) and (4.21b), one gets the canonical equations (4.1). Equation (4.21c) is a consequence of Eqs. (4.21a) and (4.21b).

In the above derivation for the GPCII (4.11), only the primary first-class constraints have been taken into account. If the dynamics of the system is generated by an extended Hamiltonian H_E, obtained by adding a linear combination of all secondary first-class constraints to the primary ones, the GPCC (4.11) can also be deduced as long as the simultaneous variations of the canonical variables satisfy the conditions (4.19) and (4.20). Thus we conclude that the necessary and sufficient condition for Eqs. (4.16) to be the canonical equations arising from an extended Hamiltonian is that the GPCII exists for such systems. Use of the GPCII enables one to write the equations of motion for a dynamical system as canonical equations arising from an extended Hamiltonian, and one sees that all the first-class constraints appear in the Hamiltonian. Here one cannot introduce any distinction between primary and secondary ones. That is to say, the existence of the GPCII for a system implies that Dirac's conjecture holds true for this system. The GPCII differs from the usual ones for a regular Lagrangian in that the variation of canonical variables must satisfy Eqs. (4.19) and (4.20). Applying this result to the example given in Sec. III, the constraints ϕ^k $(k=0,1,2,3)$ and the variations $\delta q^i_{(s)},\delta p_i^{(s)}$ must satisfy the conditions (4.19) and (4.20) for the existence of the GPCII. This requirement leads to $\Delta z\approx 0$. Owing to this restriction on canonical variables in the GPCII, one cannot deduce all the canonical equations (arising from H_E) via the GPCII even with the help of the transformation (3.18). This implies that the equivalence between the GPCII and the canonical equations (arising from H_E) is violated. Therefore, the GPCII does not exist under the condition $\Delta z\approx 0$ for an extended Hamiltonian. Thus, one cannot predict that the Dirac's conjecture is valid in that example.

V. APPLICATIONS TO THE GAUGE FIELD THEORIES

To illustrate the use of the results of Sec. II, we give some preliminary applications to the Yang-Mills theory with higher-order Lagrangians.

(i) The Yang-Mills theory with a higher-order Lagrangian was proposed by Baker *et al.* [17]. The canonical formalism for the theory was discussed by Saito *et al.* [13], whose Lagrangian is given by [13]

$$\mathcal{L}_0=-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}-\kappa D^\rho F^{\mu\nu}D_\rho F_{\mu\nu} , \tag{5.1}$$

where

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu] , \tag{5.2}$$

$$A_\mu=T_a A^a_\mu , \tag{5.3}$$

$$D_\mu B=\partial_\mu B+[A_\mu,B] , \tag{5.4}$$

κ is a constant, and T_a are generators of the gauge group. Let $\pi^{(1)}_\mu$ and π_μ denote the canonical momenta conjugate to fields $\dot A^\mu$ and A^μ, respectively. The canonical Hamiltonian and constraints were found by Saito *et al.* to be as follows [13]:

$$H_c=\int d^3x\mathcal{H}_c$$
$$=\int d^3x\left\{\frac{1}{8\kappa}(\pi_i^{(1)})^2+\pi_i^{(1)}(D_iA^0_{(1)}+[A^0,A^i_{(1)}]+[F_{0i},A^0])+\kappa D_0F_{ij}D^0F^{ij}\right.$$
$$\left.+2\kappa D_iF_{0j}D^iF^{0j}+\kappa D_iF_{jk}D^iF^{jk}+\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}+\pi_\mu A^\mu_{(1)}\right\}, \quad (5.5)$$

$$\phi^0=\pi_0^{(1)}\approx 0 , \quad (5.6)$$

$$\phi^1=-\pi_0-D^i\pi_i^{(1)}\approx 0 , \quad (5.7)$$

$$\phi^2=-D^i\pi_i+D^i[A^0,\pi_i^{(1)}]+[F^{0i},\pi_i]\approx 0 , \quad (5.8)$$

$$\phi^3=[\phi^2,A^0]\approx 0 . \quad (5.9)$$

All the constraints are first class.

We have shown that the canonical action for the Lagrangian (5.1) is invariant under the following transformation [53]:

$$\begin{aligned}\delta A^a_\mu&=D^a_{b\mu}\varepsilon^b(x) ,\\ \delta A^a_{(1)\mu}&=\partial_0D^a_{b\mu}\varepsilon^b_{(x)} ,\\ \delta\pi^\mu_a&=f^a_{bc}\pi^\mu_c\varepsilon^b(x)+f^a_{bc}\pi^{(1)\mu}_c\dot{\varepsilon}^b(x) ,\\ \delta\pi^{(1)\mu}_a&=f^a_{bc}\pi^{(1)\mu}_c\varepsilon^b(x) .\end{aligned} \quad (5.10)$$

The extended Hamiltonian for this system is given by $H_E=H_c+H'$, where

$$H'=\int d^3x(\lambda_0^a\phi_a^0+\lambda_1^a\phi_a^1+\lambda_2^a\phi_a^2+\lambda_3^a\phi_a^3) . \quad (5.11)$$

Dirac's conjecture is valid for this system. Using the GNI (2.23), (5.10), and (5.11), one gets

$$f^a_{bc}\pi^{(1)\mu}_c\frac{\delta H'}{\delta\pi^{(1)\mu}_a}+f^a_{bc}\pi^\mu_c\frac{\delta H'}{\delta\pi^\mu_a}-f^a_{bc}\partial_0\left[\pi^{(1)\mu}_c\frac{\delta H'}{\delta\pi^{(1)\mu}_a}\right]+\tilde{D}^a_{b\mu}\left[\frac{\delta H'}{\delta A^a_\mu}\right]-\partial_0\tilde{D}^a_{b\mu}\left[\frac{\delta H'}{\delta A^a_{(1)\mu}}\right]=0 , \quad (5.12)$$

$$\tilde{D}^a_{b\mu}=-\delta^a_b\partial_\mu+f^a_{cb}A^c_\mu , \quad (5.13)$$

along the dynamical trajectory of motion arising from the extended Hamiltonian; substitution of Eq. (5.11) for the above expression (5.12) gives us the relationships for Lagrange multipliers connected with first-class constraints. As is well known, in theories with second-class constraints all Lagrange multipliers connected with second-class constraints are determined by the canonical Hamiltonian and second-class constraints themselves, but in theories with first-class constraints, the Lagrange multipliers connected with first-class constraints are not determined by the stationary conditions of constraint because the Poisson bracket of the first-class constraints equal zero on the constraint hypersurface. If Dirac's conjecture holds true in a problem, along the trajectory of motion the GNI (2.23) may become a trivial equation or sometimes give us more additional relationships for these Lagrange multipliers connected with the same first-class constraints as the above expressions (5.12). Therefore, the application of the GNI in canonical formalism enables us to obtain some additional information about the Dirac constraints and the corresponding Lagrange multipliers.

(ii) In non-Abelian gauge theories, the Lagrangian without ghosts violates unitarity. Using the Faddeev-Popov "trick" through a transformation of the generating functional for the Lagrangian (5.1) in a Lorentz gauge, one can obtain the effective Lagrangian with derivatives of higher order

$$\mathcal{L}_{\text{eff}}=\mathcal{L}_0+B^a\partial_\mu A^{a\mu}+\frac{\alpha}{2}(B^a)^2-\partial_\mu\bar{C}^aD^{a\mu}_bC^b , \quad (5.14)$$

where C^a and $\bar{C}^a$ are ghost fields, B^a are additional even fields, and α is a parameter. One can also derive the effective Lagrangian (5.14) by using the Dirac theory of constrained systems through functional integrals as discussed in Ref. [28].

The effective Lagrangian (5.14) is invariant up to a divergence term under the following BRST (Becchi-Rouet-Stora-Tyutin) transformation:

$$\begin{aligned}\delta A^a_\mu&=D^a_{b\mu}C^b\tau ,\\ \delta B^a&=0 ,\\ \delta C^a&=\tfrac{1}{2}f^a_{be}C^bC^e\tau ,\\ \delta\bar{C}^a&=B^a\tau ,\end{aligned} \quad (5.15)$$

where τ is Grassmann's parameter, and

$$\delta\mathcal{L}_{\text{eff}}=\partial^\mu F_\mu\tau,\quad F_\mu=B^aD^a_{b\mu}C^b . \quad (5.16)$$

A consequence of the BRST invariance of the effective Lagrangian is the presence in the theory of conserved Noether current J^ν,

$$J^\nu=\frac{\partial\mathcal{L}_{\text{eff}}}{\partial A^a_{\mu,\nu}}D^a_{b\mu}C^b+\frac{\partial\mathcal{L}_{\text{eff}}}{\partial A^a_{\mu,\nu\rho}}\partial_\rho(D^a_{b\mu}C^b)-\partial_\rho\left[\frac{\partial\mathcal{L}_{\text{eff}}}{\partial A^a_{\mu,\nu\rho}}\right]D^a_{b\mu}C^b+\frac{\partial\mathcal{L}_{\text{eff}}}{\partial C^a_{,\nu}}\delta C^a+\frac{\partial\mathcal{L}_{\text{eff}}}{\partial\bar{C}^a_{,\nu}}\delta\bar{C}^a-B^aD^a_{b\nu}C^b$$
$$=J^\nu_1-F^{a\nu\mu}D^a_{b\mu}C^b+B^aD^{a\nu}_bC^b-\tfrac{1}{2}\partial^\nu\bar{C}^af^a_{be}C^bC^e , \quad (5.17)$$

where

$$J_1^\nu=\frac{1}{\kappa^2}\{[f_{ea}^m A^{e\rho}\partial_\rho F^{m\mu\nu}+f_{ae}^m f_{fd}^m A_\lambda^e A^{f\lambda}F^{d\mu\nu}-\partial_\rho(\partial^\rho F^{a\mu\nu}+f_{ed}^a A^{e\rho}F^{d\mu\nu})]D_{b\mu}^a C^b+(\partial^\rho F^{a\mu\nu}+f_{ed}^a A^{e\rho}F^{d\mu\nu})\partial_\rho(D_{b\mu}^a C^b)\}\ , \tag{5.18}$$

which implies the conserved BRST charge

$$Q=\int_V J^0 d^3x=\int_V d^3x[J_1^0-F^{a0\mu}D_{b\mu}^a C^b+B^a D_b^{a0}C^b-\tfrac{1}{2}\partial^0\bar{C}^a f_{be}^a C^b C^e]\ . \tag{5.19}$$

Now let us consider only the transformation of the Yang-Mills fields, fixing the ghost fields and additional even fields in the BRST transformation, i.e.,

$$\begin{aligned}&\delta A_\mu^a=D_{b\mu}^a C^b\tau,\quad \delta A_{(1)\mu}^a=\partial_0(D_{b\mu}^a C^b\tau)\ ,\\&\delta\pi_a^\mu=f_{be}^a\pi_e^\mu C^b\tau-f_{be}^a\pi_e^{(1)\mu}\dot{C}^b\tau\ ,\\&\delta\pi_a^{(1)\mu}=f_{be}^a\pi_e^{(1)\mu}C^b\tau\ ,\\&\delta C^a=\delta\bar{C}^a=\delta B^a=0,\quad \delta\pi_a=\delta\bar{\pi}_a=\delta\pi_{B^a}=0\ .\end{aligned} \tag{5.20}$$

Under the transformation (5.20), the effective Lagrangian is variant, and

$$\begin{aligned}\delta\mathcal{L}_{\text{eff}}&=F(\theta)+u_a^\mu\partial_\mu\theta^a+u_a\partial^2\theta^a\\&=F(\theta)+f_{be}^a(\partial^\mu\bar{C}^a C^e-B^a A^{e\mu})\partial_\mu\theta^b+B^a\partial^2\theta^a\ ,\end{aligned} \tag{5.21}$$

where $F(\theta)$ does not contain the derivatives of $\theta^a(\theta^a=C^a\tau)$. According to the weak conservation laws (2.37) for field theories, one obtains the conserved PBRST charge ("P" stands for the word "partial")

$$\begin{aligned}Q^{(P)}&=\int_V d^3x\left\{\left[\left[\delta_b^a\frac{\delta H_1}{\delta A_0^a}-u_b^0-u_b^{0\mu}\partial_\mu+\partial_\mu u^{0\mu}+\pi_a^\mu D_{b\mu}^a\right]C^b+\pi_a^{(1)\mu}\partial_0(D_{b\mu}^a C^b)\right]\right\}\\&=\int_V d^3x[\pi_a^\mu D_{b\mu}^a C^b+\pi_a^{(1)i}\partial_0(D_{bi}^a C^b)+f_{be}^a(\dot{C}^a C^e C^b-B^a A^{0e}C^b)+\dot{B}^a C_a-B^a\dot{C}_a]\ ,\end{aligned} \tag{5.22}$$

with

$$\pi_a^{(1)i}=\frac{1}{\kappa}D_{aj}^b F_b^{ij},\quad \pi_a^0=\frac{1}{\kappa}D_{aj}^b D_{b0}^e F_e^{j0}\ ,$$

$$\pi_a^i=\frac{1}{\kappa}(D_a^{bj}D_{bj}^e F_e^{0i}+D_{aj}^b D_{b0}^e F_e^{ij})-D_{a0}^b\pi_b^{(1)i}+F_a^{0i}\ . \tag{5.23}$$

It is easy to check that this conserved PBRST charge $Q^{(P)}$ differs significantly from the conserved BRST charge (5.19). Similarly, if we fix the gauge fields A_μ^a and change only the ghost fields, the weak conservation laws (2.37) imply a trivial identity.

We have shown that for certain cases the GNI (or strong conservation laws) in canonical formalism may be converted into the weak conserved charge along the trajectory of motion, even if the Lagrangian is not invariant under the specific local transformation. This algorithm, which deduces the conservation laws, differs from the classical first Noether theorem, where the invariance under a finite continuous group implies that there are some conservation laws.

VI. CONCLUSIONS

The GFNT and GPCII in canonical formalism for singular Nth-order Lagrangians and GNI in canonical formalism for variant systems have been deduced, and the strong and weak conservation laws are also obtained. Applying the GNI, we have established that for certain variant systems there is also a Dirac constraint. In certain cases the GNI may be converted into the conservation laws along the trajectory of motion, even if the Lagrangian of the system is not invariant under the specific local transformation. Applying the present theory to the gauge theories for a system with a singular second-order Lagrangian, a new conserved PBRST charge is obtained which differs from the BRST charge. Applying the GNI to singular second-order classical Yang-Mills theory, we obtain some relationships for Lagrange multipliers connected with first-class constraints. Deriving the GPCII with the aid of canonical action, we have carefully distinguished the conditions imposed by constraints on the simultaneous variations from those imposed on the total variations. In deriving the Poincaré-Cartan integral invariant for nonholonomic systems, we have emphasized the same distinctions of these conditions [54]. Using canonical formalism, it is easy to show that the GPCII is equivalent to canonical equations for a system with a singular Nth-order Lagrangian, which depends on time explicitly.

Based on the symmetries of the constrained Hamiltonian system, using the GFNT and GPCII we have discussed the validity of Dirac's conjecture. A counterexample is given in which there is no linearization of constraints for a system with a singular second-order Lagrangian to show that Dirac's conjecture fails. A brief discussion about the generator of gauge transformation for a constrained Hamiltonian system is given. The time evolution of the coefficients of secondary constraints connected with the coefficients of first-class constraints in the generator of gauge transformation has been obtained clearly. Some confusion in the literature has been elucidated.

The extension of the present theory to field theories for a system with a singular Nth-order Lagrangian is straightforward. The extension of theory to supersymmetry needs further discussion.

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[1] M. Ostrogradsky, Mem. Acad. St. Pet. VI **4**, 385 (1850); E. T. Whitaker, *Treatise on the Analytical Dynamics of Particles and Rigid Bodies,* 4th ed. (Cambridge University Press, Cambridge, 1959), p. 226.

[2] M. De Leon and P. R. Rodrigues, *Generalized Classical Mechanics and Field Theory* (North-Holland, Amsterdam, 1985).

[3] L. D. Faddeev and A. A. Slavnov, *Gauge Field, Introduction to Quantum Theory* (Benjamin-Cummings, New York, 1980).

[4] E. S. Fradkin and A. A. Tseyllin, Nucl. Phys. **B201**, 469 (1982).

[5] D. G. Boulware and S. Deser, Phys. Rev. D **25**, 2556 (1985).

[6] G. W. Kentwell, J. Math. Phys. **29**, 46 (1988).

[7] K. S. Stelle, Phys. Rev. D **16**, 953 (1977).

[8] P. West, Nucl. Phys. **B268**, 113 (1986).

[9] C. Battle, J. Gomis, M. Pons, and N. Roman, Lett. Math. Phys. **13**, 17 (1987).

[10] V. V. Nesterenko and S. H. Nguyen, Int. J. Mod. Phys. A **3**, 2315 (1988).

[11] C. P. A. Galvão and N. A. Lemos, J. Math. Phys. **29**, 1588 (1988).

[12] V. V. Nesterenko, J. Phys. A **22**, 1673 (1989).

[13] Y. Saito, R. Sugano, T. Ohta, and T. Kimura, J. Math. Phys. **30**, 1122 (1989).

[14] P. W. Hebda, J. Math. Phys. **31**, 2116 (1990).

[15] R. A. Moore and T. C. Scott, Phys. Rev. A **44**, 1477 (1991).

[16] A. L. Alekseev, B. A. Arbuzov, and V. A. Baikov, Teor. Mat. Fiz. (USSR) **52**, 187 (1982) [Theor. Math. Phys. **52**, 739 (1982)].

[17] M. Baker, L. Carson, J. S. Ball, and F. Zachariasen, Nucl. Phys. **B229**, 456 (1983).

[18] S. W. de Leeuw, J. W. Perram, and H. G. Peterson, J. Stat. Phys. **61**, 1203 (1990).

[19] J. Maharana, Phys. Lett. **1288**, 411 (1983).

[20] D. S. Djukic, Arch. Mech. **26**, 243 (1974).

[21] C. Ferrario and A. Passerini, J. Phys. A **23**, 506 (1990).

[22] L. Lusanna, Riv. Nuovo Cimento, **14**, 1 (1991).

[23] P. A. M. Dirac, *Lectures on Quantum Mechanics,* Monograph Series No. 2 (Belfer Graduate School of Science, Yeshiva University, New York, 1964).

[24] Z-P. Li, and X. Li, Int. J. Theor. Phys. **30**, 225 (1991).

[25] Z-P. Li, Chin. Sci. Bull. **36**, 940 (1991).

[26] Z-P. Li, J. Phys. A **24**, 4261 (1991).

[27] In Ref. [14] an example was given to argue the fact that there may be other primary constraints which are not a direct consequence of the degenerate of an extended Hessian matrix. However, the Lagrangian in that example is not admissible because this higher-order Lagrangian is trivial, equivalent to the first-order ones, and the Euler-Lagrange equations deriving from that Lagrangian imply that the variables are separate. The equations of motion for a system with a singular *N*th-order Lagrangian deriving from a total Hamiltonian are equivalent to the corresponding Lagrange equation. Therefore, the canonical equations for a system with a singular *N*th-order Lagrangian are assumed naturally to be Eqs. (2.7).

[28] D. M. Gitman and I. V. Tyutin, *Quantization of Fields with Constraints* (Springer-Verlag, Berlin, 1990).

[29] Z-P. Li, Int. J. Theor. Phys. **26**, 853 (1987); High Energy Phys. Nucl. Phys. **6**, 782 (1988).

[30] P. G. Bergmann, Phys. Rev. **83**, 1018 (1951).

[31] M. Henneaux, C. Teiteboim, and J. Zanell, Nucl. Phys. B **332**, 169 (1990).

[32] M. E. V. Costa, H. O. Girotti, and T. J. M. Simões, Phys. Rev. D **32**, 405 (1985).

[33] A. Cabo, J. Phys. A **19**, 629 (1986).

[34] R. Sugano and H. Kamo, Prog. Theor. Phys. **67**, 1966 (1982).

[35] D. M. Appleby, J. Phys. A **15**, 1191 (1982).

[36] L. Castellani, Ann. Phys. (NY) **143**, 357 (1982).

[37] X. Grácia and J. M. Pons, Ann. Phys. (NY) **1987**, 355 (1988).

[38] M. J. Gotay, J. Phys. A **16**, L141 (1983).

[39] G. R. Allcock, Philos. Trans. R. Soc. London Ser. A **297**, 33 (1975).

[40] R. Cawley, Phys. Rev. Lett. **42**, 413 (1979); Phys. Rev. D **21**, 2988 (1980).

[41] A. Frenkle, Phys. Rev. D **21**, 2986 (1980).

[42] Z. Qi, Int. J. Theor. Phys. **29**, 1309 (1990).

[43] C. P. A. Galvão and J. B. T. Boechat, J. Math. Phys. **31**, 448 (1990).

[44] Z-P. Li, Chin. Phys. Lett. **10**, 68 (1993).

[45] D. Anderson, J. Phys. A **6**, 299 (1973).

[46] Z-P. Li, Europhys. Lett. **21**, 141 (1993).

[47] F. Benavent and J. Gomis, Ann. Phys. (NY) **118**, 476 (1979).

[48] D. Dominici and J. Gomis, J. Math. Phys. **21**, 2124 (1980).

[49] D. Dominici and J. Gomis, J. Math. Phys. **23**, 256 (1982).

[50] R. Sugano, Prog. Theor. Phys. **68**, 1377 (1982).

[51] K. Sundermeyer, *Constrained Dynamics with Applications to Yang-Mills Theory. General Relativity, Classical Spin, Dual String Model* (Springer-Verlag, Berlin, 1982); Z. P. Li, J. Beijing Polytech. Univ. **19**, 1 (1993).

[52] F. Gantmacher, *Lectures in Analytical Mechanics* (Mir, Moscow, 1970).

[53] Z. P. Li and Y. C. Xie, Commun. Theor. Phys. **21**, 247 (1994).

[54] Z-P. Li and X. Li, Int. J. Theor. Phys. **29**, 765 (1990).