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Scaling of Lyapunov exponents at nonsmooth bifurcations

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In nonsmooth maps intermittency can arise when a periodic orbit loses stability by crossing a set where the mapping is nondifferentiable. Motivated by the impact oscillator, which gives rise to a discontinuous mapping with infinite stretching, we consider classes of continuous but nondifferentiable maps in one and two dimensions. We show that the largest Lyapunov exponent λ has a discontinuous jump at the bifurcation and the scaling when the bifurcation parameter ϵ is $\lambda \sim 1/|\ln \epsilon|$. For a similar class of discontinuous maps there can be no immediate transition to intermittent chaos.

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I. INTRODUCTION

The theory of smooth dynamical systems has grown explosively in recent years, but less attention has been given to dynamical systems which do not satisfy the usual smoothness assumptions. Such systems arise very naturally in many physical applications, for example, impacting systems, relaxation oscillators, and electronics. In this note we analyze a form of intermittency, identified in [1,2], which occurs when a periodic orbit, associated with a map, suddenly loses its stability by crossing a set where the map is nondifferentiable. We shall call such an event a nonsmooth bifurcation.

Intermittency is characterized by long regular excursions, known as the laminar phases, which are interrupted by short chaotic bursts and in smooth maps it has been observed and investigated by many authors. A simple mechanism which often produces intermittent behavior, identified by Pomeau and Manneville [3], is the loss of stability of a periodic orbit. Suppose that the mapping has a parameter ϵ such that when ϵ increases through 0 a periodic orbit loses its stability. For smooth mappings this can only happen when an eigenvalue (or a complex conjugate pair) crosses the unit circle. Then for small positive ϵ intermittency arises as follows. A nearby trajectory will spend a long time in the neighborhood of the periodic orbit, first moving in along the directions of the stable manifolds before being (slowly) forced away along the unstable direction. This corresponds to the laminar phase. Then in many instances, depending upon the global properties of the map, the trajectory will be reinjected into the neighborhood of the periodic orbit via a typically short, irregular motion, which is the chaotic burst. Pomeau and Manneville considered each of the ways in which an eigenvalue can cross the unit circle and showed that the intermittent behavior described above is chaotic with a positive Lyapunov exponent which scales like $\epsilon^{\frac{1}{2}}$.

However, for nonsmooth maps periodic orbits can lose stability in another way. We suppose that the map is nondifferentiable at a set of points that we shall call the discontinuity set S (later we shall distinguish between the cases where the mapping is discontinuous or merely nondifferentiable at these points). Now a periodic orbit can suddenly lose stability by colliding with the discontinuity set. This possibility was first examined in [1], where it was shown that the average length of the laminar phase scales like $1/|\ln \epsilon|$ (where again ϵ denotes the distance from the bifurcation point in parameter space).

The present study was motivated by the authors' work on the impact oscillator system, which naturally gives rise to a two-dimensional nonsmooth mapping. In [4] it was shown that intermittency of the type described above can occur. In Sec. II we describe the impact oscillator and this map and construct a simplified map which can be analyzed in detail.

In Sec. III, rather than plunge straight into the analysis of the two-dimensional case we will consider a family of closely related one-dimensional maps, first introduced in [5], and derive the scaling for the Lyapunov exponents, which, as expected from the results of [1], is $\lambda \sim 1/|\ln \epsilon|$. We also show that there are important qualitative differences between mappings which are discontinuous and those that are continuous but nondifferentiable.

In Sec. IV having clarified the important issues with the one-dimensional case we perform a similar analysis for the simplified two-dimensional map and show that the same scaling exists. Numerical calculations for the actual impact oscillator map provide good evidence for the generality of the results.

II. THE IMPACT OSCILLATOR

The impact oscillator is a mass on a linear spring, moving in one dimension subject to a sinusoidal forcing, which impacts against a rigid obstacle referred to as the wall (Fig. 1). The impact itself is modeled as an instantaneous process with coefficient of restitution r, $0 \le r \le 1$. With suitable rescaling we can write the equations of motion as

$$\begin{aligned} \ddot{x} + x &= \cos \omega t\,, \qquad x < \sigma \\ \dot{x} &\mapsto -r\dot{x}\,, \qquad x = \sigma \end{aligned} \tag{1}$$

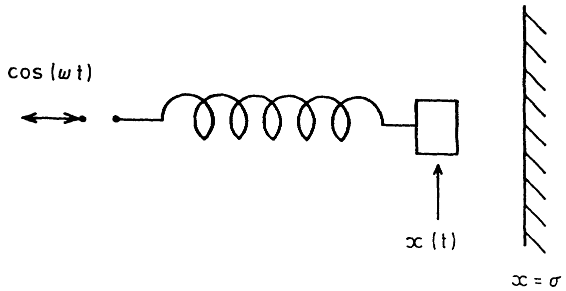

FIG. 1. The mechanical impact oscillator.

where ω is the forcing frequency (the natural frequency of spring is 1) and σ, the clearance, is the position of the wall relative to the equilibrium position of the mass. Thus the system has three parameters r, σ, and ω. The most effective way to study this system is via the impact map P, which maps one impact onto the next one. The ith impact can be described by (double) the kinetic energy $z_i = v_i^2$ of the mass immediately before the impact and the phase of the forcing cycle $\phi_i = \omega t_i \bmod 2\pi$, where t_i is the time of the ith impact. Therefore, defining $\Sigma = 2\pi S^1 \times \mathbb{R}^+$,

$$P:\ \Sigma \to \Sigma\ ,\quad P:\ (\phi_i, z_i) \mapsto (\phi_{i+1}, z_{i+1})\ .$$

The precise definition (see [4,6]) of P is more complicated since it is possible for the mass to impact infinitely often in finite time and then stick to the wall before moving away again, but these details are unimportant here.

The motion between impacts is simply that of the harmonic oscillator and implicit differentiation of the equations of motion gives the following (implicit) form for the Jacobian of P:

$$DP(\phi_0, z_0) = \begin{pmatrix} (N_0 S - r\sqrt{z_0}C)/\sqrt{z_1} & rS/2\sqrt{z_0 z_1} \\ 2(N_0 N_1 - r\sqrt{z_0 z_1})S - 2(N_0\sqrt{z_1} + rN_1\sqrt{z_0})C & r(N_1 S - \sqrt{z_1}C)/\sqrt{z_0} \end{pmatrix}\ , \tag{2}$$

where $S = \sin(t_1 - t_0)$, $C = \cos(t_1 - t_0)$, and $N_i = \cos(\omega t_i) - \sigma$, $i = 0, 1$. The N_i are the accelerations of the mass just before the respective impacts. The determinant of the Jacobian is simply r^2 and explains our (at first sight) rather strange choice of z as one of the variables.

The map P is smooth everywhere except at the preimage of the line $v = 0$ due to a phenomenon known as grazing, which is shown in Fig. 2. On trajectory B the mass approaches the wall, touches it, and is then pulled away again by the spring. Nearby trajectories will either (just) hit the wall (trajectory C) and be deflected by it or miss the wall and go on to impact at a later time (trajectory A). Surprisingly, low velocity impacts, such as those which occurred on trajectory C, strongly distort the phase space and are an important source of unpredictability and chaos in the impact oscillator. Zero velocity impacts are known as grazes and the discontinuity set S, which is the preimage of the line $v = 0$, is defined by

$$S = \{(\phi, v) : P(\phi, v) = (\psi, 0)\}\ .$$

Similarly we define the set W as the image of the line $v = 0$.

As can be seen from (2), as $v_{i+1} \to 0$ the elements of the Jacobian become arbitrarily large. To illustrate the nature of P close to the discontinuity set S we imagine a line segment I of initial conditions on Σ, which transversely crosses S (Fig. 3). Let I and S meet at point B and call the resulting two sections of I, I^+ and I^-. The particle motions corresponding to the points A, B, and C are those shown in Fig. 2. As we move along I^+ from A to B the image curve $P(I^+)$ is traced out and moves towards the line W and meets it transversely. Suddenly, at

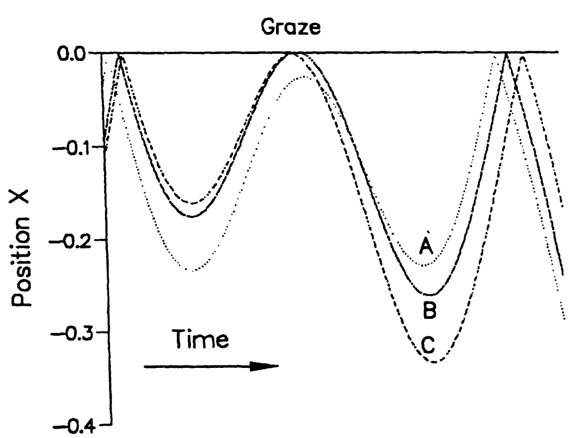

FIG. 2. Trajectories in the neighborhood of a graze.

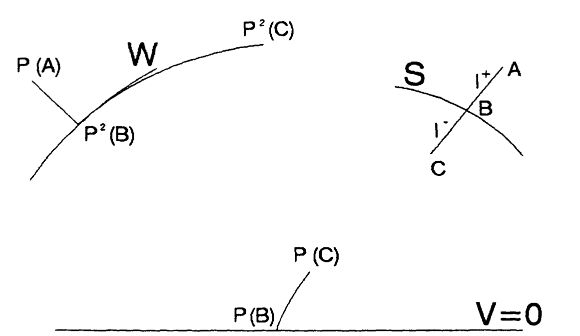

FIG. 3. The dynamics close to S in the impact oscillator.

B a graze occurs at an intermediate point of the trajectory and so $P(B)$ lies on the line $v = 0$. As we continue from B to C the next impact is now a low velocity impact. So the curve $P(I^-)$ grows out of the line $v = 0$ and the line I would appear to be split in two. However, the second iterate of I^- rejoins $P(I^+)$ meeting it at $P^2(B)$ and Whiston [7] showed that $P_I^2(I^-)$ meets W tangentially. The side of S that does not map directly to low velocity impacts is called the nonimpact side and the side that does is the impact side. The line I^- is locally stretched by P^2 by a factor of $O(\epsilon^{-\frac{1}{2}})$, where ϵ is the distance from S. Together the cutting and stretching have important implications for the dynamics. Because grazes tend to collapse trajectories onto W, W and its iterates strongly influence the overall dynamics [7]. Although the map P is technically discontinuous at S, this is not really the case since by replacing P with P^2 on the impacting side of S (and therefore ignoring the extra low velocity impact) the impact map in the neighborhood of S can be treated as nondifferentiable but continuous.

An example of a nonsmooth bifurcation, which we shall call a *grazing bifurcation*, occurs when a stable periodic orbit of P, on the nonimpact side of S, crosses S as a parameter is varied. Suddenly, the trajectory experiences an additional, low velocity impact which introduces a large degree of stretching and therefore has a sudden destabilizing influence. Following this bifurcation it is possible to observe intermittent behavior [8,4], which consists of a long laminar phase as the trajectory moves in towards the previous location of the periodic orbit and the set S from the nonimpacting side, followed by a small number of low velocity impacts (usually just one). It should be emphasized that after the bifurcation the stable periodic orbit no longer exists, but part of the stable manifold on the nonimpacting side does still exist and this accounts for the laminar phase. The stretching associated with low velocity impacts provides the reinjection mechanism which starts the trajectory off on the next laminar phase. A grazing bifurcation of a fixed point occurs at the parameter values $r = 0.8$, $\omega = 2.0$, and $\sigma = \sigma^* = -0.3312\ldots$ (see [4]) and a nearby intermittent trajectory is shown in Fig. 4.

This extremely short interruption of the laminar phase by a single low velocity impact is untypical of intermittent behavior since a chaotic burst usually involves a seemingly random excursion around different parts of the phase space. However, the existence of scalings for the average laminar length and largest Lyapunov exponent will justify our use of the word "intermittent."

The map P is difficult to analyze, because it is implicit and the location of S cannot be computed analytically. For this reason we introduce the following mapping which "straightens" out the sets S and W and models the dynamics close to the singularity set,

$$F_\epsilon(x,y) = \begin{cases} \left.\begin{aligned} x_1 &= Ay_0 \\ y_1 &= By_0 + x_0 + \epsilon \end{aligned}\right\} y_0 \leq 0 \\ \left.\begin{aligned} x_1 &= Ay_0 \\ y_1 &= B'\sqrt{y_0} + x_0 + \epsilon \end{aligned}\right\} y_0 l > 0. \end{cases} \tag{3}$$

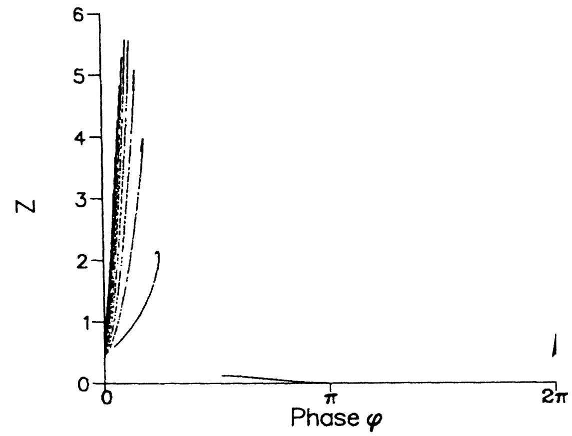

FIG. 4. The intermittent attractor for the impact oscillator with parameter values $r = 0.8$, $\omega = 2.0$, and $\sigma - \sigma^* = 0.01$.

[In [8] a similar mapping was constructed, but this was more complicated than (3) and introduced additional discontinuities.] For $\epsilon \leq 0$ there is an attracting fixed point X, which collides with the x axis at (0,0) when $\epsilon = 0$. If μ_1 and μ_2 are the two eigenvalues of X, then $A = -\mu_1\mu_2$ and $B = \mu_1 + \mu_2$. In order to observe intermittency for $\epsilon > 0$ we require $0 < \mu_2 \leq \mu_1 < 1$. This condition ensures that trajectories in region I attracted by X can approach it without immediately entering region II. The parameter $B' < 0$ reinjects the trajectory back into region I. The Jacobian of the mapping has constant determinant $-A = \mu_1\mu_2$, which corresponds to r^2 in the impact oscillator.

The line $y = 0$ gets mapped to the line $x = 0$. These two lines play the roles of S and W, respectively. Region I, $y \leq 0$, corresponds to the nonimpacting side of S in the impact oscillator and the mapping here is linear. Region II, $y > 0$, corresponds to the impacting side of S and will model the stretching and contracting close to $y = 0$ by a square-root-type singularity. This map describes the behavior of P close to S as shown in Fig. 3, but ignores the extra low velocity impact on the impacting side.

For $\epsilon < 0$ the position of X is given by

$$(A\epsilon/(1 - A - B),\ \epsilon/(1 - A - B))\ ,$$

from which we see that ϵ is proportional to the distance of the fixed point from the discontinuity set $y = 0$. For $\epsilon > 0$ this fixed point no longer exists for the map (3) but continues to exist for the related map where the linear part defined on region I is extended to the whole plane. Therefore, for more complicated systems, such as the impact oscillator where the position of the singularity set is also a function of the parameters, it is natural to define the distance of the fixed point of this related map from S as our bifurcation parameter. In fact any parameter that causes the periodic orbit to move across S transversely will suffice. An intermittent attractor for the map (3) is shown in Fig. 5.

Before examining the two-dimensional case it will be helpful to first analyze a family of one-dimensional maps with similar properties.

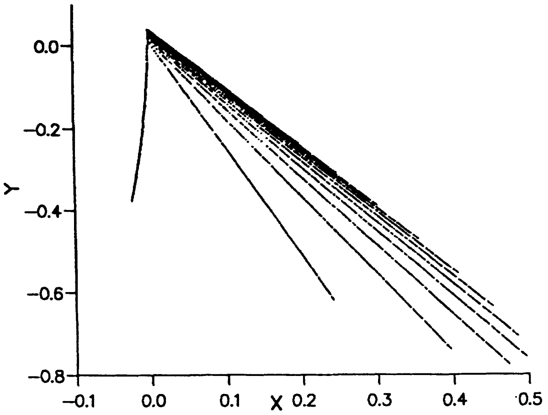

FIG. 5. The intermittent attractor for (3) with parameter values $A = -0.64$, $B = 1.6$, $B' = -2.0$, and $\epsilon = 0.01$.

III. THE ONE-DIMENSIONAL CASE

Nusse, Ott, and Yorke [5] introduced the following family of one-dimensional maps:

$$F_\epsilon(x) = \begin{cases} \alpha x + \epsilon & \text{if } x \leq 0 \\ \beta x^z + \epsilon & \text{if } x > 0, \end{cases} \tag{4}$$

where $0 < \alpha < 1$, $\beta < -1$, $0 < z < 1$, and ϵ is the bifurcation parameter which belongs to some interval I of 0. An example of such a mapping is shown in Fig. 6. This family of maps is smooth everywhere except at $x = 0$, where it is only continuous. As in the impact oscillator, the derivative at the nondifferentiable point becomes infinite on one side (corresponding to the impacting side of S). For $\epsilon < 0$, $F_\epsilon(x)$ has an attracting fixed point (with negative Lyapunov exponent $\lambda = \ln \alpha$). The parameter α corresponds loosely to the coefficient of restitution r in the impact oscillator. This fixed point collides with the point $x = 0$ when $\epsilon = 0$.

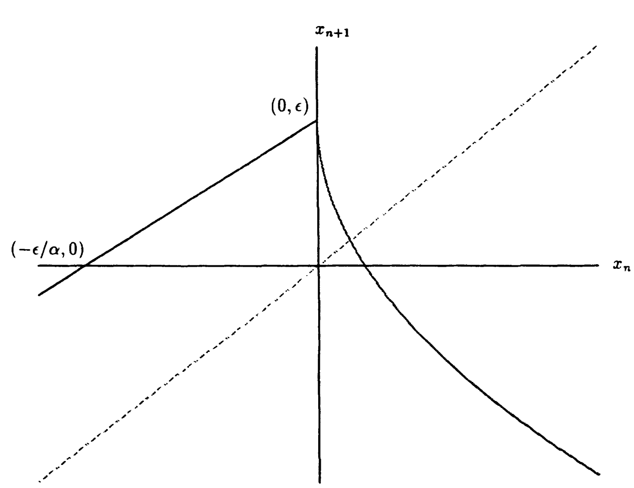

FIG. 6. The mapping of the interval defined by (4).

The laminar phase corresponds to motion in $x < 0$ and this is interrupted when the motion reaches the region $x > 0$ and the square-root singularity and is then reinjected back into the region $x < 0$. From Fig. 6 it is clear that, for $\epsilon > 0$, a trajectory started in $x < 0$ can only reach the interval $(0, \epsilon]$ in $x > 0$. In what follows we assume that the chaotic burst consists of just one iteration in the region $x > 0$ and that the distribution over the interval $(0, \epsilon]$ is uniform (these assumptions are seen to hold very well in numerical experiments). For all the mappings considered here, where the stretching on one side of the discontinuity set becomes infinite, it is shown in the Appendix that the errors in making these assumptions are of small order.

The case $z = \frac{1}{2}$ corresponds most closely to the impact oscillator, being a very natural one-dimensional analog of the family of maps (3), but we shall obtain results for all $0 < z < 1$. We start by estimating the average length of the laminar phase for $0 < \epsilon \ll 1$ (a more detailed argument is given in [1]). After an excursion into $x > 0$ the trajectory will be reinjected into $x < 0$ with a value of x that is typically $O(\epsilon^z)$. At each iteration, the trajectory moves closer to $x = 0$ by a factor of approximately α and, from Fig. 6, the laminar phase ends when the trajectory returns to within ϵ/α of the discontinuity at $x = 0$. So if we denote the average number of iterates in the laminar region by L, then for $\epsilon \ll 1$,

$$K\alpha^L \epsilon^z \approx \epsilon \ , \tag{5}$$

which gives

$$L \approx \frac{(1-z)\ln \epsilon}{\ln \alpha} + C \ , \tag{6}$$

where K and C are $O(1)$ constants, which reflect the mean position of the reinjection into $x < 0$. This scaling of the average laminar length is different to the power-law scaling usually found in smooth systems (see [3]).

Now we can calculate the Lyapunov exponent λ, which for a one-dimensional map is defined by

$$\lambda = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \ln |(F_\epsilon^i)'(x_0)| \ . \tag{7}$$

This has contributions from both parts of the map (4) that have to be evaluated separately.

Let p be the proportion of points that land in $x > 0$. Then the contribution to (7) from $x \leq 0$ is

$$(1-p)\ln \alpha \ . \tag{8}$$

But $p = 1/L$, so from (6) this is

$$\left(1 - \frac{\ln \alpha}{(1-z)\ln \epsilon + C \ln \alpha}\right) \ln \alpha \ . \tag{9}$$

Then from (7) and the assumptions that we made about the distribution on $(0, \epsilon]$ the contribution to λ from $x > 0$ is

$$\frac{\ln \alpha}{(1-z)\ln \epsilon + C \ln \alpha} \frac{1}{\epsilon} \int_0^\epsilon \ln |\beta z x^{z-1}| dx \ , \tag{10}$$

which after some elementary integration becomes

$$\frac{\ln\alpha}{(1-z)\ln\epsilon + C\ln\alpha}\ln|\beta z| - \frac{(z-1)\ln\alpha}{(1-z)\ln\epsilon + C\ln\alpha}[\ln\epsilon - 1]$$

and after rearranging and adding to (9) gives (to leading order in ϵ)

$$\lambda = \frac{1}{\ln\epsilon}\left[\ln\alpha\left(1 - \frac{\ln\alpha}{1-z} + \frac{\ln|z\beta|}{1-z} + \frac{C\ln\alpha}{1-z}\right)\right] . \quad (11)$$

Table I shows the results for the system with parameters $z = \frac{1}{2}$, $\alpha = 0.8$, and $\beta = -2$. There is good agreement between the above theory and the numerical results both for the scaling of the average laminar length and the Lyapunov exponent.

There are two things to note about (11). First, the scaling as $\epsilon \to 0^+$ is generically $\lambda \sim 1/|\ln\epsilon|$, which differs from the scaling observed close to smooth bifurcations. Pomeau and Manneville [3] showed that λ normally scales like $\epsilon^{\frac{1}{2}}$ when a fixed point of a smooth system loses stability, although other authors [9–11] have identified different power-law exponents, such as $\frac{2}{3}$ or 1 for particular systems. Thus, for $0 < z < 1$ we have a family of mappings with a different scaling for λ.

The second point is that as ϵ increases through 0, λ jumps suddenly from $\ln\alpha$ to 0. In general, for this kind of nonsmooth intermittency, λ will exhibit a discontinuous jump, but it will not necessarily jump to 0. For example, if we introduce a new parameter γ into (4) and consider the family of discontinuous maps

$$F_\epsilon(x) = \begin{cases} \alpha x + \epsilon & \text{if } x \leq 0 \\ \beta x^z + \gamma & \text{if } x > 0, \end{cases} \quad (12)$$

where γ is negative and $O(1)$, then the reinjection into $x < 0$ occurs at a value of x which is $O(1)$ and this changes the scaling behavior of L. Equation (5) now becomes

$$K\alpha^L = \epsilon \quad (13)$$

for all z. Proceeding as before we obtain the following expression for the Lyapunov exponent:

$$\lambda = z\ln\alpha + \frac{1}{\ln\epsilon}\{\ln\alpha[(1-z) - \ln\alpha + \ln|z\beta| + C\ln\alpha]\} , \quad (14)$$

where once again C is $O(1)$ and now includes a contribution from γ.

The exponent λ still increases like $1/|\ln\epsilon|$ for $\epsilon > 0$, but now it starts from a negative value rather than 0. Therefore there can be no immediate bifurcation to a nearby intermittently chaotic attractor. This is confirmed by numerical experiments where the system settles onto a stable intermittent periodic orbit.

TABLE I. Results for small ϵ for 2×10^6 iterates at parameter values $z = \frac{1}{2}$, $\alpha = 0.8$, and $\beta = -2$. As predicted, $\lambda \sim 1/|\ln(\epsilon)|$.

ϵ	λ	L	$\lambda\|\ln\epsilon\|$
10^{-2}	0.231	6.747	1.063
10^{-3}	0.141	11.339	0.974
10^{-4}	0.102	16.182	0.940
10^{-5}	0.079	21.269	0.910
10^{-6}	0.067	26.267	0.926
10^{-7}	0.058	31.126	0.935
10^{-8}	0.050	36.382	0.921

IV. THE TWO-DIMENSIONAL CASE

We now analyze the scaling close to the grazing bifurcation in two dimensions, using the map (3). The existence of the Lyapunov exponents for mappings such as (3) is guaranteed by Oseledec's multiplicative ergodic theorem. Obtaining analytic expressions for the Lyapunov exponents of higher dimensional systems is in general an extremely difficult problem because it depends upon being able to compute the eigenvalues of complicated, noncommutative, matrices. Very few results exist and even then only for perturbations of sets of commutative matrices. However, for the family of maps (3) it is possible to obtain some analytical estimates.

At the end of the ith laminar phase the trajectory will land in region II with a y value that we denote by y_i. The y_i are $O(\epsilon)$ and we make the same assumptions as before, namely, that the y_i are uniformly distributed over $(0,\epsilon]$ with the trajectory immediately returning to region I. In addition we assume that the fluctuations in the laminar lengths about the mean L are unimportant.

We start by defining the Jacobians in regions I and II

$$J_{\rm I} = \begin{pmatrix} 0 & -\mu_1\mu_2 \\ 1 & \mu_1+\mu_2 \end{pmatrix}, \quad J_{\rm II} = \begin{pmatrix} 0 & -\mu_1\mu_2 \\ 1 & \frac{1}{2}B' y_i^{-\frac{1}{2}} \end{pmatrix} , \quad (15)$$

where we have used the variables μ_1 and μ_2 instead of A and B. Once again, we can estimate the average length of the laminar phase L, which by a similar argument to the one-dimensional case is given by

$$K{\mu_1}^L \sim \sqrt{\epsilon} . \quad (16)$$

It is easily shown by induction that

$$J_{\rm I}^L = \frac{{\mu_1}^L}{\mu_1-\mu_2}\begin{pmatrix} -\mu_2\left[1-\left(\frac{\mu_2}{\mu_1}\right)^{L-1}\right] & -\mu_1\mu_2\left[1-\left(\frac{\mu_2}{\mu_1}\right)^{L}\right] \\ 1-\left(\frac{\mu_2}{\mu_1}\right)^{L} & \mu_1\left[1-\left(\frac{\mu_2}{\mu_1}\right)^{L+1}\right] \end{pmatrix} \quad (17)$$

TABLE II. Results for small ϵ for 2×10^6 iterates at parameter values $A = -0.64$, $B = 1.6$, and $B' = -2.0$.

ϵ	λ_1	L	$\lambda_1 \vert \ln \epsilon \vert$
10^{-2}	0.159	13.30	0.732
10^{-3}	0.098	20.23	0.677
10^{-4}	0.072	26.66	0.663
10^{-5}	0.057	32.82	0.656
10^{-6}	0.047	38.80	0.650
10^{-7}	0.041	44.58	0.660
10^{-8}	0.037	49.99	0.681

from which it follows that

$$J_{\rm II}J_{\rm I}^L = \frac{\mu_1^L}{\mu_1-\mu_2}\left\{\frac{1}{2}B'\sqrt{y_i}\left[1-\left(\frac{\mu_2}{\mu_1}\right)^L\right]\right.$$

$$\left.-\mu_2\left[1-\left(\frac{\mu_2}{\mu_1}\right)^{L-1}\right]\right\}$$

$$\times\begin{pmatrix} O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ 1 & O(1)\end{pmatrix} . \qquad (18)$$

Matrices of the form

$$\begin{pmatrix} O(\sqrt{\epsilon}) & O(\sqrt{\epsilon}) \\ 1 & O(1)\end{pmatrix} , \qquad (19)$$

when multiplied together, can be renormalized to yield another matrix of the same form. By keeping track of the renormalizations and neglecting terms of small order we find the following expression for the largest Lyapunov exponent λ_1:

$$\lambda_1 = \lim_{n\to\infty}\frac{1}{NL}\sum_{i=1}^{N}\left(\ln\left\{\frac{1}{2}B'\left[1-\left(\frac{\mu_2}{\mu_1}\right)^L\right]\right.\right.$$

$$\left.\left.\times\frac{{\mu_1}^L}{\sqrt{y_i}(\mu_2-\mu_1)}\right\}\right) . \qquad (20)$$

Using Eq. (16) and some more algebra we obtain the result that, assuming the attracting set is chaotic and the y_i are uniformly distributed over the interval $(0,\epsilon]$, then $\lambda_1 \sim 1/|\ln\epsilon|$.

Table II shows numerical calculations of L and λ_1 for the case shown in Fig. 5 and there is very good confirmation of these scalings.

A. The impact oscillator

Finally we examine numerically the scaling behavior of the impact oscillator close to an intermittent grazing bifurcation. In order to generate trajectories of the impact oscillator it is necessary to calculate the time of the next impact using a root-finding algorithm and this is both quick and accurate since no numerical integration is needed (although the possibility of missing extremely low velocity impacts can never be excluded). The convergence of the Lyapunov exponents, although slower than for smooth systems, is quite rapid and reliable estimates can be obtained after just a few thousand iterates.

TABLE III. Results for small ϵ for 5×10^4 iterates of the impact oscillator with parameter values $r = 0.8$, $\omega = 2.0$, and $\sigma^* = -0.3312\ldots$. Again $\lambda_1 \sim 1/|\ln\epsilon|$.

$\epsilon = \sigma - \sigma^*$	λ_1	L	$\lambda_1 \vert \ln \epsilon \vert$
10^{-3}	0.089	24.37	0.614
10^{-4}	0.070	30.63	0.644
10^{-5}	0.058	35.71	0.668
10^{-6}	0.047	41.32	0.649

For the parameter values $r = 0.8$, $\omega = 2.0$, and $\sigma = \sigma^* = -0.3312\ldots$ it is shown in [4] that there is a grazing bifurcation where a fixed point of P becomes an intermittently chaotic attractor. The small parameter is $\epsilon = \sigma^* - \sigma$. For these parameter values $\mu_1 = \mu_2 = 0.8$. The Lyapunov exponents have been calculated for small values of the bifurcation parameter and are shown in Table III. The agreement with the theory for the family of maps in (3) is very good and strongly suggests that the scaling $\lambda_1 \sim 1/|\ln\epsilon|$ also holds for more general systems.

V. CONCLUSIONS

We have considered families of continuous maps in one and two dimensions where intermittency arises close to nonsmooth bifurcations. This intermittency is unusual since the chaotic burst which separates and decorrelates the laminar phases is just a single iterate which is highly destabilizing due to the arbitrarily large stretching in these maps. We proved that the largest Lyapunov exponent λ has a discontinuous jump at the bifurcation and then scales like $1/|\ln\epsilon|$. Numerical results for the impact oscillator suggest that the results are generic in some larger class of maps. However, the situation is subtly altered if we allow the maps to be discontinuous. This changes the scaling of the average laminar length with the result that when ϵ increases through 0, λ jumps to a new value that is still negative so there cannot be an immediate transition to intermittent chaotic behavior.

We conclude by noting that the type of nonsmoothness analyzed in this paper was motivated by a particular example, the impact oscillator, and a complete classification of the various kinds of nonsmooth bifurcation may be a fruitful enterprise.

ACKNOWLEDGMENT

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APPENDIX

In Sec. III we made two assumptions. First, that for the maps (4) the distribution of impacts over the interval $[0, \epsilon]$ was uniform and second, that the "chaotic bursts" comprised just one iterate. We now show that the error in making these assumptions is small.

Let us denote the nth preimage of the interval $[0, \epsilon]$, as defined by the linear part of (4), by I_n. This gives us a set of disjoint intervals of length $O(\epsilon)$ lying in $x < 0$. The reinjection into the region $x < 0$ after a chaotic burst in $[0, \epsilon]$ can occur at any point in the interval $[\beta\epsilon^z + \epsilon, 0]$ (remember that $\beta < 0$) and may therefore lie in any one of $O(\epsilon^{z-1})$ of the intervals I_n, each of which is eventually mapped back onto $[0, \epsilon]$. This implies that the return map from the start of one chaotic burst to the next is strongly mixing for small ϵ and justifies (but does not prove) the assumption of uniformity.

We now estimate the error in the calculated Lyapunov exponent made by assuming that all the chaotic bursts are of length 1. Clearly a chaotic burst can be of any length as there is an unstable fixed point in the interval $[0, \epsilon]$ (Fig. 6). This fixed point x^* satisfies

$$\beta(x^*)^z + \epsilon = x^*$$

and so $x^* = O(\epsilon^{\frac{1}{z}})$ to leading order. Furthermore the gradient of the map at x^* is found from (4) to be of order $O(\epsilon^{1-\frac{1}{z}})$. This enables us to estimate the probability of a chaotic burst of length n, which is the probability that a point in the region $x < 0$ is mapped to a point y sufficiently close to x^* so that after n iterates it still lies in the region $x > 0$. From the estimate of the gradient at x^* this can only occur if y lies in a small neighborhood of x^* whose length l is of order

$$l = O(\epsilon^{\frac{1}{z}}\epsilon^{(\frac{1}{z}-1)(n-1)}).$$

Therefore the probability that a chaotic burst will be of length n is of leading order l/ϵ, which is of order $O(\epsilon^{n(\frac{1}{z}-1)})$.

The contribution to the Lyapunov exponent from chaotic bursts of length $n \geq 2$ is

$$\frac{\ln\alpha}{(1-z)\ln\epsilon + C\ln\alpha}\sum_{n=2}^{\infty} O\left(\epsilon^{n(\frac{1}{z}-1)} n \ln|\beta\epsilon^{\frac{1}{z}-1}|\right) .$$

This can be rewritten as

$$\sum_{n=2}^{\infty} O\left(\epsilon^{n(\frac{1}{z}-1)}\left[\ln\beta + \left(\frac{1}{z}-1\right)\ln\epsilon\right]\right) .$$

These terms are of decreasing order for increasing n and all of them are of smaller order than the leading order term in (11).

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