Generating functional approach to space- and time-dependent colored noise

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A stochastic model for time- and space-dependent colored noise is suggested based on the following assumptions: (1) the noise field is generated by a random number of point events corresponding to a correlated point process; (2) the contributions of the different events to the noise field are additive, each contribution being a random function selected from the same probability density functional. An analytical treatment of the noise field described by these assumptions is possible; the generating functional of the noise field as well as the corresponding cumulants can be computed exactly. All cumulants are explicitly evaluated when the contribution of an event is given by a diffusion equation. A detailed analysis of the asymptotic behavior is made for time-homogeneous and translationally invariant processes. A Gaussian random field colored in space and time emerges for very frequent independent events of very small intensities, provided that the central moments of the contribution of an event to the noise field are finite and fast decreasing in space. The correlations among events lead to corrections of the Gauss limit law; however, the cumulants of the random fields are also finite. If the central moments of an event are slowly decreasing in space a different type of asymptotic behavior occurs; the cumulants of the noise field become infinite and the resulting field is described by a non-Gaussian stable law of the Lévy type. The theory may be applied to the study of stochastic gravitational fluctuations in galactic systems, to the analysis of concentration fluctuations for diffusion processes in disordered systems and for the analysis of the influence of environmental fluctuations in continuum mechanics and electrodynamics.

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I. INTRODUCTION

Colored noise has been studied in the literature starting in the early 1960s [1,2] but quite recently a more general interest has appeared, with new approaches of potentially wider interest being proposed. Recent strong motivation to study the colored noise has arisen mostly in connection with the analysis of systems under the influence of external noise [3,4]. For these systems the statistical properties of the noise are unrelated to the intrinsic dynamics and as a consequence the usual justification of white noise based on the separation of time scales cannot be invoked. External colored noise is commonly used for describing a broad class of natural phenomena ranging from optics and radiophysics to population dynamics [5–12].

Traditionally the term "colored noise" is used for noise sources with a frequency-dependent spectrum. Recently this notion has been extended to space-dependent processes [13,14]; in this case the noise spectrum depends both on frequency and on the wave vector. In [13,14] a space-dependent analog of the Ornstein-Uhlenbeck process has been suggested. This generalized process has been introduced in a formal way, by relating it to an auxiliary random process which is white both in space and time. Although commonly used in stochastic dynamics this type of approach has disadvantages: it is derived from a set of "*ad hoc*" assumptions without clear physi-

cal significance and it does not say anything about the mechanism of noise generation. In contrast the early attempts at studying noise [15–18] were based on the assumption that the noise is due to a very large number of random events of very small intensity. Such an approach was used for the study of Brownian motion [15] and of shot noise [16–18]. The corresponding models are not very detailed; their analysis generated the notion of white noise. In the last 50 years the study of noise has followed two different complementary directions: (a) a microscopic approach aiming to give different detailed descriptions for different processes and (b) a mesoscopic approach which ignores the details of a given process and uses the noise sources or the master or stochastic differential equations as formal tools. Although it is usually assumed that the initial idea that the noise is due to a large number of random events of very small intensity is included in the mesoscopic description, most authors do not use it explicitly.

The purpose of this paper is to resume the above mentioned initial assumption on which the early research of noise was based and to use it for deriving a general description of multivariable space- and time-dependent colored noise. Such an approach has many advantages: it is still mesoscopic and encompasses a large class of natural phenomena; on the other hand, it is based on a physical mechanism which allows one to understand the origin of colored noise and to investigate the different types of asymptotic behavior. The structure of the paper is as follows. In Sec. II we rephrase the model of space-independent colored noise in a form which is suitable for

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generalization to space-dependent processes. In Secs. III and IV the stochastic equations for space and time-dependent colored noise are formulated based on the theory of random point processes. In Sec. V a scaling condition is introduced which is used in Sec. VI to investigate the asymptotic behavior of random noise fields generated by uncorrelated as well as correlated events. Section VII presents an application of the theory by assuming that the contribution of a source is described by a diffusion equation. Section VIII deals with another example leading to a stable noise field of the Lévy type. Finally in Sec. IX the possible applications of our approach are discussed.

II. SPACE-INDEPENDENT COLORED NOISE

We consider the temporal fluctuations of a scalar random variable $X(t)$ which obeys a stochastic functional evolution equation:

$$dX/dt = \Psi_1[X(t)] + \Psi_2[X(t)]F , \qquad (1)$$

where $\Psi_1[X(t)]$ and $\Psi_2[X(t)]$ are deterministic functionals of $X(t)$ which generally depend on all values of the function $X(t')$ for all previous times $t' = t_1, t_2, \ldots$ and F is a random noise source. Our purpose is to study the stochastic properties of the noise source F by assuming that they are due to a random succession of random events of small intensities; we do not discuss in this paper the stochastic properties of the random variable $X(t)$.

By combining the approaches of Rice [17], Bartlett [20], and Van Kampen [21] we assume that the effect at time t of a random event occurring at a previous moment t' is given by

$$\beta c \varphi(t - t') , \qquad (2)$$

where c is a random intensity selected from a probability

law with finite moments

$$p(c)dc \quad \text{with} \quad \int p(c)dc = 1 , \qquad (3)$$

β is a scaling factor used to control the intensity of an event [21], and $\varphi(t - t')$ is an attenuation function which depends only on the time difference $t - t'$; the principle of causality requires that

$$\varphi(t - t') = 0 \quad \text{for} \quad t' > t . \qquad (4)$$

By considering a large time interval T we suppose that the random events are homogeneously distributed with an average frequency ω; the number N of events corresponding to the time interval T is a random variable obeying Poisson statistics

$$P(N) = [\nu(T)]^N (N!)^{-1} \exp[-\nu(T)] , \qquad (5)$$

where the average number of events $\nu(T)$ is a linear function of T,

$$\nu(T) = \omega T . \qquad (6)$$

For a given number N of events occurring at times $t_1, \ldots, t_N$ with the intensities $c_1, \ldots, c_N$ the value of the random force $F(t)$ at time t is equal to

$$F(t) = [c_1 \varphi(t - t_1) + \cdots + c_N \varphi(t - t_N)]\beta . \qquad (7)$$

In order to characterize the stochastic properties of the random force $F(t)$ we introduce the generating functional

$$\mathfrak{L}[K(t)] = \left\langle \exp\left[i \int_0^T F(t)K(t)dt \right] \right\rangle , \qquad (8)$$

where $K(t)$ is a suitable test function and the average $\langle\ \rangle$ is computed over the values of the random intensities $c_1, \ldots, c_N$, over all possible times $t_1, \ldots, t_N$ and over the number of events N. We get

$$\mathfrak{L}[K(t)] = \sum_{N=0}^{\infty} \frac{(\omega T)^N}{N!} e^{-\omega T}$$
$$\times \int dc_1 \cdots \int dc_N \int_0^T \frac{dt_1'}{T} \cdots \int_0^T \frac{dt_N'}{T} \exp\left[i\beta \sum \int_0^T c_{N'} \varphi(t - t_{N'}')K(t)dt \right] p(c_1) \cdots p(c_N) . \qquad (9)$$

By examining Eq. (9) we note that it is the expansion of an exponential:

$$\mathfrak{L}[K(t)] = \exp\left[-\omega T + \omega \int_0^T dt' \int p(c)dc \exp\left[ic\beta \int_{t'}^T \varphi(t - t')K(t)dt \right] \right] . \qquad (10)$$

By expanding in Eq. (10) the term $\exp[ic\beta \int \varphi(t - t')K(t)dt]$ in a Taylor series and using the identity [22]

$$\int_0^T dt' \int_{t'}^T dt_1 \cdots \int_{t'}^T dt_N f(t', t_1, \ldots, t_N) = \int_0^T dt_1 \cdots \int_0^T dt_N \int_0^{t_N^*} dt' f(t', t_1, \ldots, t_N) , \qquad (11)$$

where f is a function of $t', t_1, \ldots, t_N$ and

$$t_N^* = \min(t_1, \ldots, t_N) , \qquad (12)$$

we can express the generating functional $\mathfrak{L}[K(t)]$ in the standard form of a cumulant expansion,

$$\mathfrak{L}[K(t)] = \exp\left[\omega \sum_{m=1}^{\infty} \frac{(i\beta)^m}{m!} \int_0^T dt_1 \cdots \int_0^T dt_m \langle c^m \rangle K(t_1) \cdots K(t_m) \int_0^{t_m^*} dt' \varphi(t_1 - t') \cdots \varphi(t_m - t') \right] , \qquad (13)$$

where

$$\langle c^m \rangle = \int c^m p(c)dc \tag{14}$$

are the moments of the intensity of an individual event. From Eq. (13) we can compute the cumulants of the noise source $F(t)$ by evaluating the functional derivatives

$$\langle\!\langle F(t_1) \cdots F(t_m) \rangle\!\rangle = (-i)^m \frac{\delta^m \ln\mathfrak{L}[K(t)]}{\delta K(t_1) \cdots \delta K(t_m)} \Big|_{K(t)=0} . \tag{15}$$

We obtain

$$\langle\!\langle F(t_1) \cdots F(t_m) \rangle\!\rangle = \omega\beta^m \langle c^m \rangle \int_0^{t_m^*} \varphi(t_1 - t') \cdots \\ \times \varphi(t_m - t')dt' , \tag{16}$$

where the double brackets denote the cumulant average. Equations (15) are a generalization of Campbell's theorem [16,17]. In general all cumulants of the noise source are different from zero and the corresponding stochastic process is colored and non-Gaussian. However, a Gaussian behavior may emerge for very large time intervals $(T \to \infty)$ if the average frequency of events is very large $(\omega \to \infty)$ and their effect is very small $(\beta \to 0)$. By assuming that in Eqs. (13) and (16) the integrals over t' exist and are finite the Gaussian limit corresponds to

$$\langle c \rangle = 0 , \tag{17}$$

$$\beta \to 0, \quad \omega \to \infty \quad \text{with } \beta^2\omega = \tilde{\omega} = \text{const} \ne 0 . \tag{18}$$

In this case we have

$$\langle F(t) \rangle = 0 , \tag{19}$$

$$\langle\!\langle F(t_1)F(t_2) \rangle\!\rangle = \langle F(t_1)F(t_2) \rangle \\ = \tilde{\omega} \langle c^2 \rangle \int_0^{t_2^*} \varphi(t_1 - t')\varphi(t_2 - t')dt' \tag{20}$$

and

$$\langle\!\langle F(t_1)F(t_2) \cdots F(t_m) \rangle\!\rangle = 0 \quad \text{for } m > 2 . \tag{21}$$

In this case the stochastic properties of the noise source $F(t)$ are completely determined by the second moment $\langle\!\langle F(t_1)F(t_2) \rangle\!\rangle$. The corresponding stochastic process is generally nonhomogeneous in time; the time homogeneity emerges as $T \to \infty$. In terms of $\langle\!\langle F(t_1)F(t_2) \rangle\!\rangle$ the limit $T \to \infty$ is equivalent to

$$t_1, t_2 \to \infty \quad \text{with } t_1 - t_2 = \text{const} \tag{22}$$

and Eq. (20) becomes

$$\langle\!\langle F(t_1)F(t_2) \rangle\!\rangle = \langle c^2 \rangle \tilde{\omega} \int_0^\infty \varphi(|t_1 - t_2| + \theta)\varphi(\theta)d\theta . \tag{23}$$

In particular, if the attenuation function φ is an exponential with a characteristic frequency Ω:

$$\varphi(t - t') = \Omega \exp[-\Omega(t - t')] , \tag{24}$$

in the limit (22) we recover the Ornstein-Uhlenbeck process for which

$$\langle\!\langle F(t_1)F(t_2) \rangle\!\rangle = \tfrac{1}{2}\tilde{\omega} \langle c^2 \rangle \Omega \exp[-\Omega|t_1 - t_2|] , \tag{25}$$

and all other cumulants are equal to zero. If the attenua-

tion function is δ shaped $(\Omega \to \infty)$, that is, if the effect of an event is local in time we come to the case of Gaussian and white noise.

The results presented in this section are not new. Although not identical to our approach, similar formalisms have been developed by Rice [17] and Van Kampen [21]. Our approach has the advantage that it can be easily extended for the study of space- and time-dependent colored noise.

III. SPACE- AND TIME-DEPENDENT COLORED NOISE

We consider a set of random variables

$$\mathbf{X} = (X_1, X_2, \ldots) , \tag{26}$$

depending on time t and on the position vector

$$\mathbf{r} = (r_1, r_2, \ldots) , \tag{27}$$

in a d_s-dimensional Euclidean space $(d_s = 1, 2, \ldots)$. We denote by

$$\mathbf{z} = (\mathbf{r}, t) \tag{28}$$

the position vector in space-time continuum. $\mathbf{X}(\mathbf{z}) = \mathbf{X}(\mathbf{r}, t)$ obeys a stochastic functional equation similar to Eq. (1):

$$d\mathbf{X}/dt = \mathbf{\Psi}_1[\mathbf{X}(\mathbf{z})] + \mathbf{\Psi}_2[\mathbf{X}(\mathbf{z})] \cdot \mathbf{F} , \tag{29}$$

where

$$\mathbf{F}(\mathbf{z}) = (F_1(\mathbf{z}), F_2(\mathbf{z}), \ldots) \tag{30}$$

is the vector of noise sources and $\mathbf{\Psi}_1[\mathbf{X}(\mathbf{z})]$, $\mathbf{\Psi}_2[\mathbf{X}(\mathbf{z})]$ are deterministic functionals of $\mathbf{X}(\mathbf{z})$. We assume that the stochastic behavior of the noise sources is described by the following assumptions.

(1) The noise is generated by certain (punctual) events occurring in the space-time continuum. For the Brownian motion the events are random collisions. The effect of an event occurring at a position $(\mathbf{r}, t)$ is not localized in space and time but rather distributed. The contribution to the vector $\mathbf{F}(\mathbf{r}, t)$ of an event occurring at a position $(\mathbf{r}', t')$ is given by

$$\beta\mathbf{f}(\mathbf{r} - \mathbf{r}', t - t') , \tag{31}$$

where β is a scaling factor similar to the one used in Sec. II and $\mathbf{f}(\Delta\mathbf{r}, \Delta t) = \mathbf{f}(\Delta\mathbf{z})$ is a random vector selected from a constant probability density functional

$$B[\mathbf{f}(\Delta\mathbf{z})]D[\mathbf{f}(\Delta\mathbf{z})] \\ \text{with } \iint B[\mathbf{f}(\Delta\mathbf{z})]D[\mathbf{f}(\Delta\mathbf{z})] = 1 , \tag{32}$$

where $D[\mathbf{f}(\Delta\mathbf{z})]$ is an integration measure defined over the space of functions $\mathbf{f}(\Delta\mathbf{z})$ and $\iint$ stands for the functional integral. All functions $\mathbf{f}(\Delta\mathbf{z})$ should obey the causality principle

$$\mathbf{f}(\Delta\mathbf{r}, \Delta t) = 0 \quad \text{for } \Delta t < 0 . \tag{33}$$

(2) The random punctual events occurring in the space-time continuum are generally correlated; their stochastic properties may be described by using the formalism of random point processes [21,23]. The number of

random events as well as their position in space-time continuum are described by a set of Janossy densities [21,23]:

$$Q_0, \quad Q_S(z_1, \ldots, z_S)dz_1 \cdots dz_S = Q_S(\mathbf{y}_S)d\mathbf{y}_S , \quad (34)$$

where

$$\mathbf{y}_S = (z_1, \ldots, z_S) . \quad (35)$$

$Q_S(z_1, \ldots, z_S)dz_1 \cdots dz_S$ is the probability that there are S events and that the first event occurs at a position between z_1 and $z_1 + dz_1, \ldots$, and that the Sth event occurs at a position between z_S and $z_S + dz_S$. We follow the usual convention according to which there are no restrictions concerning the relative positions of the events in the space-time continuum and thus a $1/S!$ Gibbs factor should be introduced in the normalization condition of the Janossy densities,

$$Q_0 + \sum_{S=1}^{\infty} \frac{1}{S!} \int Q_S(\mathbf{y}_S)d\mathbf{y}_S = 1 . \quad (36)$$

Our aim is to evaluate the probability density functional

$$\mathbb{G}[\mathbf{F}(z)]D[\mathbf{F}(z)] \quad \text{with} \quad \iint \mathbb{G}[\mathbf{F}(z)]D[\mathbf{F}(z)] = 1 \quad (37)$$

of the vector of noise sources or the corresponding generating functional

$$\mathfrak{L}[\mathbf{K}(z)] = \iint \exp\left[i \int \mathbf{K}(z) \cdot \mathbf{F}(z)dz \right]$$
$$\times \mathbb{G}[\mathbf{F}(z)]D[\mathbf{F}(z)] . \quad (38)$$

The probability density functional $\mathbb{G}[\mathbf{F}(z)]D[\mathbf{F}(z)]$ is an average of a Dirac δ functional symbol

$$\delta[\mathbf{F}(z) - \beta\mathbf{f}_1(z - z_1) - \cdots - \beta\mathbf{f}_N(z - z_N)]D[\mathbf{F}(z)] , \quad (39)$$

corresponding to the superposition of the contributions of N events over all possible functions $\mathbf{f}_1(z - z_1), \ldots, \mathbf{f}_N(z - z_N)$, all positions $z_1, \ldots, z_N$ of the events and over all possible number of events N:

$$\mathbb{G}[\mathbf{F}(z)]D[\mathbf{F}(z)] = \sum_{N=0}^{\infty} \iint \cdots \iint \frac{1}{N!} \int Q_N(\mathbf{y}_N)B[\mathbf{f}_1(z - z_1)]D[\mathbf{f}_1(z - z_1)] \cdots B[\mathbf{f}_N(z - z_N)D[\mathbf{f}_N(z - z_N)]$$
$$\times \delta[\mathbf{F}(z) - \beta\mathbf{f}_1(z - z_1) - \cdots - \beta\mathbf{f}_N(z - z_N)]D[\mathbf{F}(z)]d\mathbf{y}_N , \quad (40)$$

where the average is computed in terms of the Janossy densities (34). Equation (40) looks very complicated; however, it can be used to study all stochastic properties of the vector of noise sources. By making use of the formalism of random point processes [21,23] Eq. (40) leads to a closed expression for the generating functional of the noise sources which is a generalization of Eq. (10) derived in Sec. II.

IV. A GENERATING FUNCTIONAL APPROACH TO NOISE SOURCES

We introduce the joint densities [21,23]

$$\eta_N(\mathbf{y}_N)d\mathbf{y}_N = \left[\sum_{S=0}^{\infty} \frac{1}{S!} \int Q_{N+S}(\mathbf{y}_N, \mathbf{y}_S)d\mathbf{y}_S \right]d\mathbf{y}_N ,$$
$$\eta_0 = 1 . \quad (41)$$

Like the Janossy functions Q_S, the joint densities describe completely the stochastic properties of the point process; they allow us to evaluate the Janossy densities by means of the relationship [21,23]:

$$Q_S(\mathbf{y}_S)d\mathbf{y}_S = \left[\sum_{N=0}^{\infty} \frac{(-1)^N}{N!} \int \eta_{S+N}(\mathbf{y}_S, \mathbf{y}_N)d\mathbf{y}_N \right]d\mathbf{y}_S . \quad (42)$$

The main advantage of the joint densities is that they allow us to evaluate the moments of the number of events in a simple way. In particular, given a region Σ in the space-time continuum, the factorial moments of the number of events are given by [21]

$$\mathbb{C}_m = \langle N(N-1) \cdots (N-m+1) \rangle = \int_{\Sigma} \eta_N(\mathbf{y}_N)d\mathbf{y}_N . \quad (43)$$

$\eta_N d\mathbf{y}_N$ are joint densities of events rather than probability densities and thus they do not obey a normalization condition similar to Eq. (36).

In terms of Q_S and η_N we can define two different types of generating functionals: for the Janossy densities

$$\Lambda[W(z)] = \sum_{S=1}^{\infty} \frac{1}{S!} \int \cdots \int W(z_1) \cdots W(z_S)$$
$$\times Q_S(\mathbf{y}_S)d\mathbf{y}_S + Q_0 , \quad (44)$$

and for the joint densities

$$\Xi[W(z)] = 1 + \sum_{N=1}^{\infty} \frac{1}{N!} \int \cdots \int W(z_1) \cdots W(z_N)$$
$$\times \eta_N(\mathbf{y}_N)d\mathbf{y}_N , \quad (45)$$

respectively, where $W(z)$ is a suitable test function. It is easy to check that these two generating functionals are related to each other through the relationship [21]

$$\Lambda[W(z)] = \Xi[W(z) - 1] . \quad (46)$$

By performing a cumulant expansion of $\Xi[W(z)]$ we can introduce the correlation functions $g_N(\mathbf{y}_N)$ of the point process [21,23]:

$$\Xi[W(\mathbf{z})]=\exp\left[\sum_{m=1}^{\infty}\frac{1}{m!}\int\cdots\int W(\mathbf{z}_1)\cdots W(\mathbf{z}_m)\right.$$

relationships between η_N and g_N:

$$\eta_N(\mathbf{y}_N)=\sum_{\text{partitions of }N}g_{N_1}(\mathbf{z}_{j_{11}},\ldots,\mathbf{z}_{j_{1N_1}})\cdots$$

$$\left.\times g_N(\mathbf{y}_N)d\mathbf{y}_N\right]. \quad (47)$$

$$\times g_{N_l}(\mathbf{z}_{j_{l1}},\ldots,\mathbf{z}_{j_{lN_l}}) \quad (48)$$

By combining Eqs. (45)–(47) we can derive the following   and

$$g_N(\mathbf{y}_N)=\sum_{\text{partitions of }N}(-1)^{l-1}\sum_p\prod_{S=1}^{l}[\eta_{N_{p(S)}}(\mathbf{z}_{j_{p(S)1}},\ldots,\mathbf{z}_{j_{p(S)N_{p(S)}}})]. \quad (49)$$

In Eq. (48) the sum over partitions of N means a division of the integer N into l ordered subsets $(j_{11},\ldots,j_{1N_1}),\ldots,(j_{l1},\ldots,j_{lN_l})$ of sizes $N_1,\ldots,N_l$; for a given set of numbers $N_1,\ldots,N_l$ the total number of partitions is given by the well known combinatorial formula

$$\frac{N!}{\prod_{l'}[(l'!)^{N_{l'}}N_{l'}!]}\quad\text{with}\quad\sum l'N_{l'}=N. \quad (50)$$

Similarly in Eq. (49) the partition of N is the same as in Eq. (48), $j_{11}=1$, and p is a permutation of the integers $2,3,\ldots,l$ and $p(1)=1$; for a given l the total number of permutations is equal to $(l-1)!$.

Now we insert Eq. (40) into the definition (38) of the generating functional of the noise field $\mathbf{F}(\mathbf{z})$ and express the averages over $B[\mathbf{f}(\mathbf{z}-\mathbf{z}_l)]D[\mathbf{f}(\mathbf{z}-\mathbf{z}_l)]$ by simple brackets

$$\iint\cdots B[\mathbf{f}(\mathbf{z}-\mathbf{z}_l)]D[\mathbf{f}(\mathbf{z}-\mathbf{z}_l)]=\langle\cdots\rangle. \quad (51)$$

We obtain

$$\mathfrak{L}[\mathbf{K}(\mathbf{z})]=1+\sum_{N=1}^{\infty}\frac{1}{N!}\int\cdots\int d\mathbf{z}_1'\cdots d\mathbf{z}_N'Q_N(\mathbf{z}_1',\ldots,\mathbf{z}_N')\prod_{l=1}^{N}\left\langle\exp\left[i\beta\int\mathbf{K}(\mathbf{z})\cdot\mathbf{f}(\mathbf{z}-\mathbf{z}_l')d\mathbf{z}\right]\right\rangle. \quad (52)$$

By comparing Eqs. (44) and (52) for the generating functionals Λ and $\mathfrak{L}$ we note that if we choose the functions $W(\mathbf{z})$ and $\mathbf{K}(\mathbf{z})$ so that they obey the relation

$$W(\mathbf{z}')=\left\langle\exp\left[i\beta\int\mathbf{K}(\mathbf{z})\cdot\mathbf{f}(\mathbf{z}-\mathbf{z}')d\mathbf{z}\right]\right\rangle \quad (53)$$

then the functionals $\mathfrak{L}$, Λ, and Ξ are related to each other through the relationship

$$\mathfrak{L}[\mathbf{K}(\mathbf{z})]=\Lambda\left[\left\langle\exp\left[i\beta\int\mathbf{K}(\mathbf{z})\cdot\mathbf{f}(\mathbf{z}-\mathbf{z}')d\mathbf{z}\right]\right\rangle\right]$$

$$=\Xi\left[\left\langle\exp\left[i\beta\int\mathbf{K}(\mathbf{z})\cdot\mathbf{f}(\mathbf{z}-\mathbf{z}')d\mathbf{z}\right]\right\rangle-1\right]$$

$$=\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\int\cdots\int d\mathbf{z}_1'\cdots d\mathbf{z}_m'g_m(\mathbf{z}_1',\ldots,\mathbf{z}_m')\right.$$

$$\left.\times\prod_{l=1}^{m}\left[\left\langle\exp\left[i\beta\int\mathbf{K}(\mathbf{z})\cdot\mathbf{f}(\mathbf{z}-\mathbf{z}_l')d\mathbf{z}\right]\right\rangle-1\right]\right\}. \quad (54)$$

By introducing the generating functional of the contribution of an individual event to the noise field

$$L[\mathbf{K}(\Delta\mathbf{z})]=\iint\exp\left[i\int\mathbf{K}(\Delta\mathbf{z})\cdot\mathbf{f}(\Delta\mathbf{z})d\Delta\mathbf{z}\right]B[\mathbf{f}(\Delta\mathbf{z})]D[\mathbf{f}(\Delta\mathbf{z})], \quad (55)$$

we can write Eq. (54) in a simpler form,

$$\mathfrak{L}[\mathbf{K}(\mathbf{z})]=\Lambda\{L[\beta\mathbf{K}(\mathbf{z}'+\Delta\mathbf{z})]\}=\Xi\{L[\beta\mathbf{K}(\mathbf{z}'+\Delta\mathbf{z})]-1\}$$

$$=\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m!}\int\cdots\int d\mathbf{z}_1'\cdots d\mathbf{z}_m'g_m(\mathbf{z}_1',\ldots,\mathbf{z}_m')\prod_{l=1}^{m}L[\beta\mathbf{K}(\mathbf{z}_l'+\Delta\mathbf{z})]\right\}. \quad (56)$$

Equations (54) and (56) are the main results of this paper; they express all stochastic properties of the resulting noise

field $F(z)$ in terms of the stochastic properties of the point process describing the behavior of the ensemble of events and in terms of the stochastic properties of the contribution $f(\Delta z)$ of an individual event to the noise source. In particular, for

$$K(z) = K_1 \delta(z - z_1) + \cdots + K_l \delta(z - z_l) \tag{57}$$

and

$$K(\Delta z) = K_1 \delta(\Delta z - \Delta z_1) + \cdots + K_l \delta(\Delta z - \Delta z_l) , \tag{58}$$

the generating functionals $\mathfrak{L}[K(z)]$ and $L[K(\Delta z)]$ become the Fourier transforms in the $\mathbf{F}$ space of the l-point joint probability densities of the individual and total fields f and $\mathbf{F}$:

$$P_l(\mathbf{F}_1, z_1; \cdots; \mathbf{F}_l, z_l), \quad \int \cdots \int P_l d\mathbf{F}_1 \cdots d\mathbf{F}_l = 1 , \tag{59}$$

and

$$R_l(f_1, \Delta z_1; \cdots; f_l, \Delta z_l), \quad \int \cdots \int R_l df_1 \cdots df_l = 1 . \tag{60}$$

We have

$$\bar{P}_l(K_1, z_1; \cdots; K_l, z_l) = \int \cdots \int \exp(iK_1 \cdot \mathbf{F}_1 + \cdots + iK_l \cdot \mathbf{F}_l) P_l d\mathbf{F}_1 \cdots d\mathbf{F}_l$$

$$= \mathfrak{L}[K(z) = K_1 \delta(z - z_1) + \cdots + K_l \delta(z - z_l)] \tag{61}$$

and

$$\bar{R}_l(K_1, \Delta z_1; \cdots; K_l, \Delta z_l) = \int \cdots \int \exp(iK_1 \cdot f_1 + \cdots + iK_l \cdot f_l) R_l df_1 \cdots df_l$$

$$= L[K(\Delta z) = K_1 \delta(\Delta z - \Delta z_1) + \cdots + K_l(\Delta z - \Delta z_l)] , \tag{62}$$

where the overbar denotes the Fourier transformation. It follows that the l-point joint probability density of the resulting field can be evaluated as an inverse Fourier transform of $\mathfrak{L}$:

$$P_l(\mathbf{F}_1, z_1; \cdots; \mathbf{F}_l, z_l) = (2\pi)^{-Ml} \int \cdots \int \exp(-iK_1 \cdot \mathbf{F}_1 - \cdots - iK_l \cdot \mathbf{F}_l)$$

$$\times \mathfrak{L}[K(z) = K_1 \delta(z - z_1) + \cdots + K_l \delta(z - z_l)] dK_1 \cdots dK_l . \tag{63}$$

Here M is the dimension of the vector of noise sources.

The central moments and the cumulants of the resulting noise field, if they exist and are finite, can also be evaluated from the expressions of the generating functional $\mathfrak{L}$. By evaluating the functional derivatives

$$\langle F_{\alpha_1}(z_1) \cdots F_{\alpha_l}(z_l) \rangle$$

$$= (-i)^l \frac{\delta^l \mathfrak{L}[K(z)]}{\delta K_{\alpha_1}(z_1) \cdots \delta K_{\alpha_l}(z_l)} \bigg|_{K=0} , \tag{64}$$

and

$$\langle\langle F_{\alpha_1}(z_1) \cdots F_{\alpha_l}(z_l) \rangle\rangle$$

$$= (-i)^l \frac{\delta^l \ln \mathfrak{L}[K(z)]}{\delta K_{\alpha_1}(z_1) \cdots \delta K_{\alpha_l}(z_l)} \bigg|_{K=0} , \tag{65}$$

we can compute, at least in principle, all central moments $\langle F_{\alpha_1}(z_1) \cdots F_{\alpha_l}(z_l) \rangle$ and the cumulants $\langle\langle F_{\alpha_1}(z_1) \cdots F_{\alpha_l}(z_l) \rangle\rangle$ of the resulting noise field. The actual calculations are rather complicated, the complexity increasing with the order of cumulants. In Appendix A and in Secs. VI and VII we show how Eqs. (64) and (65) can be used for computing the moments.

V. SCALING CONDITIONS FOR POINT PROCESSES

In order to investigate the asymptotic behavior of the noise sources further information is necessary concerning the distribution of random events in the space-time continuum. The simplest approach corresponds to a generalization of the space-independent model discussed in Sec. II: we assume that all random events are independent and homogeneously randomly distributed in the space-time continuum. Denoting by ρ the average space density and by ω the average frequency of events, the space-time density ε of events is simply equal to

$$\varepsilon = \rho \omega . \tag{66}$$

By considering a space volume V and a time interval T the corresponding average number of events is equal to

$$\nu(V, T) = \rho \omega VT = \varepsilon \mathcal{V} , \tag{67}$$

where

$$\mathcal{V} = VT \tag{68}$$

is a space-time hypervolume. As we assume a random homogeneous distribution the average number of events corresponding to a position vector between $z = (r, t)$ and $z + dz = (r + dr, t + dt)$ is independent of position and

time and depends only on the differential element of hypervolume

$$d\mathcal{V}=dV\,dT=d\mathbf{z}=d\mathbf{r}\,dt \; ; \tag{69}$$

we have

$$\eta_1(\mathbf{z})d\mathbf{z}=\varepsilon d\mathbf{z}=\rho\omega\,d\mathbf{r}\,dt \; . \tag{70}$$

Due to statistical independence all other joint densities are products of η_1:

$$\eta_N(\mathbf{y}_N)d\mathbf{y}_N=\eta_1(\mathbf{z}_1)d\mathbf{z}_1\cdots\eta_1(\mathbf{z}_N)d\mathbf{z}_N=\varepsilon^N d\mathbf{y}_N \; . \tag{71}$$

The Janossy densities, the correlation functions, and the generating functionals Λ and Ξ can be easily evaluated from Eqs. (42), (44), (45), (49), and (71). We obtain

$$Q_S(\mathbf{y}_S)d\mathbf{y}_S=\exp(-\varepsilon\mathcal{V})\varepsilon^S d\mathbf{y}_S \; , \tag{72}$$

$$g_1(\mathbf{z})d\mathbf{z}=\eta_1(\mathbf{z})d\mathbf{z}, \quad g_m(\mathbf{z})d\mathbf{z}=0, \quad m\geq 2 \tag{73}$$

$$\Lambda[W(\mathbf{z})]=\exp\left[-\varepsilon\mathcal{V}+\varepsilon\int W(\mathbf{z})d\mathbf{z}\right] \; , \tag{74}$$

$$\Xi[W(\mathbf{z})]=\exp\left[\varepsilon\int W(\mathbf{z})d\mathbf{z}\right] \; . \tag{75}$$

Due to statistical independence all correlation functions g_m with $m\geq 2$ are equal to zero. The probability $P(N)$ that N events occur in the space-time hypervolume $\mathcal{V}$ results by integrating the expression (72) of the Janossy densities:

$$P(N)=\frac{1}{N!}\int\cdots\int Q_N(\mathbf{y}_N)d\mathbf{y}_N$$
$$=[\nu(\mathcal{V})]^N[N!]^{-1}\exp[-\nu(\mathcal{V})] \; . \tag{76}$$

Here a Gibbs factor $1/N!$ similar to the one used in the normalization condition (36) has been introduced. As expected, due to statistical independence of the events $P(N)$ is a Poissonian with an average value $\nu(\mathcal{V})$ given by Eq. (67).

For the Poissonian distribution (76) all cumulants of the number of events are proportional to the space-time hypervolume

$$\langle\langle N^l\rangle\rangle=\nu(\mathcal{V})=\varepsilon\mathcal{V}=\text{ independent of } l \; . \tag{77}$$

From Eqs. (77) it follows that the relative fluctuation of the number of events decreases with the increase of the hypervolume as $\mathcal{V}^{-1/2}$:

$$\langle\Delta N^2\rangle^{1/2}/\langle N\rangle=\langle\langle N^2\rangle\rangle^{1/2}/\langle\langle N\rangle\rangle=\varepsilon^{-1/2}\mathcal{V}^{-1/2} \; . \tag{78}$$

As $\mathcal{V}\to\infty$ the relative fluctuation tends to zero, that is, the fluctuations of the number of events do not have an intermittent behavior; although possible, the large fluctuations of N have very small probabilities of occurrence.

If the occurrence of events is described by a homogeneous random point process we can assume that the first density function η_1 is also given by Eq. (70); the other density functions, however, are no longer products of the

first density function. The most convenient way of defining such a correlated point process is to use a set of correlation functions

$$g_1(\mathbf{z})d\mathbf{z}=\eta_1(\mathbf{z})d\mathbf{z}=\varepsilon\,d\mathbf{z}, \quad g_m(\mathbf{y}_m)d\mathbf{y}_m, \quad m\geq 2 \; . \tag{79}$$

Due to the correlated behavior the functions g_m with $m\geq 2$ are generally different from zero. The joint densities and the Janossy functions can be computed from the correlation functions by applying Eqs. (48) and (42).

Our main assumption here is that the correlation functions depend only on the position vectors $\mathbf{z}_1,\mathbf{z}_2,\ldots$, on the space density ρ, and on the time frequency ω and are independent of other dimensional parameters; they do not depend on the space volume V or on the time interval T. This assumption leads to the following scaling condition for the correlation functions, which results from dimensional analysis

$$g_m(\mathbf{y}_m)d\mathbf{y}_m=\varepsilon^m\tilde{g}_m(\rho^{1/d_s}\mathbf{r}_1,\omega t_1;\cdots;\rho^{1/d_s}\mathbf{r}_m,\omega t_m)d\mathbf{y}_m \; , \tag{80}$$

where

$$\tilde{g}_m(\tilde{\mathbf{r}}_1,\tilde{t}_1;\cdots;\tilde{\mathbf{r}}_m,\tilde{t}_m) \tag{81}$$

are dimensionless correlation functions depending only on the dimensionless times $\tilde{t}_1,\ldots,\tilde{t}_m$ and dimensionless position vectors $\tilde{\mathbf{r}}_1,\ldots,\tilde{\mathbf{r}}_m$:

$$\tilde{\mathbf{r}}_\alpha=\rho^{1/d_s}\mathbf{r}_\alpha, \quad \tilde{t}_\alpha=\omega t_\alpha, \quad \alpha=1,\ldots,m \; . \tag{82}$$

We note that this scaling condition is automatically fulfilled by the first of Eqs. (79) which corresponds to

$$\tilde{g}_1=1 \; . \tag{83}$$

A simple expression for the probability $P(N)$ of the occurrence of N events is not available for arbitrary correlation functions. An expression for $P(N)$ can, however, be derived for the simplest correlated point process conceivable which corresponds to

$$g_2\neq 0, \quad g_m=0, \quad m>2 \; . \tag{84}$$

In this case we have (see Appendix B)

$$P(N)=\exp[-\nu(\mathcal{V})+\tfrac{1}{2}\mu(\mathcal{V})]$$
$$\times\sum_{k=0}^{[N/2]}\frac{[\nu(\mathcal{V})+\mu(\mathcal{V})]^{N-2k}[\mu(\mathcal{V})]^k}{2^k(N-2k)!k!} \; , \tag{85}$$

where $\nu(\mathcal{V})$ is given by Eq. (67) and

$$\mu(\mathcal{V})=\varepsilon^2\int_{\rho V}\int_{\omega T}\int_{\rho V}\int_{\omega T}\tilde{g}_2(\tilde{\mathbf{r}}_1,\tilde{t}_1;\tilde{\mathbf{r}}_2,\tilde{t}_2)$$
$$\times d\tilde{\mathbf{r}}_1 d\tilde{t}_1 d\tilde{\mathbf{r}}_2 d\tilde{t}_2 \; . \tag{86}$$

All cumulants of the number of events can be computed exactly for any correlation functions obeying the scaling condition (80) (see Appendix B):

$$\langle\langle N^l\rangle\rangle=\sum_{m=1}^{l}\varepsilon^m\mathfrak{S}_l^{(m)}\int_{\rho V}\int_{\omega T}\cdots\int_{\rho V}\int_{\omega T}\tilde{g}_m(\tilde{\mathbf{r}}_1,\tilde{t}_1;\cdots;\tilde{\mathbf{r}}_m,\tilde{t}_m)d\tilde{\mathbf{r}}_1 d\tilde{t}_1\cdots d\tilde{\mathbf{r}}_m d\tilde{t}_m \; , \tag{87}$$

where

$$\mathfrak{S}_l^{(m)} = \sum_{k=0}^{m} \frac{(-1)^{m-k} k^l}{k!(m-k)!} \tag{88}$$

are the Stirling numbers of the second kind. Unlike the case of uncorrelated processes the expressions of the cumulants cannot be used to investigate the intermittent or nonintermittent nature of the fluctuations of the number of events. This is due to the fact that in Eqs. (87) the volume and time dependence of the integrals over $\tilde{\mathbf{r}}_1, \tilde{t}_1; \cdots ; \tilde{\mathbf{r}}_m, \tilde{t}_m$ are not uniquely determined by the scaling condition (80). In Appendix C we show that by generalizing the Kubo extensivity Ansatz [19] we can introduce additional restrictions for the correlation functions which lead to a nonintermittent behavior for the fluctuations of the number of events.

The requirement that the random point process is homogeneous leads to another restriction for the correlation functions: they should not depend on the vectors $z_1, z_2, \ldots, z_m$ but rather on the differences $z_1 - z_l, \ldots, z_m - z_l$ where z_l is an arbitrary position vector selected from the set $z_1, \ldots, z_m$.

VI. ASYMPTOTIC BEHAVIOR

We begin with the study of independent point processes. In this case all correlation functions g_m of order bigger than the unity $m = 2, 3, \ldots$ are equal to zero and from Eq. (54) we come to the following expression for the generating functional of the resulting random field

$$\mathfrak{L}[\mathbf{K}, (\mathbf{z})] = \exp\left\{ \int \eta_1(\mathbf{z}')d\mathbf{z}' \left[\left\langle \exp\left[i\beta \int \mathbf{K}(\mathbf{z}) \cdot \mathbf{f}(\mathbf{z} - \mathbf{z}')d\mathbf{z} \right] \right\rangle - 1 \right] \right\}, \tag{89}$$

from which, by expanding the exponential $\exp[i\beta \int \mathbf{K}(\mathbf{z}) \cdot \mathbf{f}(\mathbf{z} - \mathbf{z}')d\mathbf{z}]$ in a Taylor series we obtain

$$\mathfrak{L}[\mathbf{K}(\mathbf{z})] = \exp\left\{ \sum_{l=1}^{\infty} \beta^l \frac{(i)^l}{l!} \int_{\mathbf{z}'} \int_{\mathbf{z}_1} \cdots \int_{\mathbf{z}_l} \sum_{\alpha_1} \cdots \sum_{\alpha_l} \eta_1(\mathbf{z}') K_{\alpha_1}(\mathbf{z}_1) \cdots K_{\alpha_l}(\mathbf{z}_l) \right.$$
$$\left. \times \langle f_{\alpha_1}(\mathbf{z}_1 - \mathbf{z}') \cdots f_{\alpha_l}(\mathbf{z}_l - \mathbf{z}') \rangle d\mathbf{z}' d\mathbf{z}_1 \cdots d\mathbf{z}_l \right\}. \tag{90}$$

If in Eq. (90) we change the order of integration of $\mathbf{z}', \mathbf{z}_1, \ldots, \mathbf{z}_l$ we can express the generating functional $\mathfrak{L}[\mathbf{K}(\mathbf{z})]$ in the standard form of a cumulant expansion. The time integrals should be handled with care because we should take the principle of causality into account; for that we should use the integral identity (11). On the other hand, no restrictions similar to those imposed by the principle of causality exist in real space so that we can change the order of integration over $\mathbf{r}', \mathbf{r}_1, \ldots, \mathbf{r}_l$ without problems. We obtain

$$\mathfrak{L}[\mathbf{K}(\mathbf{r}, t)] = \exp\left\{ \sum_{l=1}^{\infty} \frac{(i)^l}{l!} \int \cdots \int \sum_{\alpha_1} \cdots \sum_{\alpha_l} K_{\alpha_1}(\mathbf{r}_1, t_1) \cdots K_{\alpha_l}(\mathbf{r}_l, t_l) \langle\langle F_{\alpha_1}(\mathbf{r}_1, t_1) \cdots F_{\alpha_l}(\mathbf{r}_l, t_l) \rangle\rangle \right.$$
$$\left. \times d\mathbf{r}_1 dt_1 \cdots d\mathbf{r}_l dt_l \right\}, \tag{91}$$

where

$$\langle\langle F_{\alpha_1}(\mathbf{r}_1, t_1) \cdots F_{\alpha_l}(\mathbf{r}_l, t_l) \rangle\rangle = \beta^l \int_{\mathbf{r}'} \int_0^{t_l^*} \eta_1(\mathbf{r}', t') \langle f_{\alpha_1}(\mathbf{r}_1 - \mathbf{r}', t_1 - t') \cdots f_{\alpha_l}(\mathbf{r}_l - \mathbf{r}', t_l - t') \rangle d\mathbf{r}' dt' \tag{92}$$

are the cumulants of the resulting field and t_l^* is the smallest number selected from the set $(t_1, \ldots, t_l)$ [see Eq. (12)].

We note that the cumulants of the resulting noise field are averages of the central moments of the contribution of an individual event. Equations (92) are space-time analogs of the generalized Campbell relations derived in Sec. II for space-independent systems [Eqs. (16)]. The finiteness of the central moments of $\mathbf{f}$ corresponding to an individual event does not necessarily imply the finiteness of the cumulants of the resulting noise field. The cumulants are usually finite for finite V and T; as $V, T \to \infty$ they are finite only if the central moments of $\mathbf{f}$ and the density function η_1 are fast decreasing functions, for in-

stance, of the exponential type.

If the integrals in Eqs. (91) and (92) exist and are finite then a Gaussian behavior may emerge if the space-time density of events is very large ($\varepsilon \to \infty$) and the contribution of an individual event is very small ($\beta \to 0$). In order to investigate the Gaussian limit behavior we assume that the density function η_1 is given by Eq. (70) and that the average contribution of an individual event is equal to zero,

$$\langle \mathbf{f} \rangle = 0, \tag{93}$$

and introduce a limit similar to Eq. (18) used in Sec. II,

$$\beta \to 0, \quad \varepsilon \to \infty, \quad \text{with } \beta^2 \varepsilon = \bar{\varepsilon} = \text{const} \neq 0, \tag{93'}$$

where $\tilde{\epsilon}$ is an average scaled density of events in space-time continuum. In this limit if the integrals in Eqs. (91) and (92) exist and are finite and all cumulants of the resulting field excepting the second one are equal to zero.

We have

$$\langle\langle \mathbf{F} \rangle\rangle = 0, \quad \langle\langle F_{\alpha_1} F_{\alpha_2} \cdots F_{\alpha_l} \rangle\rangle = 0, \quad l = 3, 4, \ldots \quad (94)$$

and

$$\langle\langle F_{\alpha_1}(\mathbf{r}_1, t_1) F_{\alpha_2}(\mathbf{r}_2, t_2) \rangle\rangle = \tilde{\epsilon} \int_{\mathbf{r}'} \int_0^{\min(t_1, t_2)} d\mathbf{r}' dt' \langle f_{\alpha_1}(\mathbf{r}_1 - \mathbf{r}', t_1 - t') f_{\alpha_2}(\mathbf{r}_2 - \mathbf{r}', t_2 - t') \rangle . \quad (95)$$

The expression for the generating functional $\mathfrak{L}[\mathbf{K}(\mathbf{z})]$ becomes

$$\mathfrak{L}[\mathbf{K}(\mathbf{z})] = \exp\left\{ -\tfrac{1}{2} \int \int \sum_{\alpha_1} \sum_{\alpha_2} K_{\alpha_1}(\mathbf{z}_1) K_{\alpha_2}(\mathbf{z}_2) \langle\langle F_{\alpha_1}(\mathbf{z}_1) F_{\alpha_2}(\mathbf{z}_2) \rangle\rangle dz_1 dz_2 \right\} . \quad (96)$$

By using Eqs. (61) and (96) we can compute the expression of the l-point joint probability density of the resulting field. After some algebra we get an Ml-dimensional Gaussian distribution

$$P_l(\mathbf{F}_1, \mathbf{r}_1, t_1; \cdots; \mathbf{F}_l, \mathbf{r}_l, t_l)$$
$$= [\det \mathbf{C}]^{1/2} (2\pi)^{-Ml/2} \exp[-\tfrac{1}{2} \mathbf{Y} \mathbf{C}^{-1} \mathbf{Y}^\dagger] , \quad (97)$$

where

$$\mathbf{Y} = (\mathbf{F}_i)_{i=1, \ldots, l};$$

$$\mathbf{C} = \| \langle\langle \mathbf{F}^\dagger(\mathbf{r}_i, t_i) \mathbf{F}(\mathbf{r}_j, t_j) \rangle\rangle \|_{i,j=1, \ldots, l} \quad (98)$$

and the double bars stand for a matrix. In studying the asymptotic Gaussian behavior we have assumed that the point process which describes the random behavior of individual events is homogeneous; however, the limit Gaussian law given by Eqs. (95)–(98) is generally neither translationally invariant nor time homogeneous. This is due to the fact that the total volume V and the total time

interval T, although possibly large, have been assumed to be finite. The finiteness of V and T generate boundary conditions which destroy the time homogeneity and the translational invariance. By making a comparison with the space-independent model studied in Sec. II we expect that the translational invariance and the time homogeneity emerge in the limit $V, T \to \infty$. This problem will be investigated in the following section.

The asymptotic behavior for correlated systems can be discussed in a similar way. The main idea is to evaluate the nonvanishing terms which "survive" in the limit (93) in the expression (54) for the generating functional of the resulting field. We expand the terms

$$\left\langle \exp\left[i\beta \int \mathbf{K}(\mathbf{z}) \cdot \mathbf{f}(\mathbf{z} - \mathbf{z}') d\mathbf{z} \right] \right\rangle - 1 \quad (99)$$

in power series in β and change the order of integrals by using Eq. (11). The next step is to insert these expansions and the scaling conditions (80) into Eq. (54) and to order the different terms containing different powers of β. After some algebra we get an expression of the type

$$\mathfrak{L}[\mathbf{K}(\mathbf{z})] = \exp\left\{ \sum_{m=1}^{\infty} \frac{1}{m!} \epsilon^m \sum_{l=0}^{\infty} (i\beta)^{m+l} \sum_{\alpha_1} \cdots \sum_{\alpha_{m+l}} \int \cdots \int A_{\alpha_1 \cdots \alpha_{m+l}}^{(ml)} (\mathbf{y}_{m+l}) K_{\alpha_1}(\mathbf{z}_1) \cdots K_{\alpha_{m+l}}(\mathbf{z}_{m+l}) d\mathbf{y}_{m+l} \right\} , \quad (100)$$

where $A_{\alpha_1 \cdots \alpha_{m+l}}^{(ml)}(\mathbf{y}_{m+l})$ are complicated integral expressions depending on the correlation functions $\tilde{g}_m$ and on the central moments of $\mathbf{f}$. A few values of $A_{\alpha_1 \cdots \alpha_{m+l}}^{(ml)}$ are given by

$$A_{\alpha_1 \cdots \alpha_{1+l}}^{(1l)}(\mathbf{y}_{1+l}) = \frac{1}{(l+1)!} \int_{\mathbf{r}'} \int_0^{t_{1+l}^*} d\mathbf{r}' dt' \langle f_{\alpha_1}(\mathbf{r}_1 - \mathbf{r}', t_1 - t') \cdots f_{\alpha_{1+l}}(\mathbf{r}_{1+l} - \mathbf{r}', t_{1+l} - t') \rangle , \quad (101)$$

$$A_{\alpha_1 \cdots \alpha_4}^{(22)}(\mathbf{y}_4) = \tfrac{1}{4} \int_{\mathbf{r}_1'} \int_0^{\min(t_1, t_2)} \int_{\mathbf{r}_2'} \int_0^{\min(t_3, t_4)} d\mathbf{r}_1' dt_1' d\mathbf{r}_2' dt_2' \tilde{g}_2(\rho^{1/d_s} \mathbf{r}_1', \omega t_1'; \rho^{1/d_s} \mathbf{r}_2', \omega t_2')$$
$$\times \langle f_{\alpha_1}(\mathbf{r}_1 - \mathbf{r}_1', t_1 - t_1') f_{\alpha_2}(\mathbf{r}_2 - \mathbf{r}_1', t_2 - t_1') \rangle$$
$$\times \langle f_{\alpha_3}(\mathbf{r}_3 - \mathbf{r}_2', t_3 - t_2') f_{\alpha_4}(\mathbf{r}_4 - \mathbf{r}_2', t_4 - t_2') \rangle . \quad (101')$$

In general we have

$$A_{\alpha_1 \cdots \alpha_{m+l}}^{(ml)}(\mathbf{y}_{m+l}) = \int \cdots \int \tilde{g}_m(\rho^{1/d_s} \mathbf{r}_1', \omega t_1'; \cdots; \rho^{1/d_s} \mathbf{r}_m', \omega t_m')$$
$$\times J_{\alpha_1 \cdots \alpha_{m+l}}^{(ml)}(\mathbf{y}_{m+l}; \mathbf{r}_1', t_1'; \cdots; \mathbf{r}_m', t_m') d\mathbf{r}_1' dt_1' \cdots d\mathbf{r}_m' dt_m' , \quad (102)$$

where $J_{\alpha_1 \cdots \alpha_{m+l}}^{(ml)}$ are complicated expressions depending on the central moments of the contribution of a source.

By examining Eq. (100) we note that for correlated processes the limit (93) should be supplemented by the additional restrictions

$$A_{\alpha_1 \cdots \alpha_{m+l'}}^{(ml')} = 0, \quad l' = 0, \ldots, m-1, \quad m = 1, 2, \ldots . \tag{103}$$

These restrictions play a similar role to Eq. (93) in the case of uncorrelated processes; they should be introduced in order to ensure that in the limit (93) all terms of the expansion in Eq. (100) are finite. On physical grounds we expect that Eqs. (103) are satisfied when all odd moments of the contribution f of a source are equal to zero,

$$\langle f_{\alpha_1}(\mathbf{z}_1) \cdots f_{\alpha_j}(\mathbf{z}_j) \rangle = 0, \quad j = 1, 3, 5, \ldots ; \tag{104}$$

we have checked that this is indeed the case for the first values of m ($m = 1, 2, 3$); however, we have failed to give a general proof of this conjecture.

If the restrictions (103) are fulfilled then in the limit (93) the generating functional of the resulting field becomes

$$\mathfrak{L}[\mathbf{K}(\mathbf{z})] = \exp \left\{ \sum \frac{1}{m!} (\varepsilon i^2)^m \sum_{\alpha_1} \cdots \sum_{\alpha_{2m}} \int \cdots \int A_{\alpha_1 \cdots \alpha_{2m}}^{(mm)}(\mathbf{z}_1, \ldots, \mathbf{z}_{2m}) \right.$$
$$\left. \times K_{\alpha_1}(\mathbf{z}_1) \cdots K_{\alpha_{2m}}(\mathbf{z}_{2m}) d\mathbf{z}_1 \cdots d\mathbf{z}_{2m} \right\} . \tag{105}$$

Equation (105) has the standard form of a cumulant expansion; the corresponding cumulants are equal to

$$\langle\langle F_{\alpha_1}(\mathbf{z}_1) \cdots F_{\alpha_j}(\mathbf{z}_j) \rangle\rangle = 0, \quad j = 2m+1, \quad m = 0, 1, 2, \ldots \tag{106}$$

and

$$\langle\langle F_{\alpha_1}(\mathbf{z}_1) \cdots F_{\alpha_j}(\mathbf{z}_j) \rangle\rangle = \frac{(2m)!}{m!} \bar{\varepsilon}^m$$
$$\times A_{\alpha_1 \cdots \alpha_{2m}}^{(mm)}(\mathbf{z}_1, \ldots, \mathbf{z}_{2m}) ,$$
$$j = 2m, \quad m = 1, 2, \ldots . \tag{107}$$

We observe that all even cumulants are generally different from zero and thus the limit behavior of the noise sources is generally non-Gaussian. It is interesting that the cumulants of the second order are given exactly by the same expression as in the case of noncorrelated processes. To prove that we insert Eq. (101) into Eq. (107) for $l = 1$ and $j = 2$; after some calculations we recover Eq. (95).

The departure from the Gaussian behavior depends on the values of the correlation functions attached to the point process; the stochastic properties of the contribution f of an individual event are less important. In Eqs. (102) applied for $l = m$ the functions $J_{\alpha_1 \cdots \alpha_{2m}}^{(2m)}$ contain combinations of the products of the even moments of f which are always positive and thus the only possibility to have $A_{\alpha_1 \cdots \alpha_{2m}}^{(2m)} = 0$ is $g_m = 0$. If the point process is strongly correlated the correlation functions g_m are different from zero and the asymptotic behavior is very different from the Gaussian one. If the point process is weakly correlated the correlation functions $g_2, g_3, \ldots$ are close to zero and the asymptotic behavior is close to the Gaussian one; the correlation functions $g_2, g_3, \ldots$ introduce only small corrections to the Gauss limit law (97). For example, for the point processes corresponding to Eq. (84) in Eq. (105) only the functions $A_{\alpha_1 \alpha_2}^{(11)}$ and $A_{\alpha_1 \cdots \alpha_4}^{(22)}$ are different from zero and only the cumulants

of the fourth order, given by Eqs. (101) and (107) give a correction to the Gauss law.

Both the Gaussian and the non-Gaussian limit laws considered here belong to the same type: for them the cumulants of the resulting field are finite. If the central moments of f and the correlation functions of the random point process are slowly decreasing the cumulants of the resulting noise field become infinite; in this case a statistical fractal stable law of the Lévy type may emerge. This kind of asymptotic behavior is investigated in Sec. VIII.

VII. APPLICATIONS OF INDEPENDENT PROCESSES

The general results of the theory presented in the preceding sections are rather abstract. For concrete applications we need to specify the stochastic properties of the contribution f of an individual event and of the point process.

In this section we consider homogeneous independent processes described by Eqs. (70) and (71). To avoid the complications generated by the boundary conditions we discuss only the limit

$$V \to \infty, \quad T \to \infty . \tag{108}$$

By following the approach used in Sec. II we assume that the components of the vector f of the contribution of an event are given by

$$f_\alpha(\mathbf{r}, t) = c_\alpha \varphi_\alpha(\mathbf{r}, t) , \tag{109}$$

where c_α are random intensities with zero average values

$$\langle c_\alpha \rangle = 0, \quad \alpha = 1, \ldots, M , \tag{110}$$

selected from a probability law with finite moments

$$p(\mathbf{c})d\mathbf{c} , \tag{111}$$

with

$$\int p(\mathbf{c})d\mathbf{c} = 1, \quad \mathbf{c} = (c_j)_{j=1, \ldots, M} \tag{112}$$

and φ_α are deterministic attenuation functions which obey the causality principle

$$\varphi_\alpha(\mathbf{r}, t < 0) = 0 \ . \tag{113}$$

From these assumptions it follows that the probability density functional $B[f(\Delta z)]D[f(\Delta z)]$ and the generating functional $L[K(\Delta z)]$ are equal to

$$B[f(\Delta z)]D[f(\Delta z)] = \int \prod_\alpha \delta[f_\alpha - c_\alpha \varphi_\alpha(\Delta \mathbf{r}, \Delta t)]$$

$$\times D[f(\Delta \mathbf{r}, \Delta t)]p(\mathbf{c})d\mathbf{c} \quad (114)$$

and

$$L[K(\Delta z)] = \int p(\mathbf{c})d\mathbf{c}$$

$$\times \exp\left[i \sum_\alpha \int c_\alpha K_\alpha(\Delta z)\varphi_\alpha(\Delta z)dz\right] \ .$$

$$\tag{115}$$

In order to derive a set of expressions for the attenuation functions φ_α we start out from the homogeneous case discussed in Sec. II. By generalizing Eq. (24) for multivariable homogeneous systems we have

$$\varphi_\alpha(t) = \Omega_\alpha \exp(-\Omega_\alpha t) \ , \tag{116}$$

where Ω_α is a characteristic frequency attached to the lth component of the noise source. Equations (116) are the normalized solutions of the linear differential equations

$$d\varphi_\alpha(t)/dt = -\Omega_\alpha \varphi_\alpha(t) \ , \tag{117}$$

with

$$\int_0^\infty \varphi_\alpha(t)dt = 1 \ . \tag{118}$$

The simplest possible generalization of Eqs. (117) for space-dependent systems is of the form (see also [13,14])

$$\partial\varphi_\alpha(\mathbf{r}, t)/\partial t = -\Omega_\alpha(1 - \lambda_\alpha^2 \nabla^2)\varphi_\alpha(\mathbf{r}, t) \ , \tag{119}$$

where λ_α is a wavelength attached to the αth component of the noise sources. Equations (119) may be obtained by replacing in Eqs. (117) Ω_α by a set of translationally in-

variant space-dependent integral operators, performing a functional Taylor expansion, requiring that $\varphi_\alpha(\mathbf{r}, t)$ are symmetric functions of $r = |\mathbf{r}|$ and keeping the first non-vanishing terms in the expansion.

We are searching for a set of functions $\varphi_\alpha(\mathbf{r}, t)$ obeying Eqs. (119) and which fulfills a set of normalization conditions similar to Eqs. (118):

$$\int \int \varphi_\alpha(\mathbf{r}, t)d\mathbf{r}\,dt = 1 \ . \tag{120}$$

The simplest approach is to look for separable solutions of Eqs. (119) of the type

$$\varphi_\alpha(\mathbf{r}, t) = \varphi_\alpha^{(0)}(t)\varphi_\alpha^{(1)}(\mathbf{r}|t) \ , \tag{121}$$

where $\varphi_\alpha^{(0)}(t)$ are normalized space-independent solutions of the type (116):

$$\varphi_\alpha^{(0)}(t) = \Omega_\alpha \exp(-\Omega_\alpha t) \tag{122}$$

and the functions $\varphi_\alpha^{(1)}(\mathbf{r}|t)$ obey the space normalization conditions

$$\int \varphi_\alpha^{(1)}(\mathbf{r}|t)d\mathbf{r} = 1 \ . \tag{123}$$

By inserting Eqs. (121) into Eqs. (119) and using Eqs. (122) it follows that $\varphi_\alpha^{(1)}$ obey the diffusion equations

$$\partial\varphi_\alpha^{(1)}(\mathbf{r}|t)/\partial t = \Omega_\alpha \lambda_\alpha^2 \nabla^2 \varphi_\alpha^{(1)}(\mathbf{r}|t) \ . \tag{124}$$

As $V \to \infty$ the solutions of Eqs. (124) which fulfill the normalization conditions (123) are given by the Green function of the diffusion equation for an infinite medium

$$\varphi_\alpha^{(1)}(\mathbf{r}|t) = (4\pi\lambda_\alpha^2 \Omega_\alpha t)^{-d_s/2} \exp[-\mathbf{r}^2/(4\Omega_\alpha \lambda_\alpha^2 t)] \ . \tag{125}$$

Now we can evaluate the cumulants of the noise source; for independent events they are given by Eqs. (92). In these equations as $V \to \infty$ the spatial integration variables run from $-\infty$ to $+\infty$. The integration over the time variable is more complicated; due to the causality principle we should first evaluate the integral between the limits prescribed by Eqs. (92) and then pass to the limit $T \to \infty$.

Equations (92) become

$$\langle\langle F_{\alpha_1}(\mathbf{r}_1, t_1) \cdots F_{\alpha_l}(\mathbf{r}_l, t_l)\rangle\rangle = \varepsilon\beta^l \langle c_{\alpha_1} \cdots c_{\alpha_l}\rangle \Omega_{\alpha_1} \cdots \Omega_{\alpha_l}$$

$$\times \int_0^{t_l^*} dt' \varphi_{\alpha_1}^{(0)}(t_1 - t') \cdots \varphi_{\alpha_l}^{(0)}(t_l - t')$$

$$\times \int_{-\infty}^{+\infty} d\mathbf{r}' \varphi_{\alpha_1}^{(1)}(\mathbf{r}_1 - \mathbf{r}'|t_1 - t') \cdots \varphi_{\alpha_l}^{(1)}(\mathbf{r}_l - \mathbf{r}'|t_l - t') \ . \tag{126}$$

In these equations the space integrals can be evaluated exactly. By inserting the expressions (122) and (125) for $\varphi_\alpha^{(0)}$ and $\varphi_\alpha^{(1)}$ after tedious algebra we come to (see Appendix D)

$$\langle\langle F_{\alpha_1}(\mathbf{r}_1, t_1) \cdots F_{\alpha_l}(\mathbf{r}_l, t_l)\rangle\rangle = \varepsilon\beta^l \langle c_{\alpha_1} \cdots c_{\alpha_l}\rangle \Omega_{\alpha_1} \cdots \Omega_{\alpha_l}$$

$$\times \int_0^{t_l^*} \frac{\left[\sum_j \dfrac{1}{4\lambda_{\alpha_j}^2 \Omega_{\alpha_j}(t_j - t')}\right]^{-d_s/2}}{\pi^{(l-1)d_s/2} \prod_j [4\lambda_{\alpha_j}^2 \Omega_{\alpha_j}(t_j - t')]^{d_s/2}}$$

$$\times \exp\left\{ -\sum_j \Omega_{\alpha_j}(t_j - t') - \left[\sum_j \frac{1}{4\lambda_{\alpha_j}^2 \Omega_{\alpha_j}(t_j - t')}\right]^{-1} \right.$$

$$\left. \times \sum_{u>v} \frac{(\mathbf{r}_u - \mathbf{r}_v)^2}{16\Omega_{\alpha_u}\Omega_{\alpha_v}(\lambda_{\alpha_u}\lambda_{\alpha_v})^2(t_u - t')(t_v - t')} \right\} dt' . \tag{127}$$

As expected the expressions (127) depend only on the differences $\mathbf{r}_u - \mathbf{r}_v$ and not on the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_l$, that is, they are translationally invariant; however, they are not time homogeneous.

For simplicity, we evaluate the time integral in Eq. (127) only for the cumulants of second order; for independent events these are the only cumulants which "survive" in the limit (93) of very dense events with very small intensities. For $l = 2$ Eqs. (127) become

$$\langle\langle F_{\alpha_1}(\mathbf{r}_1, t_1) F_{\alpha_2}(\mathbf{r}_2, t_2) \rangle\rangle = \bar{\epsilon}\langle c_{\alpha_1} c_{\alpha_2}\rangle \int_0^{\min(t_1, t_2)} \frac{\exp[-\Omega_{\alpha_1}(t_1 - t') - \Omega_{\alpha_2}(t_2 - t')]}{[4\pi\lambda_{\alpha_1}^2\Omega_{\alpha_1}(t_1 - t') + 4\pi\lambda_{\alpha_2}^2\Omega_{\alpha_2}(t_2 - t')]^{d_s/2}}$$

$$\times \exp\left[-\frac{(\mathbf{r}_1 - \mathbf{r}_2)^2}{4\lambda_{\alpha_1}^2\Omega_{\alpha_1}(t_1 - t') + 4\lambda_{\alpha_2}^2\Omega_{\alpha_2}(t_2 - t')} \right] dt' . \tag{128}$$

The translational invariance of Eqs. (128) allows us to introduce the Fourier transforms

$$\overline{\langle\langle F_{\alpha_1}(t_1) F_{\alpha_2}(t_2)\rangle\rangle}(\mathbf{k}) = \int \exp[i\mathbf{k}\cdot(\mathbf{r}_1 - \mathbf{r}_2)]\langle\langle F_{\alpha_1}(\mathbf{r}_1, t_1)F_{\alpha_2}(\mathbf{r}_2, t_2)\rangle\rangle d(\mathbf{r}_1 - \mathbf{r}_2)$$

$$= \bar{\epsilon}\langle c_{\alpha_1} c_{\alpha_2}\rangle \Omega_{\alpha_1}\Omega_{\alpha_2} \int_0^{\min(t_1, t_2)} \exp[-\Omega_{\alpha_1}(1 + \lambda_{\alpha_1}^2|\mathbf{k}|^2)(t_1 - t') - \Omega_{\alpha_2}(1 + \lambda_{\alpha_2}^2|\mathbf{k}|^2)(t_2 - t')]dt' . \tag{129}$$

Now the time integral can be easily evaluated, resulting in

$$\overline{\langle\langle F_{\alpha_1}(t_1) F_{\alpha_2}(t_2)\rangle\rangle}(\mathbf{k}) = \frac{\bar{\epsilon}\langle c_{\alpha_1} c_{\alpha_2}\rangle \Omega_{\alpha_1}\Omega_{\alpha_2}}{\Omega_{\alpha_1}(1 + \lambda_{\alpha_1}^2|\mathbf{k}|^2) + \Omega_{\alpha_2}(1 + \lambda_{\alpha_2}^2|\mathbf{k}|^2)}$$

$$\times (\exp\{-\Omega_{\alpha_1}(1 + \lambda_{\alpha_1}^2|\mathbf{k}|^2)[t_1 - \min(t_1, t_2)] - \Omega_{\alpha_2}(1 + \lambda_{\alpha_2}^2|\mathbf{k}|^2)[t_2 - \min(t_1, t_2)]\}$$

$$- \exp[-\Omega_{\alpha_1}(1 + \lambda_{\alpha_1}^2|\mathbf{k}|^2)t_1 - \Omega_{\alpha_2}(1 + \lambda_{\alpha_2}^2|\mathbf{k}|^2)t_2]) . \tag{130}$$

We evaluate the behavior of Eqs. (130) in the limit (22) introduced in Sec. II which for the cumulants of the second order is equivalent to $T \to \infty$. Equations (130) become

$$\overline{\langle\langle F_{\alpha_1} F_{\alpha_2}\rangle\rangle}(t_1 - t_2, \mathbf{k}) = \frac{\bar{\epsilon}\langle c_{\alpha_1} c_{\alpha_2}\rangle \Omega_{\alpha_1}\Omega_{\alpha_2}}{\Omega_{\alpha_1}(1 + \lambda_{\alpha_1}^2|\mathbf{k}|^2) + \Omega_{\alpha_2}(1 + \lambda_{\alpha_2}^2|\mathbf{k}|^2)} \{h(t_1 - t_2)\exp[-|t_1 - t_2|\Omega_{\alpha_1}(1 + \lambda_{\alpha_1}|\mathbf{k}|^2)]$$

$$+ h(t_2 - t_1)\exp[-|t_1 - t_2|\Omega_{\alpha_2}(1 + \lambda_{\alpha_2}^2|\mathbf{k}|^2)]\} , \tag{131}$$

where $h(b)$ is the usual Heaviside function. Equations (131) are new; in the Gaussian limit they represent the multivariable space-time analog of the Ornstein-Uhlenbeck colored noise. We notice that for these equations the condition of microscopic reversibility

$$\overline{\langle\langle F_{\alpha_1} F_{\alpha_2}\rangle\rangle}(\mathbf{k}, t_1 - t_2) = \overline{\langle\langle F_{\alpha_1} F_{\alpha_2}\rangle\rangle}(\mathbf{k}, t_2 - t_1) \tag{132}$$

is generally not fulfilled; it is valid only if the characteristic frequencies Ω_{α_1} and Ω_{α_2} and the characteristic wavelengths λ_{α_1} and λ_{α_2} are equal to each other,

$$\Omega_{\alpha_1} = \Omega_{\alpha_2}, \quad \lambda_{\alpha_1} = \lambda_{\alpha_2} . \tag{133}$$

For a single noise source ($M = 1$) Eqs. (131) reduce to

$$\overline{\langle\langle FF\rangle\rangle}(t_1 - t_2, \mathbf{k}) = \frac{\bar{\epsilon}\langle c^2\rangle \Omega}{2(1 + \lambda^2|\mathbf{k}|^2)}$$

$$\times \exp[-|t_1 - t_2|\Omega(1 + \lambda^2|\mathbf{k}|^2)] . \tag{134}$$

Equation (134) is identical (up to a constant proportionality factor) with the equation derived by Lam and Bagayoko [14] by describing the properties of colored noise in terms of an auxiliary random process which is white both in space and time.

VIII. NONANALYTIC BEHAVIOR

The computations presented in Secs. VI and VII are based on the assumption that the central moments of the contribution of an individual event to the noise source are finite and fast decreasing and thus all cumulants of the resulting field are finite. We have seen that a set of fast decreasing central moments of the exponential type may be generated by using the assumption of analyticity. The evolution equations (119) which are responsible for the fast decrease of $\varphi_\alpha(\mathbf{r}, t)$ have been derived by means of a functional Taylor expansion which is possible only for an analytic description. The aim of this section is to investigate the stochastic properties of the noise sources generated by nonanalytic and slowly decreasing attenuation functions.

To save space we discuss the case of a single noise source $(M=1)$ and assume that only the spatial part $\varphi^{(1)}(\mathbf{r}|t)$ of the attenuation function is slowly decreasing and nonanalytic. We suppose that $\varphi^{(1)}(\mathbf{r}|t)$ is given by a symmetrical statistical fractal law [24]

$$\varphi^{(1)}(\mathbf{r}|t) = \mathcal{M}/r^\sigma = \text{independent of } t, \quad r = |\mathbf{r}| , \tag{135}$$

where $\mathcal{M}$ is a positive proportionality constant and σ is a positive fractal exponent. The temporal part $\varphi^{(0)}(t)$ of the attenuation function is supposed to be an analytic and fast decreasing exponential function of the type (122)

$$\varphi^{(0)}(t) = \Omega \exp(-\Omega t) . \tag{136}$$

The total attenuation function $\varphi(\mathbf{r}, t)$ is the product of $\varphi^{(0)}(t)$ and $\varphi^{(1)}(\mathbf{r}|t)$,

$$\varphi(\mathbf{r}, t) = \mathcal{M}\Omega r^{-\sigma} \exp(-\Omega t) . \tag{137}$$

The probability density $p(c)dc$ of the intensity of an individual event is assumed to be symmetrical,

$$p(c) = p(-c) , \tag{138}$$

which implies that all odd moments of c are equal to zero,

$$\langle c^{2m+1} \rangle = 0, \quad m = 0, 1, 2, \ldots . \tag{139}$$

We also consider that the moment of order d_s/σ of the absolute value of intensity exists and is finite,

$$\langle |c|^{d_s/\sigma} \rangle = \int_{-\infty}^{\infty} |c|^{d_s/\sigma} p(c)dc = \text{finite} . \tag{140}$$

Concerning the individual events we assume that they are independent and uniformly randomly distributed in the space-time continuum with an average density $\varepsilon = \rho\omega$.

We discuss only the evaluation of the one-point probability density $P_1(F, \mathbf{r}_1, t_1)dF$ of the noise source at a given position in space-time continuum in the limit $V, T \to \infty$. The generating functional $\mathfrak{L}[K(\mathbf{r}, t)]$ of the noise field can be computed from Eq. (89) where $\eta_1 = \varepsilon$ and the average over the values of the contribution f of an individual event is evaluated in terms of the probability density functional $B[f(\Delta\mathbf{z})]D[f(\Delta\mathbf{z})]$ given by Eq. (114) applied for $M = 1$. We obtain

$$\mathfrak{L}[K(\mathbf{r}, t)] = \exp\left\{\varepsilon \int_V \int_T \int_{-\infty}^{+\infty} p(c) \left[\exp\left[i\beta c \int \int K(\mathbf{r}', t')\varphi^{(0)}(t'-t)\right. \right. \right.$$
$$\left. \left. \left. \times \varphi^{(1)}(\mathbf{r}'-\mathbf{r})d\mathbf{r}'dt'\right] - 1\right] d\mathbf{r}\, dt\, dc\right\} . \tag{141}$$

The Fourier transform of the probability $P_1(F; \mathbf{r}, t)$,

$$\bar{P}_1(K; \mathbf{r}, t) = \int_{-\infty}^{+\infty} \exp(iKF)P_1(F; \mathbf{r}, t)dF , \tag{142}$$

can be computed from Eqs. (61) and (141). As we analyze the stochastic properties of the noise source F at a single position in space-time continuum the principle of causality does not lead to restrictions for the time variable and thus we can consider the limit $V, T \to \infty$ without problems. We get

$$\bar{P}_1(K) = \exp\left\{-\varepsilon \int_0^\infty dt \int_{-\infty}^\infty d\mathbf{r} \int_{-\infty}^\infty dc\, p(c)\{1 - \exp[iK\varphi^{(0)}(t)r^{-\sigma}c\beta]\}\right\} . \tag{143}$$

As expected in the limit $V, T \to \infty$, $\bar{P}_1(K)$ is independent of $\mathbf{r}_1, t_1$ and the stochastic properties of the noise source are the same anywhere in the space-time continuum.

Due to the symmetry of $p(c)$ [Eq. (138)] the average value of a function $I(c)$ of c can be evaluated as

$$\langle I(c) \rangle = \int_{-\infty}^{\infty} I(c)p(c)dc = \int_0^\infty [I(c) + I(-c)]p(c)dc . \tag{144}$$

By using Eq. (144) the expression (143) of $\bar{P}_1(K)$ becomes

$$\bar{P}_1(K) = \exp\left\{-2\varepsilon \int_0^\infty dt \int_{-\infty}^\infty d\mathbf{r} \int_0^\infty dc\{1 - \cos[|K|\varphi^{(0)}(t)\mathcal{M}r^{-\sigma}\beta c]\}p(c)\right\} . \tag{145}$$

As the integrand in Eq. (145) depends only on the absolute value $r = |\mathbf{r}|$ of the position vector we can express the position vector in polar coordinates in d_s-dimensional space and integrate over the angular variables. We obtain

$$\int_{-\infty}^{\infty} \cdots d\mathbf{r} = \frac{d_s \pi^{d_s/2}}{\Gamma(1+d_s/2)} \int_0^{\infty} \cdots r^{d_s-1} dr \ , \tag{146}$$

where $\Gamma(a) = \int_0^{\infty} e^{-t} t^{a-1} dt$ is the complete gamma function. By using Eq. (146) we can reduce the integral over the position vector $\mathbf{r}$ to an integral over its absolute value r; in terms of r we can introduce a new integration variable

$$x = \beta c \varphi^{(0)}(t) |K| \mathcal{M} r^{-\sigma} \ . \tag{147}$$

By means of the substitution (147) the Fourier transform $\bar{P}_1(K)$ of the one-point probability density of the noise source can be expressed in terms of three independent integrals:

$$\bar{P}_1(K) = \exp\left\{ -\frac{2\varepsilon d_s \pi^{d_s/2}}{\sigma \Gamma(1+d_s/2)} (\Omega \mathcal{M} \beta)^{d_s/\sigma} \left[\int_0^{\infty} \exp\left(-\frac{d_s \Omega t}{\sigma} \right) dt \right] \left[\int_0^{\infty} c^{d_s/\sigma} p(c) dc \right] \right.$$

$$\left. \times \left[\int_0^{\infty} [1-\cos(|K|x)] x^{-1-(d_s/\sigma)} dx \right] \right\}$$

$$= \exp[-(b|K|)^H] \ , \tag{148}$$

where

$$H = d_s / \sigma \tag{149}$$

is a positive dimensionless parameter and

$$b = \frac{\pi^{(\sigma/d_s)+(\sigma/2)} \Omega^{1-(\sigma/d_s)} (\varepsilon \langle |c|^{d_s/\sigma} \rangle)^{\sigma/d_s} \mathcal{M} \beta}{[2\Gamma(1+d_s/2)\Gamma(1+d_s/\sigma)\sin(\pi d_s/2\sigma)]^{\sigma/d_s}} \tag{150}$$

is another positive parameter having the physical dimension of the random force F.

By evaluating from Eqs. (63) and (148) the probability density $P_1(F)$ of the noise source we observe that it can be expressed in terms of a symmetric Lévy stable law [25–27] with a scaling parameter H:

$$P_1(F) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp[-iKF - (b|K|)^H] dK$$

$$= b^{-1} \mathcal{L}_H(F/b) \ , \tag{151}$$

where $\mathcal{L}_H(x)$ is the Lévy symmetrical law of order H [25–27]:

$$\mathcal{L}_H(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp[-iKx - |K|^H] dK$$

$$= (\pi)^{-1} \int_0^{\infty} \cos(Kx) \exp(-K^H) dK \ . \tag{152}$$

The Lévy stable laws can be expressed in terms of the Fox functions [27]; the corresponding expressions are complicated and not very useful. More appropriate is the use of the series expansion [26]

$$\mathcal{L}_H(x) = \frac{1}{\pi} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l!} \frac{\Gamma(1+Hl)}{|x|^{1+Hl}} \sin(\tfrac{1}{2}Hl\pi) \ , \tag{153}$$

which for $1 > H > 0$ converges for any real values of x different from zero. By using this expansion it follows that $P_1(F)$ has the following asymptotic behavior as $|F| \to \infty$:

$$P_1(F) \sim \pi^{-1} b^H \sin(\tfrac{1}{2}H\pi) |F|^{-(H+1)} \Gamma(1+H)$$

$$= \frac{\varepsilon(\mathcal{M}\beta)^{d_s/\sigma} \Omega^{(d_s/\sigma)-1} \langle |c|^{d_s/\sigma} \rangle \pi^{d_s/2}}{2\Gamma(1+d_s/2)|F|^{1+(d_s/\sigma)}}$$

as $|F| \to \infty$, (154)

that is, for $1 > H > 0$ the asymptotic behavior of $P_1(F)$ is described by a statistical fractal law with a scaling exponent $H = d_s / \sigma$.

In order to understand the physical mechanism of generation of the long tails of the inverse power law type of $P_1(F)$, we compute the probability density $\mathfrak{F}(F)dF$ of the random force which is generated by the closest random event to a given point of observation in space-time continuum. This probability density can be expressed in terms of the probability density

$$\mathcal{A}(r,t) dr\, dt \quad \text{with} \quad \int_0^{\infty} \int_0^{\infty} \mathcal{A}(r,t) dr\, dt = 1 \ , \tag{155}$$

of the distance r from the nearest event to the point of observation, and of the time t that elapsed from the occurrence of this event. For given values of the intensity of an event c, of the distance r, and of the time t the value of the random force is

$$F = \beta c \varphi(\mathbf{r}, t) = \beta c \varphi^{(0)}(t) \mathcal{M} r^{-\sigma} \ . \tag{156}$$

It follows that $\mathfrak{F}(F)$ can be expressed as an average of a δ function corresponding to Eq. (156):

$$\mathfrak{F}(F) = \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \mathcal{A}(r,t) p(c) \delta(F - \beta c \varphi^{(0)}(t) \mathcal{M} r^{-\sigma})$$

$$\times h(F) dr\, dt\, dc$$

$$+ \int_0^{\infty} \int_0^{\infty} \int_{-\infty}^0 \mathcal{A}(r,t) p(c) \delta(F - \beta c \varphi^{(0)}(t) \mathcal{M} r^{-\sigma})$$

$$\times h(-F) dr\, dt\, dc \ , \tag{157}$$

where $h(F)$ is the Heaviside function. As $p(c) = p(-c)$ [Eq. (138)] this relationship can be rewritten in a simpler form:

$$\mathfrak{F}(F) = \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \mathcal{A}(r,t) p(c)$$

$$\times \delta(|F| - \beta c \varphi^{(0)}(t) \mathcal{M} r^{-\sigma}) dr\, dt\, dc \ . \tag{158}$$

The position in which the random force is evaluated is surrounded by a hypersphere of radius r in which no

events occur. The corresponding volume is equal to

$$V(r)=\int_{|\mathbf{r}|\le r}d\mathbf{r}=\frac{\pi^{d_s/2}r^{d_s}}{\Gamma(1+d_s/2)} \tag{159}$$

[see also Eq. (146)]. In the space-time continuum we can also define a space-time hypervolume $\mathcal{V}(r,t)$ which is empty, that is, in which no events occur. $\mathcal{V}(r,t)$ is simply equal to

$$\mathcal{V}(r,t)=tV(r) . \tag{160}$$

The probability $\mathcal{A}(r,t)dr\,dt$ can be expressed as

$$\mathcal{A}(r,t)dr\,dt=\mathcal{E}[\mathcal{V}(r,t)]\varepsilon\frac{\partial V(r)}{\partial r}dr\,dt , \tag{161}$$

where $\mathcal{E}(\mathcal{V})$ is the probability that the space-time hypervolume $\mathcal{V}(r,t)$ is empty. $\mathcal{E}(\mathcal{V})$ obeys the balance equation

$$\mathcal{E}(\mathcal{V}+\Delta\mathcal{V})=\mathcal{E}(\mathcal{V})(1-\varepsilon\Delta\mathcal{V}) , \tag{162}$$

which, for $\Delta\mathcal{V}\to0$, leads to a differential equation in $\mathcal{E}(\mathcal{V})$. By integrating this equation with the initial condition $\mathcal{E}(0)=1$ we get

$$\mathcal{E}(\mathcal{V})=\exp(-\varepsilon\mathcal{V}) . \tag{163}$$

By combining Eqs. (159)–(161) and (163) we get the following equation for the probability density $\mathcal{A}(r,t)$:

$$\mathcal{A}(r,t)dr\,dt=\frac{\pi^{d_s/2}\varepsilon d_s}{\Gamma(1+d_s/2)}\exp\left[-\frac{\pi^{d_s/2}\varepsilon r^{d_s}t}{\Gamma(1+d_s/2)}\right]$$
$$\times r^{d_s-1}dr\,dt . \tag{164}$$

By inserting Eq. (164) into Eq. (158) and integrating over r we can get rid of the δ function, resulting in

$$\mathfrak{F}(F)=\frac{\varepsilon d_s\pi^{d_s/2}(\beta\mathcal{M}\Omega)^{d_s/\sigma}}{\sigma\Gamma(1+d_s/2)|F|^{1+(d_s/\sigma)}}$$

$$\times\int_0^\infty\int_0^\infty c^{d_s/\sigma}p(c)\exp\left\{-\frac{d_s\Omega t}{\sigma}-\frac{\varepsilon\pi^{d_s/2}(\beta\mathcal{M}\Omega)^{d_s/\sigma}}{\Gamma(1+d_s/2)}t\left[\frac{c}{|F|}\right]^{d_s/\sigma}\exp\left[-\frac{d_s\Omega t}{\sigma}\right]\right\}dc\,dt . \tag{165}$$

In Eq. (165) the integrals over c and t cannot be expressed in a closed form. The asymptotic behavior of $\mathfrak{F}(F)$, however, can be evaluated analytically. As $|F|\to\infty$ the F-dependent term in the exponential tends to zero and thus can be neglected; by computing the integrals over c and t we come to

$$\mathfrak{F}(F)\sim\frac{\varepsilon(\mathcal{M}\beta)^{d_s/\sigma}\Omega^{(d_s/\sigma)-1}\langle|c|^{d_s/\sigma}\rangle\pi^{d_s/2}}{2\Gamma(1+d_s/2)|F|^{1+(d_s/\sigma)}}$$

$$\text{as }|F|\to\infty . \tag{166}$$

By comparing Eqs. (154) and (166) we note that as $|F|\to\infty$ the behavior of the probability densities $P_1(F)$ and $\mathfrak{F}(F)$ is exactly the same. The physical explanation of this result is simple: the very large random forces are generated by events which are very close (in space and time) to the point at which the noise is evaluated. For the closest event the attenuation effect expressed by the function φ is very small and its contribution outweighs the contributions of remote events. For not very large values of F, however, all events contribute to the noise source and the probability density $\mathfrak{F}(F)$ is a poor approximation of the probability density $P_1(F)$ of the noise source. This interpretation is similar to the one ascribed to the Holtsmark theorem from spectroscopy [28] and astrophysics [29]. From Eqs. (151) and (153) it follows that for $1>H>0$ all even moments of the noise source are infinite,

$$\langle F^{2m}\rangle=\infty, \quad m=1,2,\ldots, \quad 1>H>0 . \tag{167}$$

Due to the symmetry of $\mathcal{L}_H(x)$ all odd moments of the noise source are equal to zero,

$$\langle F^{2m+1}\rangle=0, \quad m=0,1,2,\ldots, \quad 1>H>0 . \tag{168}$$

The behavior of the moments of the absolute value of the noise force is independent of the symmetry of the Lévy function $\mathcal{L}_H(x)$ and for $1>H>0$ all positive and integer moments are infinite,

$$\langle|F|^m\rangle=\infty, \quad m=1,2,\ldots, \quad 1>H>0 . \tag{169}$$

The fractional moments of the absolute value of the random force

$$\langle|F|^a\rangle=2b^a\int_0^\infty\mathcal{L}_H(x)x^a dx, \quad a>0 \tag{170}$$

where a is a positive number, not necessarily an integer, have a different behavior. From Eqs. (153) we note that the integral in Eqs. (170) is divergent as $x\to\infty$ only for $a\ge H$. In Appendix E we show that for $a<H$ the fractional moments exist and are finite. We have

$$\langle|F|^a\rangle=\infty, \quad 1>H>0, \quad a\ge H \tag{171}$$

$$\langle|F|^a\rangle=2\pi^{-1}b^a\Gamma(a)\Gamma[1-(a/H)]\sin(\tfrac{1}{2}a\pi) ,$$

$$1>H>0, \quad a<H . \tag{172}$$

The computations presented in this section are based on the assumption that the individual events are independent and homogeneously randomly distributed in space-time continuum with a constant average density ε. The results of our computation can be easily extended for the case when the space distribution of events is random and uniform not in a d_s-dimensional Euclidean space but in a

d_f-dimensional fractal structure embedded in it ($d_f \leq d_s$). Such a fractal model is of interest in astrophysics [30]. By following a commonly used heuristic approach [31,32] we assume that the fractal hypervolume $V^*(r)$ of a hyper-sphere of radius r can be computed by Eq. (159) where the Euclidean dimension d_s is replaced by the fractal dimension d_f:

$$V^*(r) = \frac{\pi^{d_f/2} r^{d_f}}{\Gamma(1+d_f/2)} .$$ (173)

The numerical factor in Eq. (173) has no deep theoretical significance; it has the advantage that for $d_s = d_f$ the fractal hypervolume $V^*(r)$ is identical with the Euclidean volume $V(r)$. In terms of $V^*(r)$ we can introduce a space-time fractal hypervolume

$$\mathcal{V}^*(r,t) = t V^*(r) ,$$ (174)

and the fractal space-time density of events

$$\varepsilon^* = \partial \nu / \partial \mathcal{V}^* ,$$ (175)

where ν is the average number of events enclosed in the fractal space-time hypervolume $\mathcal{V}^*$. In Appendix F we show that if the fractal space-time density of events ε^* is constant then all relations derived in this section for the probability densities $P_1(F)$ and $\mathfrak{F}(F)$ remain valid provided that we make the substitutions

$$d_s \to d_f, \quad \varepsilon \to \varepsilon^* .$$ (176)

IX. DISCUSSION

Now we summarize the main results presented in this paper. We have suggested a generating functional approach for multivariable time- and space-dependent colored noise. The model is based on the assumption that the noise field is generated by the additive contributions of a random number of correlated point events occurring in space-time continuum; the contribution of each event to the noise field is a random function selected from the same probability density functional. By using the formalism of random point processes we have derived a closed equation for the generating functional $\mathcal{L}[\mathbf{K}(\mathbf{r},t)]$ of the random noise field; this equation expresses $\mathcal{L}[\mathbf{K}(\mathbf{r},t)]$ in terms of the generating functionals of the point process which describes the relationships among the different events and the generating functional attached to an individual event. The generating functional $\mathcal{L}[\mathbf{K}(\mathbf{r},t)]$ of the noise field plays a role similar to the partition functions in equilibrium statistical mechanics: by evaluating its functional derivatives we can compute all cumulants of the noise field.

We have discussed the asymptotic behavior of random noise fields in the limit of very frequent events of very small intensities. For independent events a Gaussian random field colored in space and time emerges if the central moments attached to an event are finite and fast decreasing. The correlations among events lead to deviations from the Gaussian behavior; however, the limit law is not far from a Gaussian if the central moments of an event

are finite and fast decreasing. If the central moments of an event are slowly decaying then the asymptotic behavior is different: it corresponds to a Lévy stable law with infinite moments. For a Lévy distribution the probability density of the noise source at a given point in space-time continuum has a long tail of the inverse power law type. We have shown that this long tail is generated by the closest event to the position at which the noise source is evaluated.

Concerning the validity limits of our approach, the assumption that the process takes place in an infinite volume in space-time continuum is not compulsory. The limit $\mathcal{V} \to \infty$ is not necessary for the occurrence of the Gaussian behavior; it has mainly a practical importance: it removes the boundary conditions leading to a simplified version of the model. On the other hand, although the Lévy random fields have been analyzed for a random uniform distribution of events in Euclidean space the results can be easily extended for random uniform distributions of events in fractal structures embedded in Euclidean space.

The general theory has been used to derive a multivariable generalization of the Ornstein-Uhlenbeck process colored in space and time. This generalized Ornstein-Uhlenbeck process is based on the assumption that the events are independent and the contribution of an event to the noise field is the product between a set of random intensity factors selected from a constant probability law with a set of deterministic attenuation functions which obey a set of diffusion equations. For one-variable systems our model reduces to the model of Lam and Bagayoko [14] which describes the properties of colored noise in terms of an auxiliary random process which is white both in space and time.

In comparison to other approaches to external noise presented in the literature our method has many advantages. It shares some features with the microscopic description by assuming that the noise is due to a large number of events of very small intensity; the mechanism, however, is not very detailed: no specific assumptions are made concerning the physical nature of the individual events. This is, however, an advantage rather than a disadvantage: by making suitable assumptions about the dynamics of individual events the method can be applied to different problems.

Another advantage of the theory is that it gives a unified description of both analytical and nonanalytical regimes of colored noise. The traditional approaches to colored noise [1–10,13,14,21] usually deal with the moments of the noise source which are implicitly assumed to be finite and thus they cannot be applied to the nonanalytical regime.

Recently an alternative type of fractal stochastic processes has been introduced by Koyama and Hara [12] in seismology and by Vlad [33,34] in connection with the theory of line shape. For these processes the statistical fractal features are displayed by the moments of random variable which have long tails and not by the probability density. The probability distribution is Gaussian or close to a Gaussian and has a short tail. In contrast, for our approach the statistical fractal features may be displayed

only by the probability density of the noise source and not by the moments.

Concerning the relationships of our approach with other generating functional methods presented in the literature we outline some analogies with two papers written by one of the authors dealing with the stochastic gravitational fluctuations [30] and with random spiral shapes [22]. In [30] the stochastic gravitational fluctuations in galactic systems are investigated by assuming that the galaxies are uniformly randomly distributed in a fractal structure embedded in tridimensional space; such a model is supported by observational evidence [35]. The gravitational field is random because the mass as well as the position of a galaxy are random variables. For these stochastic gravitational fluctuations a Gaussian limit behavior does not exist; due to the fact that the gravitational force decreases slowly with the distance the stochastic gravitational fluctuations are described by a Lévy stable law with infinite moments. The main difference between the noise theory presented in Sec. VIII and the stochastic gravitational fluctuations is that the mean lifetime of a galaxy is much larger than the time scale of gravitational fluctuations and thus the time variable is not taken explicitly into account by the theory. In contrast, in Sec. VIII we have assumed that the effect of an event on the value of the noise source decreases in time: the temporal component of the attenuation function is given by an exponential. The time dependence of the attenuation function generates some computational difficulties which are missing in [30]. In paper [22] the random spiral shapes are discussed in polar coordinates: the ray length is a nondecreasing random function of the polar angle which plays the role of an independent random variable. A generating functional for the ray length is introduced which is analog to the generating functional of a one-dimensional, one-variable noise field. The theory of point processes is not used: instead the random variations of the ray length are described in terms of a Markovian process or of a non-Markovian process of the continuous time random walk (CTRW [36]) or of the age-dependent master equation (ADME [37]) type. Although the mathematical formalisms and the physical problems are different the description of spiral shapes may be rephrased in a way which corresponds to a particular case of the present approach.

The possible relationships with the functional approaches used in random continuum mechanics are also of interest. In this field [38–40] a generating functional is introduced for describing systems characterized by deterministic evolution equations but for which the initial conditions are random. The relations between such a description and our approach are not yet clarified. Another open question is related to the relations with the generating functional approaches in quantum field theory, especially with the method of stochastic quantization [41].

Our approach may be applied to a broad class of natural phenomena for which the external noise may occur, for instance, the growth of a population in a random environment [42], the wave propagation in random media [4,43], the propagation of excitations in a neural network

[44], or in antiseismic civil engineering [45]. Moreover, the generating functional approach developed here paves the way for deriving a generating functional description of random fields due to intrinsic fluctuations, for instance, for the study of the influence of thermal fluctuations on reaction diffusion systems [46] or for the study of non-Markovian chemical dynamics in condensed matter systems [47]. The problem of intrinsic noise is more complicated because there is a connection between the stochastic properties of the noise sources and the intrinsic dynamics of the process.

In this paper we have studied only the stochastic properties of the noise sources. Our ultimate goal is, however, to evaluate the stochastic behavior of the random variables $\mathbf{X}=(X_1, X_2, \ldots, X_M)$ which obey the evolution equations (29). We are investigating the particular case of the influence of additive noise on a system of linear nonlocal equations with distributed delays of the type

$$\partial X_i(\mathbf{r}, t)/\partial t = \sum_j \int \int \chi_{ij}(\mathbf{r}-\mathbf{r}', t-t')X_j(\mathbf{r}', t')d\mathbf{r}'dt'$$

$$+ F_j(\mathbf{r}, t) , \qquad (177)$$

where the noise sources $F_j(\mathbf{r}, t)$ are described by the present model. Equations (177) include as a particular case the reaction-diffusion equations with distributed delays and additive noise

$$\partial X_i(\mathbf{r}, t)/\partial t = \sum_j \int D_{ij}(t-t')\nabla^2 X_j(\mathbf{r}, t')dt'$$

$$+ \sum_j \int \mu_{ij}(t-t')X_j(\mathbf{r}, t')dt'$$

$$+ F_i(\mathbf{r}, t) . \qquad (178)$$

By assuming that the evolution equations (177) are compatible with a stable steady state we have derived a set of fluctuation dissipation relations which relates the cumulants of the noise sources F_i to the delay kernels $\chi_{ij}(\mathbf{r}-\mathbf{r}', t-t')$. These fluctuation dissipation relations may be used to make a connection with the thermodynamic and stochastic theory of nonequilibrium processes by Ross and co-workers [48–50]; work on this problem is in progress and it is planned to be presented elsewhere.

Note added in proof. After submitting this paper for publication we have learned that J. E. Vitela and L. Zogaib [Phys. Rev. E **47**, 3900 (1993)] have used a one-time characteristic functional for the study of one-dimensional population fluctuations in electrical discharges. We also outline some formal analogies between our approach and a recent paper by S. J. Fraser and R. Kapral [Phys. Rev. A **45**, 3412 (1992)] dealing with the study of periodic colored dichotomous noise. These last two authors, however, do not use generating functionals for the description of fluctuations.

ACKNOWLEDGMENTS

APPENDIX A

We introduce the notations

$$\mathcal{A}_v[\mathbf{K}(z)] = \left\langle \exp\left[i\int \beta\mathbf{K}(z)\cdot\mathbf{f}(z-z'_v)dz\right]\right\rangle - 1 , \tag{A1}$$

$$\mathcal{B}_m[\mathbf{K}(z)] = \prod_{v=1}^{m}\mathcal{A}_v[\mathbf{K}(z)] , \tag{A2}$$

and rewrite Eq. (54) in the following form:

$$\mathcal{L}[\mathbf{K}(z)] = \exp\left\{\sum_1^\infty \frac{1}{m!}\int d\mathbf{y}'_m g_m(\mathbf{y}'_m)\mathcal{B}_m[\mathbf{K}(z)]\right\} . \tag{A3}$$

By combining Eqs. (65) and (A3) we can express the cumulants of the noise field in terms of the functional derivatives of $\mathcal{B}_m[\mathbf{K}(z)]$,

$$\langle\!\langle F_{\alpha_1}(\mathbf{z}_1)\cdots F_{\alpha_l}(\mathbf{z}_l)\rangle\!\rangle = \sum_{m=1}^{l}\frac{1}{m!}\int g_m(\mathbf{y}'_m)(-i)^l\frac{\delta^l\mathcal{B}_m[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)\cdots\delta K_{\alpha_l}(\mathbf{z}_l)}\bigg|_{\mathbf{K}=0}d\mathbf{y}_m . \tag{A4}$$

In Eqs. (A4) the maximum value l of the summation label m is due to the fact that

$$\mathcal{A}_v[\mathbf{K}(z)\equiv 0] = 0, \quad v=1,2,\ldots . \tag{A5}$$

It follows that the cumulants of the lth order depend only on the first l correlation functions $g_1,\ldots,g_l$.

The functional derivatives of $\mathcal{A}_v[\mathbf{K}(z)]$ can be easily evaluated from Eqs. (A1). We have

$$\frac{\delta^u\mathcal{A}_v[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)\cdots\delta K_{\alpha_u}(\mathbf{z}_u)}\bigg|_{\mathbf{K}=0} = (i\beta)^u\langle f_{\alpha_1}(\mathbf{z}_1-z'_v)\cdots f_{\alpha_u}(\mathbf{z}_u-z'_v)\rangle . \tag{A6}$$

The functional derivatives of $\mathcal{B}_m[\mathbf{K}(z)]$ can be expressed in terms of the functional derivatives of $\mathcal{A}_v[\mathbf{K}(z)]$ by repeated functional differentiation of Eqs. (A2). We get

$$\left[\frac{\delta\mathcal{B}_m[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)}\right]_{\mathbf{K}\equiv 0} = \left[\sum_{v=1}^{m}\left[\prod_{v'\neq v}^{m}A_{v'}[\mathbf{K}(z)]\right]\left[\frac{\delta\mathcal{A}_v[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)}\right]\right]_{\mathbf{K}=0}$$

$$= \delta_{m1}\left[\frac{\delta\mathcal{A}_1[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)}\right]_{\mathbf{K}=0} , \tag{A7}$$

$$\left[\frac{\delta^2\mathcal{B}_m[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)\delta K_{\alpha_2}(\mathbf{z}_2)}\right]_{\mathbf{K}=0} = \left[\sum_{v=1}^{m}\left[\prod_{v'\neq v}^{m}\mathcal{A}_{v'}[\mathbf{K}(z)]\right]\left[\frac{\delta^2\mathcal{A}_v[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)\delta K_{\alpha_2}(\mathbf{z}_2)}\right]\right]_{\mathbf{K}=0}$$

$$+ \left[\sum_{v_1}\sum_{v_2\neq v_1}\left[\prod_{v'\neq v_1,v_2}^{m}\mathcal{A}_{v'}[\mathbf{K}(z)]\right]\left[\frac{\delta\mathcal{A}_{v_1}[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)}\right]\left[\frac{\delta\mathcal{A}_{v_2}[\mathbf{K}(z)]}{\delta K_{\alpha_2}(\mathbf{z}_2)}\right]\right]_{\mathbf{K}=0}$$

$$= \delta_{m1}\left[\frac{\delta^2\mathcal{A}_1[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)\delta K_{\alpha_2}(\mathbf{z}_2)}\right]_{\mathbf{K}=0} + \delta_{m2}\left[\left[\frac{\delta\mathcal{A}_1[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)}\right]_{\mathbf{K}=0}\left[\frac{\delta\mathcal{A}_2[\mathbf{K}(z)]}{\delta K_{\alpha_2}(\mathbf{z}_2)}\right]_{\mathbf{K}=0}\right.$$

$$\left.+ \left[\frac{\delta\mathcal{A}_2[\mathbf{K}(z)]}{\delta K_{\alpha_1}(\mathbf{z}_1)}\right]_{\mathbf{K}=0}\left[\frac{\delta\mathcal{A}_1[\mathbf{K}(z)]}{\delta K_{\alpha_2}(\mathbf{z}_2)}\right]_{\mathbf{K}=0}\right], \text{ etc.} \tag{A8}$$

By combining Eqs. (A4)–(A8) we obtain

$$\langle\!\langle F_{\alpha_1}(\mathbf{z}_1)\rangle\!\rangle = \beta\int g_1(z'_1)dz'_1\langle f_{\alpha_1}(\mathbf{z}_1-z'_1)\rangle , \tag{A9}$$

$$\langle\!\langle F_{\alpha_1}(\mathbf{z}_1)F_{\alpha_2}(\mathbf{z}_2)\rangle\!\rangle = \beta^2\int g_1(z'_1)dz'_1\langle f_{\alpha_1}(\mathbf{z}_1-z'_1)f_{\alpha_2}(\mathbf{z}_2-z'_1)\rangle$$

$$+ \tfrac{1}{2}\beta^2\int\int g_2(z'_1,z'_2)dz'_1dz'_2[\langle f_{\alpha_1}(\mathbf{z}_1-z'_1)\rangle\langle f_{\alpha_2}(\mathbf{z}_2-z'_2)\rangle + \langle f_{\alpha_1}(\mathbf{z}_1-z'_2)\rangle\langle f_{\alpha_2}(\mathbf{z}_2-z'_1)\rangle], \text{ etc.}$$

$$\tag{A10}$$

In these equations the integration limits for the time variables should be established by using the principle of causality as it has been shown in Secs. II, VI, and VII.

The superior cumulants can be computed in a similar way, the complexity of computations increasing with the index l. For independent processes we can derive a general formula for all cumulants of the noise sources. In this case

$g_2, g_3, \ldots = 0$ and Eqs. (A1)–(A6) lead to

$$\langle\langle F_{\alpha_1}(\mathbf{z}_1) \cdots F_{\alpha_l}(\mathbf{z}_l) \rangle\rangle = (-i)^l \int d\mathbf{z}_1' g_1(\mathbf{z}_1') \left[\frac{\delta^l \mathcal{A}_1[\mathbf{K}(\mathbf{z})]}{\delta K_{\alpha_1}(\mathbf{z}_1) \cdots \delta K_{\alpha_l}(\mathbf{z}_l)} \right]_{\mathbf{K}=0}$$

$$= \int d\mathbf{z}_1' g_1(\mathbf{z}_1') \langle f_{\alpha_1}(\mathbf{z}_1 - \mathbf{z}_1') \cdots f_{\alpha_l}(\mathbf{z}_l - \mathbf{z}_1') \rangle \ . \tag{A11}$$

Equations (A11) are equivalent to Eqs. (92) used in Secs. VI and VII.

APPENDIX B

For correlated processes the probability $P(N)$ that N events occur in the space-time hypervolume $\mathcal{V}$ can be evaluated by integrating the Janossy densities

$$P(N) = \frac{1}{N!} \int Q_N(\mathbf{y}_N) d\mathbf{y}_N \ . \tag{B1}$$

Equation (B1) is similar to the relationship (76) derived for independent processes; in this case, however, Eqs. (72) are no longer valid and the probability $P(N)$ given by Eq. (B1) is generally non-Poissonian.

We introduce the characteristic function of $P(N)$,

$$G(b) = \sum P(N) \exp(iNb) \ , \tag{B2}$$

where the variable b plays a similar role to the test function $W(z)$ introduced in Sec. IV. The cumulants $\langle\langle N^l \rangle\rangle$ of the number of events are defined in terms of a cumulant expansion of $G(b)$,

$$G(b) = \exp\left\{ \sum_{l=1}^{\infty} \frac{i^l}{l!} b^l \langle\langle N^l \rangle\rangle \right\} \ . \tag{B3}$$

In order to compute the cumulants $\langle\langle N^l \rangle\rangle$ of the number of events we should express the characteristic function $G(b)$ in terms of the correlation functions $g_1, g_2 \ldots$. By inserting Eq. (B1) into Eq. (B2) we get

$$G(b) = \sum_{N=0}^{\infty} \exp(iNb) \frac{1}{N!} \int Q_N(\mathbf{y}_N) d\mathbf{y}_N \ . \tag{B4}$$

By comparing Eqs. (B4) and (44) we note that $G(b)$ can be expressed in terms of the generating functional $\Lambda[W(\mathbf{z})]$ of the Janossy densities. We have

$$G(b) = \Lambda[W(\mathbf{z}) = \exp(ib)] \ . \tag{B5}$$

Equations (46), (47), and (B5) lead to

$$G(b) = \Xi[W(\mathbf{z}) = \exp(ib) - 1]$$

$$= \exp\left[\sum_{m=1}^{\infty} \frac{1}{m!} (e^{ib} - 1)^m \int g_m(\mathbf{y}_m) d\mathbf{y}_m \right]$$

$$= \exp\left[\sum_{m=1}^{\infty} \sum_{k=0}^{m} \sum_{l=0}^{\infty} \frac{(-1)^{m-k}}{l! k! (m-k)!} \right.$$

$$\left. \times (ikb)^l \int g_m(\mathbf{y}_m) d\mathbf{y}_m \right] \ . \tag{B6}$$

In Eq. (B6) we reorder the different terms of the triple sum with respect to the powers of b and compare the result with the cumulant expansion (B3) of $G(b)$. We come to

$$\langle\langle N^l \rangle\rangle = \sum_{m=0}^{l} \mathfrak{S}_l^{(m)} \int g_m(\mathbf{y}_m) d\mathbf{y}_m \ , \tag{B7}$$

where $\mathfrak{S}_l^{(m)}$ are the Stirling numbers of the second kind defined by Eqs. (88). By inserting the scaling condition (80) into Eqs. (B7) we obtain Eqs. (87).

If all correlation functions of order bigger than 2 are equal to zero [Eqs. (84)] the expression (B6) for the characteristic function $G(b)$ becomes

$$G(b) = \exp[(e^{ib} - 1) \int g_1(\mathbf{z}_1) d\mathbf{z}_1$$

$$+ \tfrac{1}{2}(e^{ib} - 1)^2 \int g_2(\mathbf{y}_2) d\mathbf{y}_2] \ , \tag{B8}$$

or, by using the scaling condition (80)

$$G(b) = \exp[\nu(\mathcal{V})(e^{ib} - 1) + \tfrac{1}{2}\mu(\mathcal{V})(e^{ib} - 1)^2] \ , \tag{B9}$$

where $\nu(\mathcal{V})$ and $\mu(\mathcal{V})$ are given by Eqs. (67) and (86). By expanding Eq. (89) in a power series in $s = \exp(ib)$ and comparing the result with the definition (B2) of $G(b)$ we get Eqs. (85).

APPENDIX C

We introduce the following limit:

$$\mathcal{V} \to \infty, \quad N \to \infty \quad \text{with} \quad n = N/\mathcal{V} = \text{const} \ . \tag{C1}$$

This limit is a space-time analog of the thermodynamic limit. Here n is a fluctuating space-time density of events; its average value is equal to ε:

$$\langle n \rangle = \varepsilon \ . \tag{C2}$$

In the limit (C1) the Poisson law (76) characteristic for independent events has the following asymptotic behavior:

$$P(N) \sim \exp\{ -\mathcal{V}[\phi(n) + O(\mathcal{V}^{-1})] \} \ , \tag{C3}$$

where

$$\phi(n) = n \ln(n/\varepsilon) - n + \varepsilon \tag{C4}$$

is a stochastic potential which depends only on the fluctuating and average densities of events but is independent of the space-time hypervolume. $\phi(n)$ has an only minimum for $n = \varepsilon$ and its derivative is an increasing function for any positive values of n:

$$\partial\phi(\varepsilon)/\partial n = 0 \ , \tag{C5}$$

$$\partial^2\phi(n)/\partial n^2 > 0 \ . \tag{C6}$$

The asymptotic behavior (C3) ensures that the large fluctuations are exponentially rare and thus the intermittent behavior is missing.

Equation (C3) is a particular case of a scaling condition for stochastic systems far from critical states introduced

by Kubo, Matsuo, and Kitahara [19]; for independent systems it is a consequence of the Poissonian behavior. In this appendix we introduce a particular class of correlated processes which fulfill a scaling condition of the type (C3). We assume that $P(N)$, although generally non-Poissonian, obeys the scaling law (C3), where the potential $\phi(n)$ is generally not given by Eq. (C4) but still fulfills the conditions (C5) and (C6). For simplicity, without loss of generality, we assume that in Eq. (C3) the preexponential normalization constant is chosen so that

$$\phi(\varepsilon)=0 . \tag{C7}$$

In this case Eq. (C3) is a postulate rather than a consequence of a model; it is introduced in order that the fluctuations of the number of events have a nonintermittent behavior.

In the limit (C1) the characteristic function $G(b)$ is given by

$$G(b)\sim \sum_N \exp\{\mathcal{V}[ibn-\phi(n)]\}, \quad \mathcal{V}\to\infty . \tag{C8}$$

As $\mathcal{V}\to\infty$ we can evaluate the sum over N by applying the method of steepest descent. We obtain

$$G(b)\sim\exp\{\mathcal{V}[\psi(ib)+O(\mathcal{V}^{-1})]\}, \quad \mathcal{V}\to\infty \tag{C9}$$

where

$$\psi(ib)=ib\xi(ib)-\phi(\xi(ib)) , \tag{C10}$$

and

$$\xi(x)=[\partial\phi/\partial n]^{(-1)} ,$$
$$\partial\phi(\xi(x))/\partial\xi(x)=x, \quad x=ib \tag{C11}$$

is the inverse function of $\partial\phi(n)/\partial n$. Due to the condition (C6) $\xi(x)$ is unique. As $\partial\phi(\varepsilon)/\partial n=0$ [Eq. (C5)] it follows that $\xi(0)=\varepsilon$ and thus from Eqs. (C7) and (C10) we have

$$\psi(0)=0 . \tag{C12}$$

By expanding in Eq. (C10) the function $\psi(ib)$ in a Taylor series, using Eq. (C12), and comparing the result with the cumulant expansion (B3) it follows that

$$\langle\langle N^l\rangle\rangle \sim \mathcal{V}\partial^l\psi(0)/\partial x^l \quad \text{as } \mathcal{V}\to\infty . \tag{C12'}$$

It turns out that all cumulants of the number of events N are proportional to the space-time hypervolume; in particular the relative fluctuation has the same type of asymptotic behavior as in the Poissonian case, i.e., it decreases to zero as $\mathcal{V}^{-1/2}$ for $\mathcal{V}\to\infty$:

$$\langle\langle N^2\rangle\rangle^{1/2}/\langle\langle N\rangle\rangle \sim [\partial^2\psi(0)/\partial x^2]^{1/2}[\partial\psi(0)/\partial x]^{-1}\mathcal{V}^{-1/2}$$
$$\text{as } \mathcal{V}\to\infty . \tag{C13}$$

By comparing Eqs. (C12′) with Eqs. (B7) we note that for correlated point processes obeying the scaling condition (C3) the correlation functions should fulfill the restrictions

$$\int g_m(\mathbf{y}_m)d\mathbf{y}_m \sim \mathcal{V}, \quad \mathcal{V}\to\infty , \tag{C14}$$

for any values of $m=1,2,\ldots$.

In order to illustrate the situations in which the scaling condition (C3) may be fulfilled for correlated systems we consider a translationally invariant and time-homogeneous point process for which only the first two correlation functions g_1 and g_2 are different from zero. We consider that g_1 is simply equal to ε [Eq. (70)] and that g_2 is a fast decreasing symmetrical function of $|t_1-t_2|$ and $|\mathbf{r}_1-\mathbf{r}_2|$ and which obeys the scaling condition (80); the simplest possible choice for g_2 is

$$g_2(\mathbf{y}_2)=A\omega^2\rho^2\exp[-a_1\omega|t_1-t_2|$$
$$-a_2\rho^{2/d_s}(\mathbf{r}_1-\mathbf{r}_2)^2] , \tag{C15}$$

where A, a_1, and a_2 are dimensionless constants. By considering a time interval T and a hypercube of volume V and linear dimension V^{1/d_s} we have

$$\int g_1(\mathbf{z}_1)d\mathbf{z}_1=\varepsilon\mathcal{V} , \tag{C16}$$

$$\int g_2(\mathbf{y}_2)d\mathbf{y}_2=VT\frac{2A\rho\omega}{a_1}\left[\frac{\pi}{a_2}\right]^{d_s/2}\left[1-\frac{1-\exp(-a_1\omega T)}{a_1 T\omega}\right]$$
$$\times\left[1-\text{erfc}[(V\rho)^{1/d_s}\sqrt{a_2}]-\frac{1-\exp[-a_2(\rho V)^{2/d_s}]}{\sqrt{\pi a_2}(\rho V)^{1/d_s}}\right]^{d_s} , \tag{C17}$$

where

$$\text{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-x^2)dx \tag{C18}$$

is the complementary error function. We note that g_1 obeys the restrictions (C14) for any values of V and T. For finite values of V and T g_2 violates the restrictions (C14); however, as $V,T\to\infty$ in Eq. (C17) the expressions in square brackets tend to unity and thus the restrictions (C14) are fulfilled,

$$\int g_2(\mathbf{y}_2)d\mathbf{y}_2\to\varepsilon\chi\mathcal{V} \text{ as } V,T\to\infty , \tag{C19}$$

where

$$\chi=\frac{2A}{a_1}\left[\frac{\pi}{a_2}\right]^{d_s/2} \tag{C20}$$

is a dimensionless constant.

As $\mathcal{V}\to\infty$ the cumulants of the number of events can be evaluated from Eqs. (88), (B7), (C16), and (C17). We restrict ourselves to the study of not very strongly correlated systems for which

$$\chi\leq 1 . \tag{C21}$$

We obtain

$$\langle\langle N^l\rangle\rangle=\mathcal{V}\varepsilon(1-\chi+\chi 2^{l-1}) . \tag{C22}$$

As the cumulants are proportional to the partial derivatives of ψ [see Eqs. (C12)] we have

$$\partial\psi(0)/\partial x^l = \varepsilon(1-\chi+\chi 2^{l-1}) ,\qquad \text{(C23)}$$

and thus we get the following expression for $\psi(ib)$:

$$\psi(ib) = \sum_{l=0}^{\infty} (ib)^l \frac{1}{l!}[\partial^l\psi(0)/\partial x^l]$$

$$= \varepsilon(1-\chi)(e^{ib}-1) + \tfrac{1}{2}\varepsilon\chi(e^{2ib}-1) . \qquad \text{(C24)}$$

Equation (C24) is consistent with Eqs. (B9) and (85). By computing the function $\mu(\mathcal{V})$ from Eqs. (86) and (C19) we get

$$\mu(\mathcal{V}) = \mathcal{V}\chi\varepsilon . \qquad \text{(C25)}$$

By inserting Eqs. (67) and (C25) into Eq. (C24) we recover Eq. (B9). On the other hand, the asymptotic behavior of $P(N)$ as $\mathcal{V}\to\infty$ can be analyzed from Eq. (85) by approximating the factorials by Stirling formula and evaluating the sum over k by means of the method of the steepest descent. After lengthy calculations we come to a scaling law of the type (C3) where the potential $\phi(n)$ is given by

$$\phi(n) = n\ln(n/\varepsilon) - n + \varepsilon + \phi^{\text{corr}}(n) , \qquad \text{(C26)}$$

where

$$\phi^{\text{corr}}(n) = \frac{2\chi n^2}{\varepsilon(1-\chi)^2}\left[\left[1+\frac{4\chi n}{\varepsilon(1-\chi)^2}\right]^{1/2}+1\right]^{-2} - n\ln\left\{\frac{(1-\chi)}{2}\left[\left[1+\frac{4\chi n}{\varepsilon(1-\chi)^2}\right]^{1/2}+1\right]\right\} - \tfrac{1}{2}\chi\varepsilon \qquad \text{(C27)}$$

is a contribution due to the correlated behavior. The first two derivatives of the potential $\phi(n)$ are equal to

$$\partial\phi(n)/\partial n = \ln\left\{\frac{2n}{\varepsilon(1-\chi)}\left[\left[1+\frac{4\chi n}{\varepsilon(1-\chi)^2}\right]^{1/2}+1\right]^{-1}\right\} \qquad \text{(C28)}$$

and

$$\partial^2\phi(n)/\partial n^2 = \frac{1}{2n}\left\{1+\left[1+\frac{4\chi n}{\varepsilon(1-\chi)^2}\right]^{-1/2}\right\} > 0 . \qquad \text{(C29)}$$

From these equations we note that $\phi(n)$ has a single minimum for $n=\varepsilon$, $\phi(\varepsilon)=0$ and its second derivative is positive definite for all positive values of n and thus Eqs. (C5), (C6) are fulfilled.

APPENDIX D

We insert Eqs. (121) and (125) into Eqs. (126); we get

$$\langle\langle F_{\alpha_1}(\mathbf{r}_1,t_1)\cdots F_{\alpha_l}(\mathbf{r}_l,t_l)\rangle\rangle = \varepsilon\beta^l\langle c_{\alpha_1}\cdots c_{\alpha_l}\rangle\Omega_{\alpha_1}\cdots\Omega_{\alpha_l}$$

$$\times\int_0^{t_l^*}\exp\left[-\sum_u\Omega_{\alpha_u}(t_u-t')\right]$$

$$\times\prod_{j=1}^{d_s}\left\{\int_{-\infty}^{\infty}dr_j'(\gamma_{\alpha_1}\cdots\gamma_{\alpha_l})^{-1/2}\pi^{-1/2}\exp\left[-\sum_u\frac{(r_j^{(u)}-r_j')^2}{\gamma_{\alpha_u}}\right]\right\}dt' , \qquad \text{(D1)}$$

where

$$\gamma_{\alpha_l} = 4\lambda_{\alpha_l}^2\Omega_{\alpha_l}(t_l-t') , \qquad \text{(D2)}$$

$$\mathbf{r}' = (r_1',\ldots,r_{d_s}'), \quad \mathbf{r}_u = (r_1^{(u)},\ldots,r_{d_s}^{(u)}) . \qquad \text{(D3)}$$

In Eq. (D1) the integral over $\mathbf{r}'$ factorizes in d_s independent Gaussian integrals which can be computed analytically. We obtain

$$\langle\langle F_{\alpha_1}(\mathbf{r}_1,t_1)\cdots F_{\alpha_l}(\mathbf{r}_l,t_l)\rangle\rangle = \varepsilon\beta^l\langle c_{\alpha_1}\cdots c_{\alpha_l}\rangle\Omega_{\alpha_1}\cdots\Omega_{\alpha_l}$$

$$\times\int_0^{t_l^*}\exp\left[-\sum_u\Omega_{\alpha_u}(t_u-t')\right]\left[\sum_u\frac{1}{\gamma_{\alpha_u}}\right]^{-d_s/2}\pi^{-d_s(l-1)/2}$$

$$\times(\gamma_{\alpha_1}\cdots\gamma_{\alpha_l})^{-d_s/2}\exp\left[-\frac{\sum_j\sum_{u_1}\sum_{u_2}\tau_{ju_1u_2}}{\sum_u\frac{1}{\gamma_{\alpha_u}}}\right]dt' , \qquad \text{(D4)}$$

where

$$\tau_{ju_1u_2} = \frac{r_j^{(u_1)}r_j^{(u_2)} - (r_j^{(u_1)})^2}{\gamma_{\alpha_{u_1}}\gamma_{\alpha_{u_2}}} \ . \tag{D5}$$

The triple sum over j, u_1, and u_2 can be rearranged in a form which displays translational invariance,

$$\sum_j \sum_{u_1} \sum_{u_2} \tau_{ju_1u_2} = \sum_j \left[\sum_{u_1 > u_2} (\tau_{ju_1u_2} + \tau_{ju_2u_1}) + \sum_u \tau_{juu} \right]$$

$$= \sum_j \sum_{u_1 > u_2} \frac{1}{\gamma_{\alpha_{u_1}}\gamma_{\alpha_{u_2}}} [(r_j^{(u_1)})^2 + (r_j^{(u_2)})^2 - 2r_j^{(u_1)}r_j^{(u_2)}]$$

$$= \sum_{u_1 > u_2} \frac{1}{\gamma_{\alpha_{u_1}}\gamma_{\alpha_{u_2}}} (\mathbf{r}_{u_1} - \mathbf{r}_{u_2})^2 \ . \tag{D6}$$

By inserting Eq. (D6) into Eqs. (D4) we get the expression (127).

APPENDIX E

In order to avoid the occurrence of divergent integrals in the expression of the moments $\langle |F|^a \rangle$ we perform two partial integrations in Eq. (152):

$$\mathcal{L}_H(x) = (\pi)^{-1} \int_0^\infty \cos(Kx) \exp(-K^H) dK$$

$$= H(\pi)^{-1} x^{-2} \int_0^\infty [1 - \cos(Kx)]$$

$$\times [HK^{2H-2} - (H-1)K^{H-2}]$$

$$\times \exp(-K^H) dK \ . \tag{E1}$$

By inserting Eq. (E1) into Eq. (170) we come to

$$\langle |F|^a \rangle = 2b^a H(\pi)^{-1} \int_0^\infty \int_0^\infty dK \, dx \, x^{a-2} [1 - \cos(Kx)]$$

$$\times [HK^{2H-2} - (H-1)K^{H-2}]. \tag{E2}$$

Now we introduce the integration variables

$$y = Kx, \quad z = K^{1/H} \ . \tag{E3}$$

By means of the substitution (E3) for $H > a > 0$ we can reduce Eq. (E2) to a product of two independent integrals which can be expressed in terms of the Γ function. After some calculations we obtain Eq. (172).

APPENDIX F

In Sec. VIII we have assumed that the space-dependent component of the attenuation function depends on the absolute value r of the position vector $\mathbf{r}$. It follows that for independent processes all necessary information concerning the space-time distribution of events is contained in the averaged density

$$\bar{\eta}_1(r,t) dr \, dt = dr \, dt \int \eta_1(\mathbf{r},t) d\omega \ . \tag{F1}$$

In Eq. (F1) we have expressed the position vector $\mathbf{r}$ in d_s-dimensional polar coordinates $\mathbf{r} = (r, \omega_1, \ldots, \omega_{d_s-1})$

and integrated over the angular variables $\omega = (\omega_1, \ldots, \omega_{d_s-1})$. For independent events homogeneously randomly distributed in d_s-dimensional Euclidean space the density function η_1 is equal to the average density ε and the integral in Eq. (F1) can be evaluated by using the relationship (146); we have

$$\bar{\eta}_1(r,t) dr \, dt = \frac{\pi^{d_s/2}\varepsilon}{\Gamma(1+d_s/2)} d_s r^{d_s-1} dr \, dt \ . \tag{F2}$$

Now we compute the averaged density function $\eta_1(r,t) dr \, dt$ for a homogeneous random distribution of events in a d_f-dimensional fractal structure embedded in d_s-dimensional Euclidean space. We introduce the probability

$$p(\mathbf{r},t) d\mathbf{r} \, dt \quad \text{with} \quad \int \int p(\mathbf{r},t) d\mathbf{r} \, dt = 1 \tag{F3}$$

that an event occurs at a position between $\mathbf{r},t$ and $\mathbf{r} + d\mathbf{r}, t + dt$. By considering a time interval T and a hyperspherical spatial domain of radius R we have

$$p(\mathbf{r},t) d\mathbf{r} \, dt = \begin{cases} (d\mathbf{r})_{\text{fractal}} dt / [TV^*(R)] & \text{for } t < T, \ r < R \\ 0 & \text{for } t \geq T, \ r \geq R \ , \end{cases} \tag{F4}$$

where the fractal hypervolume $V^*(R)$ is given by Eq. (173) applied for $r = R$ and $(d\mathbf{r})_{\text{fractal}}$ is a fractal analog of the infinitesimal Euclidean element of volume $d\mathbf{r}$. For a homogeneous fractal structure we have [30]

$$(d\mathbf{r})_{\text{fractal}} / d\mathbf{r} = dV^*(r) / dV(r) \ , \tag{F5}$$

where $V^*(r)$ and $V(r)$ are given by Eqs. (173) and (159); Eq. (F5) can be rewritten in the following form:

$$(d\mathbf{r})_{\text{fractal}} = \frac{\pi^{(d_f-d_s)/2}}{d_s \Gamma(1+d_f/2)} \Gamma(1+d_s/2) d_f r^{d_f-d_s} d\mathbf{r} \ . \tag{F6}$$

If the fractal space-time density of events ε^* is constant

the density function is given by

$$\eta_1(\mathbf{r},t)d\mathbf{r}\,dt = TV^*(R)\varepsilon^*p(\mathbf{r},t)d\mathbf{r}\,dt \ . \tag{F7}$$

Now we consider the limit $R,T \to \infty$ and compute the averaged density function $\bar{\eta}_1(r,t)dr\,dt$ by means of the relationships (F1), (F4), (F6), and (F7); we obtain

$$\bar{\eta}_1(r,t) = \frac{\pi^{d_f/2}\varepsilon^*}{\Gamma(1+d_f/2)}d_f r^{d_f-1}dr\,dt \ , \tag{F8}$$

which has exactly the same form as Eq. (F2) with the difference that ε and d_s are replaced by ε^* and d_f, respectively; it follows that all relations derived in Sec. VIII in the Euclidean case can also be applied in the fractal case.

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