Structure and parameters of clusters in traffic flow

B. S. Kerner and P. Konhäuser
Research Institute, Daimler-Benz AG, F1V/VT, HPC G210, 70546 Stuttgart, Germany
(Received 24 November 1993; revised manuscript received 16 March 1994)

The nonlinear theory of the cluster effect in a traffic flow [B. S. Kerner and P. Konhäuser, Phys. Rev. E **48**, 2335 (1993)], i.e., the effect of the appearance of a region of high density and low average velocity of vehicles in an initially homogeneous flow, is presented. The structures of a stationary moving cluster are derived. It is found that the density, the average velocities of vehicles inside and outside the cluster, and also the velocity of the cluster are the characteristic parameters of the traffic flow. The dependencies of the cluster structure and parameters on the density of vehicles in the initially homogeneous flow and on the length of the road are investigated. It is found that the cluster can appear within regions of density of vehicles which correspond to a stable homogeneous flow. It is shown that an appearance of a localized perturbation, having a finite amplitude, in the stable homogeneous flow can lead to a self-formation of a local cluster of vehicles which is surrounded by the homogeneous traffic flow. The parameters of the local cluster do not depend on the amplitude of this perturbation but only on the parameters of the flow.

PACS number(s): 05.40.+j, 47.54.+r, 89.40.+k

I. INTRODUCTION

Almost every driver moving on a long road has at some time met the phenomenon of "phantom traffic jam," i.e., traffic congestion without obvious reason. At first, drivers could move with some relatively high velocity in an average homogeneous traffic flow. Then, drivers suddenly find themselves in a region with a very high density of vehicles where they could move at very low velocity only or could not move at all. After some time, drivers could move again at a high velocity in the homogeneous traffic flow, but, to their great surprise, nothing had happened on the road which could have caused the traffic jam (for the results of the experimental investigations see, e.g., [2,3]).

Traffic congestion in an initially homogeneous traffic flow can be explained by a cluster effect in a traffic flow [1]. Indeed, it has been found that, if the density of vehicles exceeds some critical value, the initially homogeneous traffic flow loses its stability with respect to a growth of a long-wavelength nonhomogeneous perturbation. The numerical calculations of the development of this critical fluctuation [1] show that the growth of this perturbation on a circular road of some circumference leads to the appearance of a stationary moving cluster of vehicles. A stationary moving cluster of vehicles represents a region which moves with a constant velocity, where the density of vehicles can be considerably higher and the average velocity of vehicles can be considerably lower than in the initial flow and outside the cluster.

In this paper, the results of the investigations on the nonlinear structures and the parameters of the cluster of vehicles, which can appear in an initially homogeneous traffic flow, will be presented. Sections II and III are related to the numerical analysis of the structure and the parameters of clusters. In Sec. II the main equations of the kinetic model of traffic flow and the conditions of stability of this flow [1] will be written, the features of the kinetics of the formation of a stationary moving cluster and its structure will be investigated, the effect of the appearance in a traffic flow of a local cluster of vehicles will be presented, and the structure of this local cluster will be found. It will be also found here that, if a perturbation of a finite amplitude occurs, the stationary moving cluster can appear in the regions of density of vehicles where the initially homogeneous traffic flow is stable. The dependence of the structure and of the parameters of the stationary moving cluster on the density of vehicles in the initially homogeneous flow and on the length of the road will be investigated in Sec. III. In Sec. IV the qualitative nonlinear theory of the stationary moving cluster will be presented. Based on this theory the explanation of the main results of the numerical investigations of the structure and of the parameters of the stationary moving cluster will be given.

II. STRUCTURE OF CLUSTERS

A. Kinetic model of traffic flow: Basic equations

The kinetic model of a traffic flow on a circular road of some circumference L, the motivations for it, and the conditions where it can apply, have been considered in [1]. This model includes the continuity equation

$$\frac{\partial \rho}{\partial t}+\frac{\partial q}{\partial x}=0 , \tag{1}$$

the equation of motion

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=\frac{V(\rho)-v}{\tau}-\frac{c_0^2}{\rho}\frac{\partial \rho}{\partial x}+\frac{l^2}{\rho}\frac{\partial^2 v}{\partial x^2} , \tag{2}$$

the integral condition

$$\int_0^L \rho(x,t)dx=N , \tag{3}$$

and the boundary conditions

$$v(0,t)=v(L,t), \quad \partial v/\partial x|_0=\partial v/\partial x|_L \ . \tag{4}$$

In (1)–(4),

$$q(x,t)=\rho(x,t)v(x,t)$$

is the flux, $\rho(x,t)$ is the density $(0<\rho\le\hat{\rho})$, and $v(x,t)$ is the average velocity of vehicles $(v\ge 0)$. N=const is the total number of vehicles on the road $(N\gg 1)$; L is the length of the road; V is a safe ("maximal and out of danger") velocity which is achieved in a both time-independent and homogeneous traffic flow. $V(\rho)$ is a monotonous decreasing function of ρ, i.e., its derivative

$$\xi(\rho)=dV(\rho)/d\rho<0 \ ; \tag{5}$$

c_0=const, τ=const, $l=\sqrt{\mu\hat{\rho}^{-1}\tau}$, μ=const, and $\hat{\rho}$ is the maximal possible density of vehicles on the road (for a road with n lanes $\hat{\rho}=n/\hat{a}$, where $\hat{a}$ is the average length of vehicles).

It will be recalled that the equation of motion (2) follows from the well-known Navier-Stokes equations, which for a one-dimensional compressible flow read as

$$\rho\left[\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}\right]=X-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left[\mu\frac{\partial v}{\partial x}\right], \tag{6}$$

where for the given case the sum of all inner forces, which appears due to interactions between individual vehicles, is given by

$$X=\rho\frac{V(\rho)-v}{\tau}, \tag{7}$$

μ is the viscosity [4] and $p=\rho c_0^2$ is the local pressure. The integral condition (3) expresses the fact that if one seeks some new distribution for $\rho(x,t)$ and $v(x,t)$, the initial quantity of the total number of vehicles on the road has to be taken into account.

For given values of N and L, there is only one homogeneous state ρ_h, v_h for the traffic flow under consideration:

$$\rho_h=N/L, \quad v_h=V(\rho_h) \ . \tag{8}$$

The corresponding value of the flux is $q_h=v_h\rho_h$. As follows from previous investigations [1], this homogeneous state loses its stability with respect to a growth of the nonhomogeneous fluctuation of the density of vehicles. On a threshold of this instability a critical perturbation is

$$\delta\rho(x)=\delta\rho_0\exp(2\pi ix/L), \quad \delta v(x)=\delta v_0\exp(2\pi ix/L) \ . \tag{9a}$$

It represents the greatest long-wavelength nonhomogeneous perturbation in a homogeneous traffic flow, which moves with the phase velocity

$$v_p=v_h-c_0 \ . \tag{9b}$$

The instability of the initially homogeneous flow occurs in some interval of the value ρ_h,

$$\rho_{c1}<\rho_h<\rho_{c2} \ , \tag{10a}$$

if inside this interval the value ξ (5) satisfies the condition

$$\xi<-[1+(2\pi/L)^2\rho_h^{-1}](c_0/\rho_h) \ . \tag{10b}$$

As it follows from (10b), one can find the critical values $\rho_h=\rho_{c1}$ and $\rho_h=\rho_{c2}$ (10a) from the equation

$$[-1-(\rho_{ci}/c_0)\xi(\rho_{ci})]\rho_{ci}=(2\pi/L)^2 \quad (i=1,2) \ . \tag{10c}$$

Due to the instability of the traffic flow, a stationary cluster of vehicles, which moves with some constant velocity v_g along the road, can spontaneously appear in the initially homogeneous flow.

For the numerical investigations of the cluster effect presented below, the problem (1)–(4) has been solved. For this reason, the functions $w(x,t)=\partial v/\partial x$ and $\tilde{n}(x,t)$, where $\partial\tilde{n}/\partial x=\rho(x,t)-\rho_h$, had been introduced and the problem (1)–(4) had been written as a system of four first order differential equations for the functions $\tilde{n}(x,t)$, $\rho(x,t)$, $v(x,t)$, $w(x,t)$ with the corresponding conditions $\tilde{n}(0,t)=0$, $\tilde{n}(L,t)=0$, $v(0,t)=v(L,t)$, $w(0,t)=w(L,t)$. The algorithm of the numerical solution of this problem is described in [1].

B. Structure of cluster and fundamental diagram

From many experimental investigations on traffic flow it was found (see the results in, e.g., [2–7]) that a dependence of the average velocity of vehicles for a homogeneous and time-independent traffic flow on the density of vehicles has a decreasing character [Fig. 1(a)]. The related dependence on ρ of the flux $Q(\rho)=\rho V(\rho)$ is obviously a function with only one maximum [Fig. 1(b)]. This curve is called the fundamental diagram (e.g., [2–7]). The function $V(\rho)$ is determined by the average balance between safety requirements and risk readiness of the drivers as well as legal traffic regulations and road conditions, i.e., $V(\rho)$ and, consequently, $Q(\rho)$, are phenomenological functions (e.g., [2–7]). In order to understand the reason why the value V decreases as ρ increases, notice that the drivers must decrease their average velocity if the headway to the vehicle in front of them is reduced. At the limit, where the density ρ reaches the maximal value $\hat{\rho}$, vehicles cannot move at all, and for this reason $V(\rho)|_{\rho\to\hat{\rho}}\to 0$. On the other hand, at small enough values of ρ, there is almost no interaction between vehicles, and they can move with some average velocity $v_f=V(\rho)|_{\rho\to 0}$ [Fig. 1(a)].

The given value of the total number of vehicles on the road N determines the definite density of vehicles in the homogeneous traffic flow $\rho=\rho_h$ (8). In other words, the given N determines the one point on the fundamental diagram $Q(\rho)$: $q_h=Q(\rho_h)$ [or the one point on the speed density relationship $V(\rho)$: $v_h=V(\rho_h)$].

Because a lot of experimental data have been collected and plotted in the (ρ,q) phase plane, it is helpful to use this plane for the investigation of cluster formation to understand its structure. In [1] it has been shown that the given value N can represent not only the homogeneous solution (8) but also the nonhomogeneous solutions in the form of clusters of vehicles. In the cluster, the values of density ρ, average velocity v, and, consequently, the flux of vehicles $q=\rho v$ depend on x and t. It means that in the (ρ,q) phase plane the dependence $q(\rho)$, which corre-

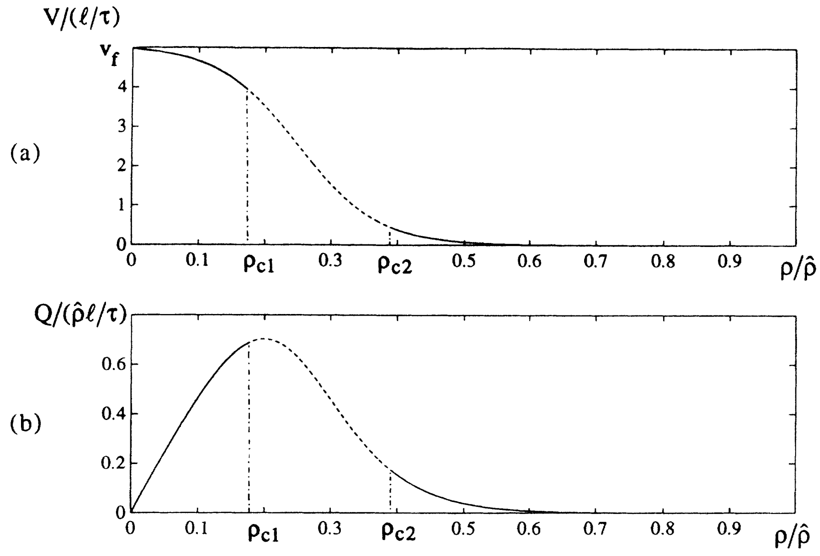

FIG. 1. The example of the function $V(\rho)$ (a) and of the corresponding fundamental diagram $Q(\rho)$ (b). Results of the numerical computations for $c_0=2.48445 l/\tau$, $V(\rho)=5.0461\ [(1+\exp\{[(\rho/\hat{\rho})-0.25]/0.06\})^{-1}-3.72\times10^{-6}] l/\tau$. The found critical values of the density for $L=800l$ are $\rho_{c1}\cong0.17335\hat{\rho}$, $\rho_{c2}\cong0.3955\hat{\rho}$.

sponds to the distributions $v(x)$ and $\rho(x)$ in the cluster of vehicles at some fixed moment of time t, represents some closed curve. The coordinates of any point on this phase curve are the values ρ and q in the cluster of vehicles for the corresponding value x.

1. *Features of kinetics of cluster formation*

In [1] it has been found that the growth of a critical perturbation (9a) at the values ρ_h, which exceed the critical value ρ_{c1} (10c) but are rather close to it, leads directly to the formation of a stationary moving cluster of vehicles. The kinetics of the formation of this cluster has some characteristic features. In particular, one can see these features clearly if an appearance of a cluster on a road with long enough length $L \gg l$ is investigated [Fig. 2(a)]:

(i) At first, the amplitude of the critical perturbation increases very slowly in time but the previous form of the perturbation (9a) is gradually deformed [Fig. 2(b),

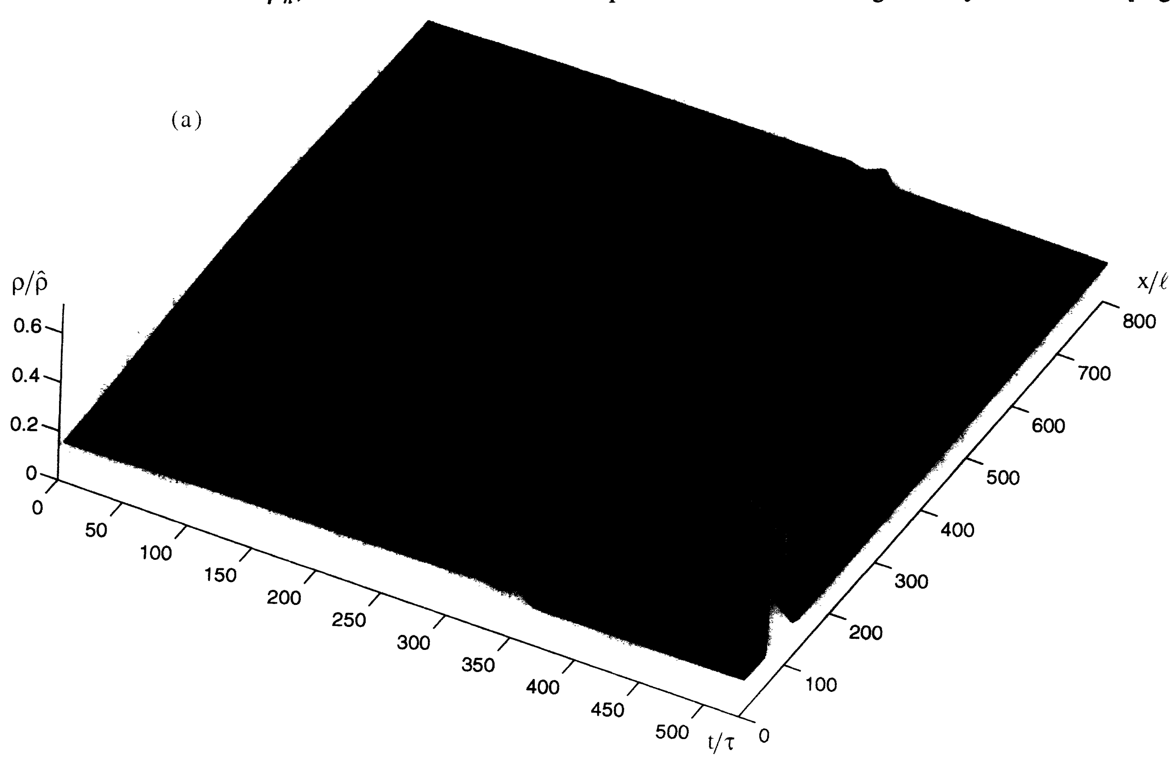

FIG. 2. The kinetics of the cluster formation: (a) the dependence $\rho(x,t)$, (b)–(e) the distributions $\rho(x)$ and $v(x)$ in the intermediate moments of time [(b) $t_1=100\tau$, (c) $t_2=150\tau$, (d) $t_3=216\tau$, (e) $t_4=300\tau$]; (f) the distributions $\rho(x)$ and $v(x)$ in the stationary cluster (at $t=490\tau$), which moves with the constant velocity $v_g^m\cong-1.09 l/\tau$ [for a visual demonstration, the functions $\rho(x)$ and $v(x)$ in (f) are shifted to the center of the road]. The initial distribution $\rho(x,0)=\rho_h+\delta\rho(x,0)$ with $\delta\rho$ (9a), $\delta\rho_0=0.02$; $\rho_h=0.174\hat{\rho}$. The other parameters are the same as in Fig. 1.

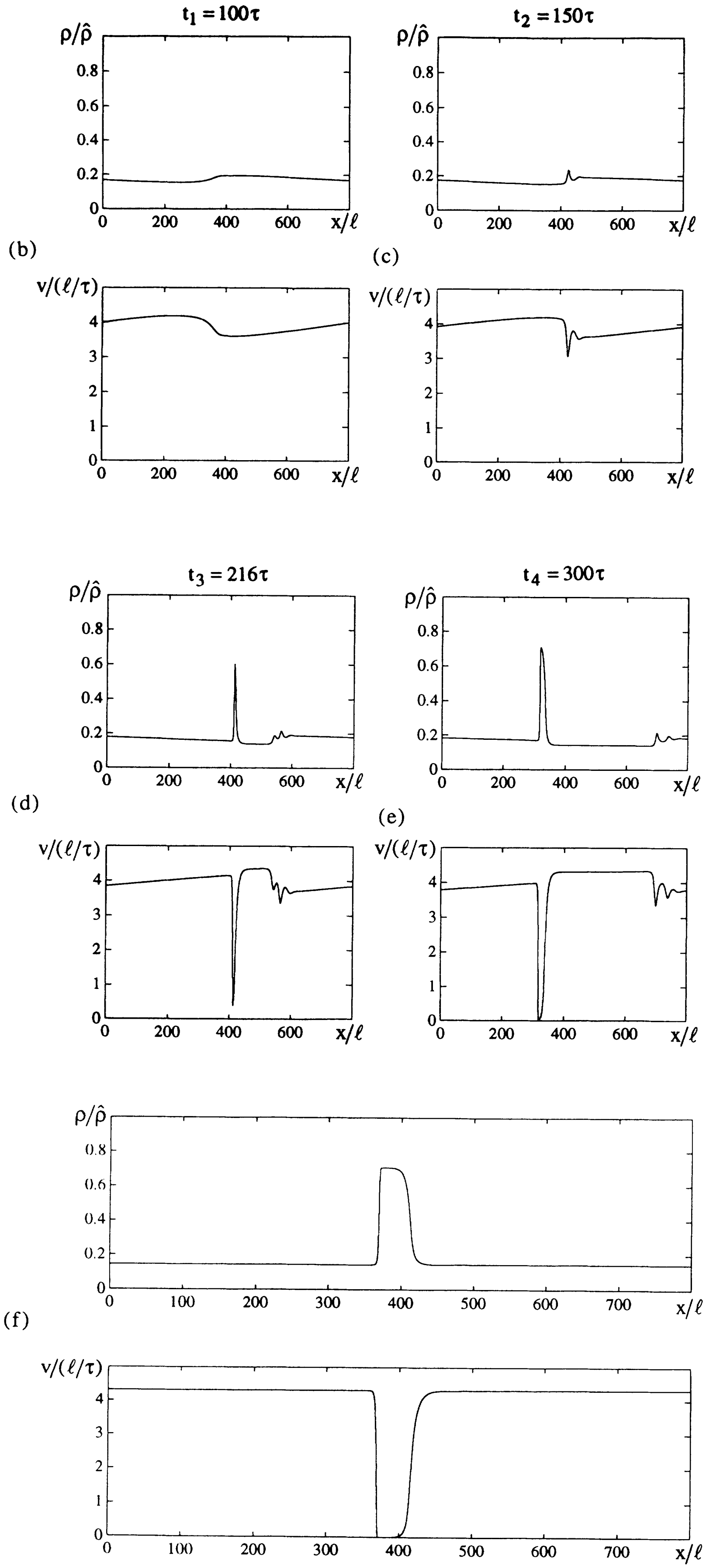

FIG. 2 *(Continued).*

$t = 100\tau$]. In the (p,q) phase plane, the considered distributions are, however, practically disposed on the fundamental diagram [Fig. 3(a)].

(ii) At some time in the vicinity of the maximum of this distribution $\rho(x)$, a local perturbation of density and average velocity is formed by itself. This local perturbation has a shape of the following one after another local region with larger and lower density and, consequently, lower and larger average velocity of vehicles [Fig. 2(c)]. On the (ρ,q) phase plane this local perturbation corresponds to a curve which sharply deviates from the fundamental diagram [Fig. 3(b)].

(iii) The further development of the distributions of the density and of the average velocity of vehicles on the whole road is almost entirely governed by the growth of the amplitude of this self-formed local perturbation. Indeed, the growth of the amplitude of the local perturbation represents an avalanchelike process [Fig. 2(a), $t \approx (140-180)\tau$] as the amplitude of the distributions $\rho(x)$ and $v(x)$ outside of the local perturbation changes insignificantly. As a result, a *local cluster of vehicles* is formed by itself, i.e., a cluster which is surrounded downstream as well as upstream by only slightly perturbed homogeneous traffic flow [Figs. 2(d) and 2(e)]. The cluster represents a local region of large density and low average velocity of vehicles (up to a standstill), followed downstream by a local region of low density of vehicles. The more the amplitude of the local cluster increases, the more the deviation of the curve, which corresponds to this cluster on the (ρ,q) phase plane, from the fundamental diagram grows [Fig. 3(c)].

(iv) The region of the large density of vehicles in this local cluster moves upstream. Contrary to it, the transition layer between the region of low density of vehicles in the local cluster and the slightly disturbed homogeneous flow downstream moves in the opposite direction. It means that between the region of the large density and the transition layer a region of relative low density is formed, whose width quickly increases in time [Fig. 2(a), $t \approx (250-290)\tau$].

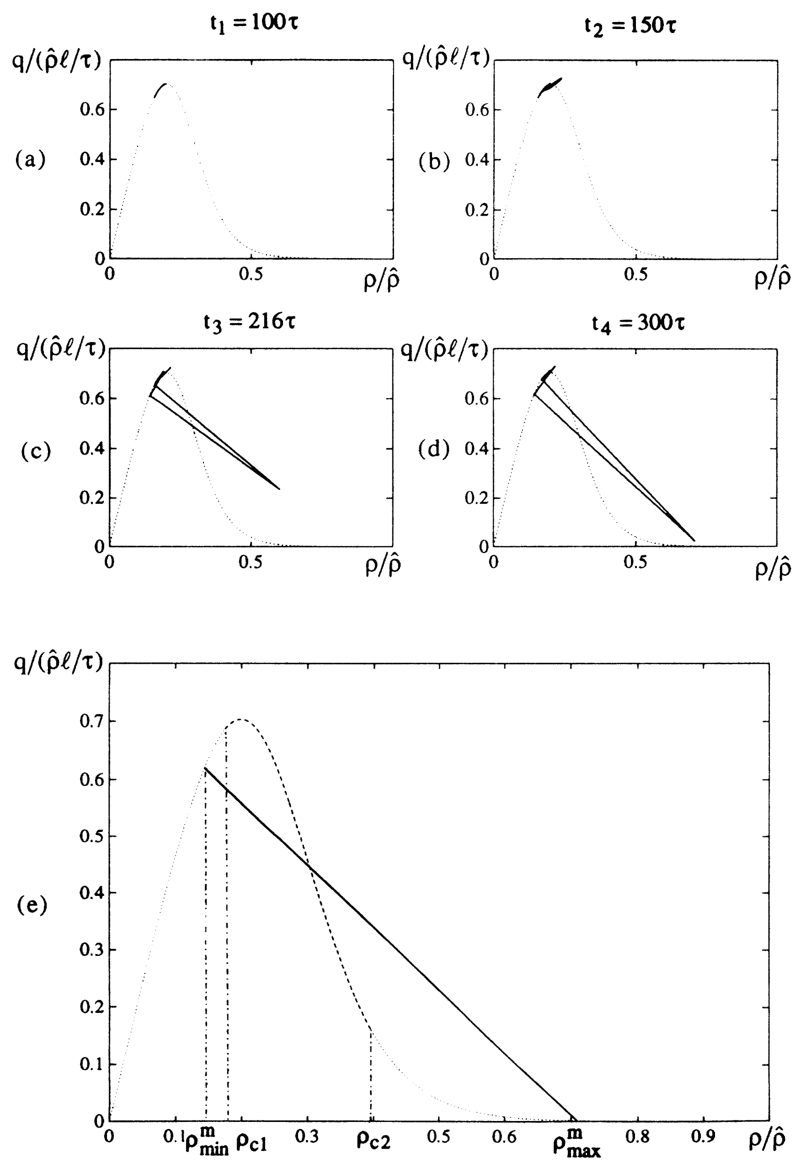

FIG. 3. The kinetics of the cluster formation in the (ρ,q) phase plane: (a)–(d) the curves $q(\rho)$ at the same moment of time as shown in Figs. 2(b)–2(e), respectively, and (e) for the stationary moving cluster in Fig. 2(f). The other parameters are the same as in Figs. 1 and 2. Dotted lines represent the fundamental diagram from Fig. 1(b), the dashed line in (e) corresponds to the region of the density, where the homogeneous traffic flow is unstable.

(v) When the width of the latter region becomes wide enough, the density and the average velocity of vehicles in this region of a low density of vehicles do not change in space, practically, and in time anymore. So, the local cluster of vehicles transforms into a local structure which is surrounded by the slightly disturbed initial traffic flow [blue depicted in Fig. 2(a)]. This local structure consists of three parts: (a) A *proper cluster* of vehicles [green, yellow, and orange depicted in Fig. 2(a)]. (b) The following (downstream) *region of a new almost homogeneous flow* (red) which is caused by this cluster. (c) The *transition layer* between this region and the slightly disturbed initial traffic flow [Fig. 2(e)]. In the (ρ,q) phase plane, the local cluster corresponds to the curve which has roughly the shape of a "triangle" [Fig. 3(d)]. Two sides of this triangle represent the left (upstream) and the right (downstream) fronts of the cluster. The third side represents the distributions $\rho(x)$ and $v(x)$ in the transition layer.

(vi) By the time $t\approx 300\tau$, the amplitude of the local cluster reaches the almost invariable value ($\rho^m_{\max}\cong 0.709\hat{\rho}$) [Figs. 2(a) and 2(e)]. The width of the cluster monotones increases in time as the left (upstream) front of the cluster moves with a higher negative velocity ($v_{gl}\approx -1.22l/\tau$) compared to the right (downstream) front ($v_{gr}\approx -1.05l/\tau$). Therefore, the flux q of vehicles through the left front of the cluster is stronger than through the right front. This means that the cluster collects more and more vehicles and acts as some new moving local source of vehicles on the road which forms behind it (downstream) a new almost homogeneous traffic flow with smaller density [red in Fig. 2(a)]. For this reason, downstream from the cluster moves a transition layer with positive velocity ($v_w\approx 1.53l/\tau$) between the slightly disturbed initially homogeneous traffic flow (blue) and the new homogeneous traffic flow (red) formed by the cluster of vehicles.

(vii) By the time $t\approx 330\tau$ the transition layer [see (v)] reaches the boundary of the road ($x=L$). Due to the periodic boundary conditions (4) used in the model, this transition layer consequently appears at the upstream boundary of the road ($x=0$). Later on, the transition layer reaches the cluster and gradually merges with it [Fig. 2(a), $t\approx(400-430)\tau$]. As a result, the stationary moving cluster on the circular road under consideration is formed [Fig. 2(a), $t>450\tau$]. The form and the velocity of this cluster do not change in time anymore.

2. *Structure of stationary moving cluster*

The stationary moving cluster formed represents a local region of large density ($\rho^m_{\max}$) and low average velocity of vehicles [$v^m_{\min}\cong V(\rho^m_{\max})$], where the traffic flow is almost homogeneous [as $v^m_{\min}\cong 0$, it is practically at a standstill; Fig. 2(f)]. This region is limited upstream and downstream by two fronts, where both the density and the average velocity of vehicles sharply change in space. These fronts move upstream with the same velocity (v^m_g) equal to the velocity of the cluster. The cluster is surrounded by another homogeneous traffic flow, where the density is lower and the average velocity of vehicles is higher than inside the cluster [Fig. 2(f)]. Both the homogeneous flows inside and outside the cluster are different from the initially homogeneous traffic flow. These new homogeneous flows produced by the cluster are stable, i.e., the density of vehicles inside the cluster is $\rho^m_{\max}>\rho_{c2}$ and outside the cluster it is $\rho^m_{\min}<\rho_{c1}$.

In the (ρ,q) phase plane, the stationary moving cluster corresponds to a segment of a line. The ends of this segment correspond to the stable homogeneous flows inside and outside the cluster. For this reason they are disposed on the fundamental diagram and outside the interval (ρ_{c1},ρ_{c2}) of the instability of traffic flow [Fig. 3(e)].

C. Local clusters in homogeneous traffic flow

From the formation process of the stationary cluster discussed above one can have a conclusion of a decisive role of a local perturbation which is formed by itself in the process of the evolution of the initially "global" perturbation on the whole road (9a). This self-formed local perturbation represents, in fact, two local regions, following downstream, one after another: A region of large and a region of low density of vehicles (Fig. 4).

From these properties of the self-formed local perturbation it can be expected that if, instead of the global per-

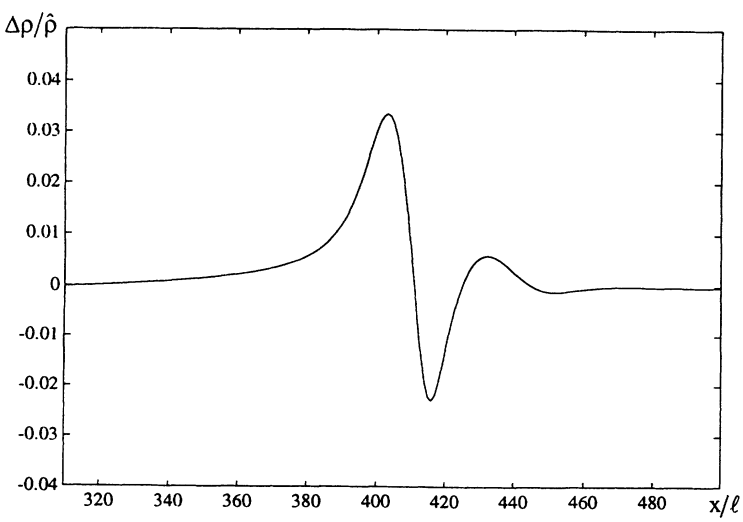

FIG. 4. The shape of the self-formed local perturbation $\Delta\rho(x)$: The difference between two distributions of the density of vehicles in Fig. 2(a), $\Delta\rho(x)=\rho(x,140\tau)-\rho(x,130\tau)$. The chosen moments of time, $t=130\tau$ and $t=140\tau$, are the moments before and after the local perturbation is formed by itself, respectively.

turbation (9a), *only* a local perturbation, which has a form similar to that shown in Fig. 4, appears, then a local cluster of vehicles *which is surrounded by the initially homogeneous flow* can be spontaneously formed on a road that is long enough. Until the local cluster reaches one of the boundaries of the road, the latter can be considered as an "open" system.

1. Local cluster in stable homogeneous flow

This supposition about an occurrence of the local cluster is confirmed by numerical analysis, where it was also found that a local cluster can appear in a stable [with respect to a global perturbation of small amplitude (9a)] homogeneous traffic flow, i.e., when $\rho_h < \rho_{c1}$ (Fig. 5).

In the numerical analysis made (Fig. 5), this stable initial homogeneous traffic flow is disturbed at $t=0$ by the local perturbation

$$\Delta\rho(x)=\Delta\rho_m\{\cosh^{-2}[0.2(x-x_0)] -0.25\cosh^{-2}[0.05(x-25l-x_0)]\} . \tag{11}$$

This perturbation has the form similar to the one of the self-formed local perturbations shown in Fig. 4. If the amplitude $\Delta\rho_m$ of the local perturbation (11) exceeds some critical value $\Delta\rho_c$, the amplitude of the initial local perturbation (11) grows in time and a local cluster appears on a road. The shape and the properties of the local cluster formed do not depend on the amplitude of this local perturbation. For the case shown in Fig. 5, this critical amplitude is $\Delta\rho_c \cong 0.06\hat{\rho}$. On the contrary, if the amplitude of the initial local perturbation $\Delta\rho_m$ is lower than $\Delta\rho_c$, the amplitude of the initial local perturbation fades in time. Notice that a *local cluster effect,* i.e., the appearance of a local cluster of vehicles surrounded upstream and downstream by the initially homogeneous flow, can only be realized on a long road—a road which is long enough so that a region of the localization of the cluster is less than the length of the road.

The local initial perturbation of the density of vehicles, which is situated at $t=0$ on the distance $x_0=250l$ from the beginning of the road [Fig. 5(a)], first moves in the direction of the flow with only slightly increasing amplitude. After some time ($t\simeq 35\tau$), this local perturbation stops and its amplitude begins to grow rapidly, forming the cluster of vehicles of large amplitude [green, yellow, and orange depicted in Fig. 5(a)] moving into the opposite direction to the initially homogeneous traffic flow (blue). By the time $t\approx 140\tau$, the amplitude of the cluster reaches the almost invariable value ($\rho_{\max}\cong 0.709\hat{\rho}$) but the width of the cluster L_s, i.e., the distance between the cluster's fronts, where the density and average velocity sharply change in space, increases in time. It occurs because the left (upstream) front of the cluster moves with a higher negative velocity ($v_{gl}\approx -1.22l/\tau$) compared to the right (downstream) front ($v_{gr}\approx -1.06l/\tau$). The cluster forms downstream from it a new almost homogeneous traffic flow with lower density ($\rho_{\min}\cong 0.14\hat{\rho}$) [red in Fig. 5(a)]. For this reason, downstream from the cluster

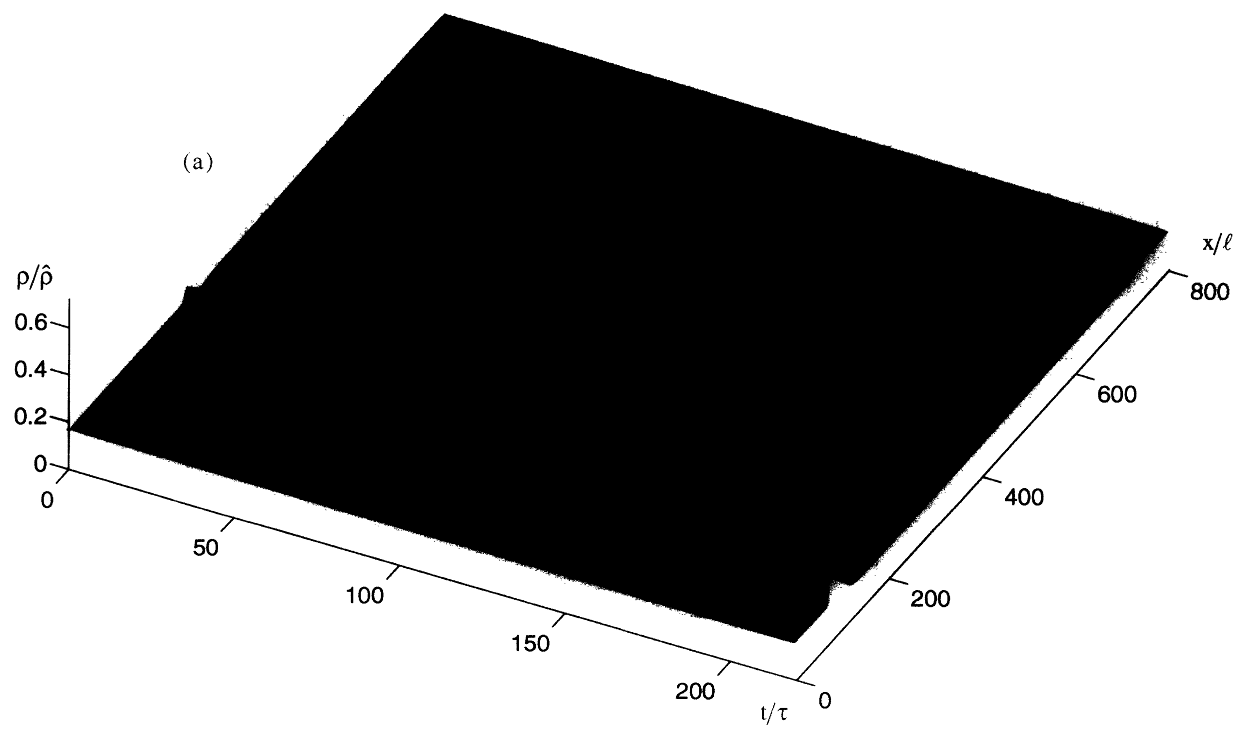

FIG. 5. The kinetics of the local cluster formation in the stable homogeneous traffic flow: (a) the dependence $\rho(x,t)$; (b) and (c) the distributions of the functions $\rho(x)$ and $v(x)$ (b) and the corresponding curve $q(\rho)$ (c) at $t=144\tau$; (d) the vehicle trajectories corresponding to (a). $\rho_h=0.17\hat{\rho}$. The initial local perturbation $\Delta\rho(x)$ (11), $\Delta\rho_m=0.06\hat{\rho}$, $x_0=250l$. The other parameters are the same as in Fig. 1.

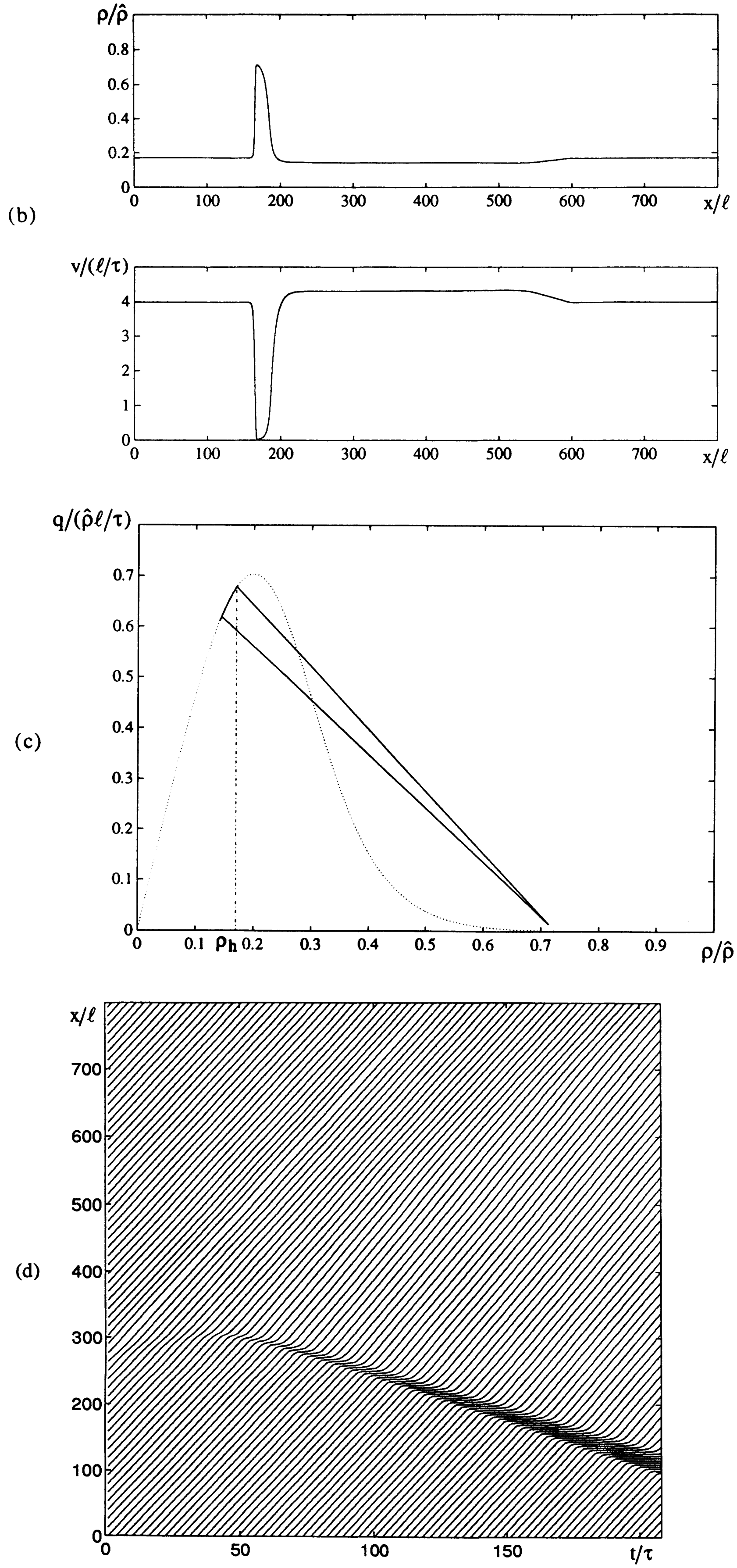

FIG. 5 *(Continued).*

moves a transition layer with positive velocity ($v_w \approx 2.3 l/\tau$) between the initially homogeneous traffic flow (blue) and the new homogeneous traffic flow (red) formed by the cluster of vehicles. In the (ρ,q) phase plane, the local cluster represents a "triangle" [Fig. 5(c)]. Two sides of this triangle represent the upstream (upper line with negative slope) and the downstream fronts (lower line with negative slope) of the cluster. The third side represents the transition layer.

When the width of the cluster L_s is large enough [$t > 180\tau$, Fig. 5(a)], the average velocity of vehicles in the cluster $v_{\min} \cong 0$ and consequently the flux $q_{\min} = v_{\min}\rho_{\max} \cong 0$, i.e., the cluster corresponds practically to a standstill. At the upstream front of the cluster vehicles are entering the cluster and must stop. Therefore the upstream front is moving with the velocity v_{gl} against the flow. At the downstream front, the drivers realize the low density region in front of them. They can accelerate and leave the cluster. Therefore the downstream front of the cluster moves with the velocity v_{gr} against the flow. Taking into account Fig. 5(c) and that in the cluster the flux $q_{\min} \cong 0$, one can write the approximate formulas for the velocities of the cluster's fronts:

$$v_{gr} = -q_{\mathrm{out}}(\rho_{\max} - \rho_{\min})^{-1}, \quad v_{gl} = -q_h(\rho_{\max} - \rho_h)^{-1}$$

where $q_{\mathrm{out}} = v_{\max}\rho_{\min}$ is the flux directly downstream from the cluster, i.e., the flux out of the cluster; $\rho_{\min}$ and $v_{\max}$ are the density and the average velocity of vehicles in the new homogeneous traffic flow [red depicted in Fig. 5(a)] formed by the cluster; $q_h = v_h\rho_h$. As $\rho_h > \rho_{\min}$ [Fig. 5(a)] and consequently $q_h > q_{\mathrm{out}}$, from these formulas it follows that $|v_{gl}| > |v_{gr}|$, corresponding to the results of the numerical calculations.

Let us compare Fig. 2(a) for the time interval between the self-formation of the local perturbation and the local cluster [$t \approx (140-315)\tau$] with Fig. 5(a). One can draw the conclusion that both kinetics of cluster formations are very similar to one another. There is only one qualitative difference between these two processes: Outside the local cluster in Fig. 2(a) there is a slightly nonhomogeneous flow which appears due to the disturbance of the initially homogeneous traffic flow by the global perturbation (9a). In Fig. 5(a), however, there is the undisturbed initially homogeneous flow. A similar conclusion follows from the comparison of Fig. 2(e) with Fig. 5(b) and Fig. 3(d) with Fig. 5(c).

Thus the local cluster of vehicles is a nonstationary localized structure which is surrounded by the initial homogeneous traffic flow. As in the kinetics of the formation of the stationary moving cluster in Fig. 2(a), this local structure consists of the following: (a) the proper cluster of vehicles (as of now called "cluster") moving upstream [green, yellow, and orange depicted in Fig. 5(a)]; (b) the new almost homogeneous traffic flow (red) formed by the cluster; and (c) the transition layer between this new and the initially homogeneous flows moving downstream [Fig. 5(b)].

As in the example considered, the cluster of vehicles and the transition layer between the two different homogeneous flows move in opposite directions and the width of a region of localization of the whole nonstationary structure increases in time. However, on the long road the drivers upstream of the cluster and downstream from the transition layer move in the initially homogeneous traffic flow [blue depicted in Fig. 5(a)] and therefore do not know about the existence of the local cluster described.

The same conclusion can be drawn from the consideration of the vehicle trajectories on the (t,x) plane [Fig. 5(d)], corresponding to the local cluster shown in Fig. 5(a). One can see in Fig. 5(d) that the driver which has caused the local perturbation at $t=0$ and $x=250l$ can always travel with high velocity in the initially homogeneous flow. Indeed, at time $t \simeq 35\tau$ when the local perturbation stops at position $x \cong 300l$ and its amplitude begins to grow rapidly forming the traffic jam, this driver is at position $x \simeq 416l$ and is therefore far away from the traffic jam he is responsible for. On the contrary, drivers which have been at $t=0$ at positions below $x \cong 100l$ must reduce their velocity sharply at the time when they reach the traffic jam. The width of the traffic jam increases during the course of time. For this reason, the later a driver approaches the jam, the longer he has to wait until the traffic jam, due to its negative velocity, has passed him. This fact can be seen in Fig. 5(d), where the vehicle trajectories are considerably denser in the cluster moving upstream, where also the velocity of vehicles, i.e., the slope of the vehicle trajectories, is practically equal to zero. Such a situation is usual for many experimental observations of a "phantom traffic jam" on a long road without any off and on ramps, far from other heterogeneities, i.e., under "pure" conditions, which correspond to the model under consideration.

2. *Physics of appearance of local cluster: Process of random appearance and disappearance of local clusters*

In order to understand the physics of the self-formation of the local cluster discussed above, let us now qualitatively consider the behavior of a local increase $\Delta\rho(x) > 0$ on the density in time. In this local region, the value V decreases: $\Delta V(x) = [dV(\rho)/d\rho]\Delta\rho(x) < 0$, as $dV/d\rho < 0$ (5). This decrease of the "maximum and out of danger" velocity V forces drivers to decrease their average velocity v sharply if the value $|dV/d\rho|$ is large enough. On the other hand, from continuity Eq. (1), it follows that a local decrease in v is accompanied by a local increase in ρ and vice versa. Therefore, the decrease in v in the local region under consideration causes a further increase in the local value of ρ in this region and, therefore, leads to further subsequent decreases in the values V and v and also to an increase in the value ρ in this local region. This avalanchelike process of the decrease in the average velocity and the corresponding increase in the density of vehicles in the local region of traffic flow one can consider as some kind of active process, which causes the spontaneous appearance of local cluster in traffic flow.

On the other hand, the local increase considered above in the density and the corresponding local decrease in the average velocity could disappear due to the diffusion (viscosity) process, influence of the gradient of pressure, and also to the relaxation process of the average velocity v to the "maximum and out of danger" velocity V. Therefore, there is a competition between these two kinds of processes: the *active processes,* trying to increase the amplitude of the local perturbation, and the *damping processes,* acting in the opposite direction.

The active process becomes weaker if the density ρ_h decreases. Indeed, for low density the interactions between vehicles seldom occur and the value $|dV/d\rho|$, which characterizes the strength of the active process, is small [Fig. 1(a)]. For this reason, there should be some boundary value of the density ρ_b, i.e., in a homogeneous flow the boundary flux $q_b=V(\rho_b)\rho_b$. At $\rho_h<\rho_b$ (i.e., at $q_h<q_b$), a local perturbation of any amplitude fades in time, i.e., a traffic jam cannot develop (Sec. II C 3).

The avalanchelike process of the growth of an amplitude of the local perturbation in the traffic flow discussed above can be observed from Fig. 5(a) at $t\approx 45\tau$. From the integral condition (3) it follows that downstream from the local region under consideration, where the density is higher, there should be a consequent decrease in the density of vehicles on the road. This consequence can be seen in Fig. 5(b). It also follows from (3) that the localized perturbation at $t=0$, which leads to the self-formation of the local cluster, should be the following one after another a local increase and a local decrease in the density that is taken into account in (11).

As the amplitude of the cluster in the course of time increases, the average velocity of vehicles decreases more and more inside the cluster, practically to a standstill [Fig. 5(b)]. As a result, in the course of time, the cluster "stores" more and more vehicles. This means that the flux of vehicles q_{out} out of the cluster (downstream) should be less than the flux q_{in} into it (upstream), where $q_{\text{in}}=q_h$. It also becomes clear why downstream the cluster the new almost homogeneous flow develops, where the density $\rho_{\min}$ is lower than in the initial flow ρ_h (Fig. 5). As $q_{\text{in}}>q_{\text{out}}$, the width of the cluster L_s monotonously increases in time [Figs. 5(a) and 5(d)].

On the other hand, from this qualitative consideration it follows that if upstream from the cluster the density $\rho<\rho_{\min}$, the local cluster should disappear in the course of time. Indeed, in this case $q_{\text{in}}<q_{\text{out}}$ and, consequently, the width of the cluster L_s should monotonously decrease. Therefore, the boundary density ρ_b of an excitation of the local cluster should be close to the density $\rho_{\min}$ in the new flow formed by the cluster directly downstream from the cluster: $\rho_b\cong\rho_{\min}$. Consequently, the boundary flux q_b of an excitation of the local cluster is close to the flux q_{out}: $q_b\cong q_{\text{out}}$. These conclusions from the qualitative consideration of the physics of the local cluster formation is supported by the results of the numerical calculations (Sec. II C 3).

Let us stress the possible effect of random appearance and disappearance of local clusters on a long road if an initial traffic flow is nonhomogeneous. Such an effect can be observed if the density in the regions, where the traffic flow is dense, is higher than ρ_b and the density in the diluted regions is lower than ρ_b. If the extension of these regions of different density is large enough to consider traffic flow inside each of them as almost homogeneous, local clusters, as has been described above, can spontaneously appear in regions of high density, where $\rho>\rho_b$. When a cluster, which moves against the flow [see Fig. 5(a)], finds itself in a region of low density, where $\rho<\rho_b$, it will disappear if the extension of the region of low density is large enough. Indeed, the flux of vehicles q_{out}, formed by the cluster downstream, exceeds the flux in front of cluster q_{in} in the region with $\rho<\rho_b$ and therefore the width of the cluster L_s gradually decreases.

Qualitatively the same process of random appearance and disappearance of local clusters can be realized on a multilane road even if the flow is initially homogeneous on each lane, but the density on one lane is $\rho>\rho_b$ and on the other lanes the densities are $\rho<\rho_b$. Under those circumstances a local cluster can spontaneously appear on the lane with high density $\rho>\rho_b$ and then can disappear in the course of time due to lane changing of vehicles to the other lanes with low densities upstream of the cluster or/and inside the cluster. The other possibility of the process of random appearance and disappearance of local clusters on a long road, which can be realized in traffic flow, will be discussed in Sec. II C 4.

3. *Boundary of excitation of local cluster*

From the numerical analysis it is found that the lower the density of vehicles in the initially homogeneous flow ρ_h (i) the more the amplitude of the initial local perturbation $\Delta\rho_m$ in (11) needs a local cluster to be excited [Fig. 6(a)]; (ii) the less the density of vehicles, which is formed directly behind (downstream) the cluster, differs from the value ρ_h [Fig. 6(c)]. Consequently, the segment of a line on the (ρ,q) phase plane, which corresponds to the downstream front of the cluster [lower line with negative slope in Fig. 6(e)], becomes closer to the segment of a line for the upstream front of the cluster [upper line with negative slope in Fig. 6(e)].

Because the flux out of the cluster q_{out} at $\rho_h>\rho_b$ is always less than the flux into it q_{in}, its width increases during the course of time. However, its amplitude is limited and does not exceed some maximum value, which coincides practically to the value $\rho_{\max}^m$ for a wide stationary moving cluster. Because the length of the road is finite, there is not enough time for the full development of this maximum amplitude, if the density of the initial homogeneous flow ρ_h is only slightly higher than the boundary density ρ_b. For this reason, for a given length of the road L, the amplitude of the local cluster $\rho_{\max}$ is an increasing function of ρ_h [Fig. 6(b)], as long as ρ_h is below some value. The latter is a decreasing function of the length L and tends to ρ_b if $L\to\infty$. This means that in the limit $L\to\infty$ the amplitude of the local cluster can reach the maximum value $\rho_{\max}\cong\rho_{\max}^m$ for all $\rho_h>\rho_b$.

As has been mentioned, there is a boundary value of density ρ_b [Figs. 6(a) and 6(b)]: At $\rho_h<\rho_b$, a local cluster cannot be excited independently on the amplitude of an initial local perturbation (11). The boundary value ρ_b is less than the critical value ρ_{c1} which determines the boundary of the stability of the initially homogeneous flow with respect to the global critical perturbation (9a) of a small amplitude.

At $\rho_h = \rho_b$, the formed cluster has the smallest width $L_b = L_s(\rho_b)$ and the lowest possible amplitude [Fig. 6(b)]. The density of vehicles in the new almost homogeneous flow formed directly behind (downstream) this cluster $\rho_{\min}$ practically coincides with ρ_h. Because of that, the segments of a line on the (ρ, q) phase plane, which correspond to the first and the second fronts of the cluster, practically coincide with one another [Fig. 6(f)] and the transition layer almost disappears. As a result, the structure of the local cluster represents the proper cluster, moving upstream and the "anticluster," moving downstream [Fig. 6(d)]. The total increase in the number of vehicles in the cluster is compensated for by the decrease in the number of vehicles in the anticluster, as is required by the integral condition (3).

4. *Critical amplitude of local perturbation*

As has been mentioned above, there is the boundary density of vehicles in the initially homogeneous flow ρ_b for the existence of a local cluster in this flow. The critical amplitude of a local perturbation (11) $\Delta\rho_c$, which is needed for an appearance of a local cluster, is maximal at this critical point (Sec. II C3).

There is also some other critical value ρ_{cr} which is higher than ρ_{c1}. At $\rho_h > \rho_{cr}$ any amplitude of the local perturbation, which has an arbitrary small initial amplitude, grows on the long road in time. This means that the critical amplitude of the local perturbation (11) is a monotonously falling function of the values ρ_h in the interval $\rho_b \leq \rho_h \leq \rho_{cr}$ [Fig. 6(a)]. Therefore, in the interval $\rho_b < \rho_h < \rho_{cr}$ the homogeneous traffic flow is in a "meta-

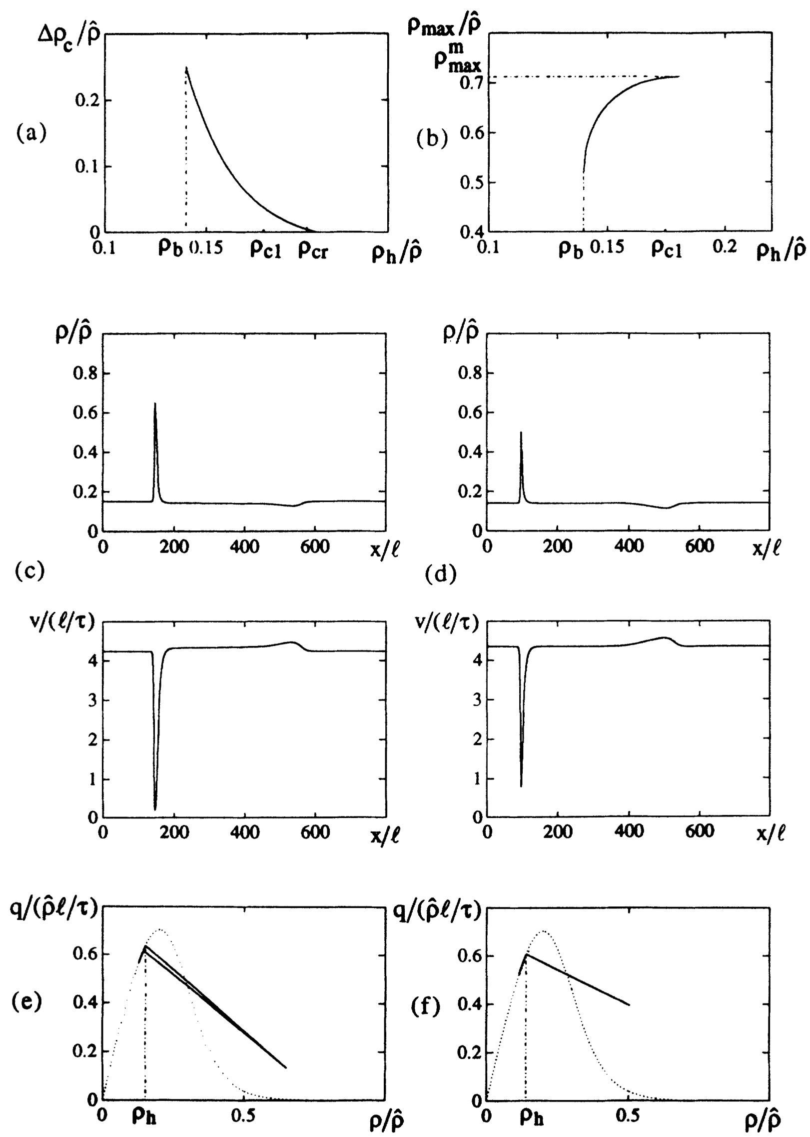

FIG. 6. The characteristics of the local cluster: (a) the critical amplitude of the initial local perturbation $\Delta\rho_c$, (b) the maximal density in the formed local clusters $\rho_{\max}$, (c) and (d) the distributions $\rho(x)$ and $v(x)$ as the functions of the density ρ_h, (e) and (f) the curves $q(\rho)$ for the local clusters shown in (c) and (d), respectively. The initial local perturbation $\Delta\rho(x)$ (11); $\rho_h = 0.15\hat\rho$, $\Delta\rho_m = 0.2\hat\rho$, $x_0 = 220l$ for (c) and (e) and $\rho_h = 0.14\hat\rho$, $\Delta\rho_m = 0.25\hat\rho$, $x_0 = 150l$ for (d) and (f); $t = 90\tau$. The other parameters are the same as in Fig. 1. The found characteristic values are $\rho_b \cong 0.14\hat\rho$, $\rho_{cr} \cong 0.2\hat\rho$, $\rho^m_{\max} \cong 0.709\hat\rho$.

stable" state: a random appearance of a localized critical fluctuation (whose amplitude exceeds a critical value) causes an appearance of a local cluster of vehicles.

On the other hand, a random appearance of a fluctuation, whose amplitude is large enough, can cause a disappearance of this cluster and a return of traffic flow into the homogeneous state, as the latter is stable in the interval of density under consideration with respect to local fluctuations of a small amplitude. In other words, there are different "metastable" states of traffic flow in the interval $\rho_b < \rho_h < \rho_{cr}$, which are not stable with respect to local perturbations of large enough amplitude: (a) the homogeneous traffic flow and (b) a nonhomogeneous state of traffic flow with one or a few local clusters.

It is known from the investigations of many nonlinear systems, due to a noise-active escape from a metastable state, that the random transitions between different metastable states are possible (e.g., [8]). For the case under consideration this means that in the interval of density $\rho_b < \rho_h < \rho_{cr}$ [Fig. 6(a)] a process of random appearance and disappearance of different local clusters on the road can occur. It is also known (e.g., [8,9]) that an average time of a noise-activated escape from a metastable state is decreased if the dispersion of fluctuations in the system is increased and also if the value of a bifurcation parameter of the system comes nearer to the critical points. These critical points are ρ_{cr} and ρ_b for the homogeneous state of traffic flow and for a state with a local cluster, correspondingly. Therefore, if the interval (ρ_b, ρ_{cr}) is not very wide and the dispersion of fluctuations in traffic flow is large enough, one can suppose that there is a considerable probability to find in this interval of density a process of random appearance and of disappearance of different local clusters on the road.

5. *Structure of transition layer*

For the values ρ_h, which are appreciably less than the critical value ρ_{c1} (10c), the transition layer between the new homogeneous flow formed by the cluster and the initial homogeneous flow has a simple structure [Fig. 5(b)]. This transition layer is represented on the (ρ, q) phase plane by a curve which practically coincides with the fundamental diagram [Fig. 5(c)].

The kinetics of appearance of the local cluster shown in Fig. 5(a) correspond to $\rho_h < \rho_{c1}$. As the value ρ_h approaches the value ρ_{c1}, exceeds it and increasingly approaches ρ_{cr}, the kinetics of self-formation of a local cluster disclose qualitatively new peculiarities.

If the value ρ_h increases and approaches the critical value ρ_{c1} (10c), the form of the transition layer changes qualitatively: A spatial oscillation of the density and the average velocity appears at that place [Fig. 7(a)]. The largest peak of density ρ is, as a rule, next to the cluster [Fig. 7(a)], but can also change its position in the transition layer in time. Nevertheless, the peaks of ρ in the spatial oscillation finally fade in space (downstream) [Fig. 7(a)]. When the oscillation in the transition layer appears, the curve, which represents this transition layer on the (ρ, q) phase plane, becomes longer and can jut out the maximal value q on the fundamental diagram [dotted line in Fig. 7(b)] which is defined only for homogeneous flows. Notice that, as expected, the qualitatively spoken same oscillation appears at $\rho_h = \rho_{c1}$ in the kinetics formation of the stationary cluster for the time interval from $t \approx 150\tau$ to $t \approx 300\tau$, i.e., when the local cluster exists on the road [Figs. 2(a), 2(d), and 2(e)].

6. *Reasons of appearance of sequence of clusters in traffic flow*

The higher the value ρ_h, the larger the amplitude of the peaks in the oscillations in the transition layer. It is clear from here that, at some value ρ_h, the amplitude of the largest peak can reach a critical value in the course of time, which is large enough to start the avalanchelike process discussed in Sec. II C3. This process leads to the spontaneous self-formation of a new cluster downstream of the first cluster (Fig. 8). In other words, the largest peak in the distribution of the density in the transition layer between the new almost homogeneous traffic flow formed by the first cluster and the initially homogeneous flow acts as in the initially local perturbation in the flow.

The new (second) cluster develops the next almost homogeneous traffic flow downstream from it (Fig. 8). Although this flow corresponds to a lower density than the density in front (upstream) of the second cluster, the difference between both values of the density is very small. Apparently, because of this reason, this second cluster is the narrower one (Fig. 8).

If the value ρ_h is increased further, the avalanchelike process of cluster formation can start in many peaks in the distribution of the density in the transition layer. Therefore, a sequence of clusters, which appears subsequently in space and time, is created. The clusters in this sequence have, as a rule, different amplitudes, different widths, different velocities, and are not situated periodically in space [Fig. 9(a)]. Some of these clusters [the first and the second, shown in Fig. 9(a)] can catch up with one another [Fig. 9(b)] and then merge into one cluster with a larger width [Fig. 9(c)]. This localized structure of the sequence of clusters on the (ρ, q) phase plane represents the complex picture of many bound segments of lines with different slopes corresponding to the different velocities of the clusters [Fig. 9(d)].

The local clusters shown in Figs. 5–9 appear due to the occurrence of only one local perturbation in traffic flow. As a lot of local perturbations of traffic flow on the long enough road can arise independently, a lot of local clusters can almost simultaneously appear in flow. The possibility of this process of the appearance of a lot of clusters, which can completely cover the flow, is essentially large, when the value ρ_h approaches the value ρ_{cr}, because the value of the critical amplitude of a local perturbation is decreased [Fig. 6(a)]. As in the latter case the clusters can have different amplitudes, different widths, different velocities, and may not situated periodically in space [1], the sequence of such clusters on the (ρ, q) phase plane can represent, as follows from the numerical investigations made, a very complex picture. The corresponding functions $\rho(x_0, t)$ and $q(x_0, t)$, i.e., the density and the flux at

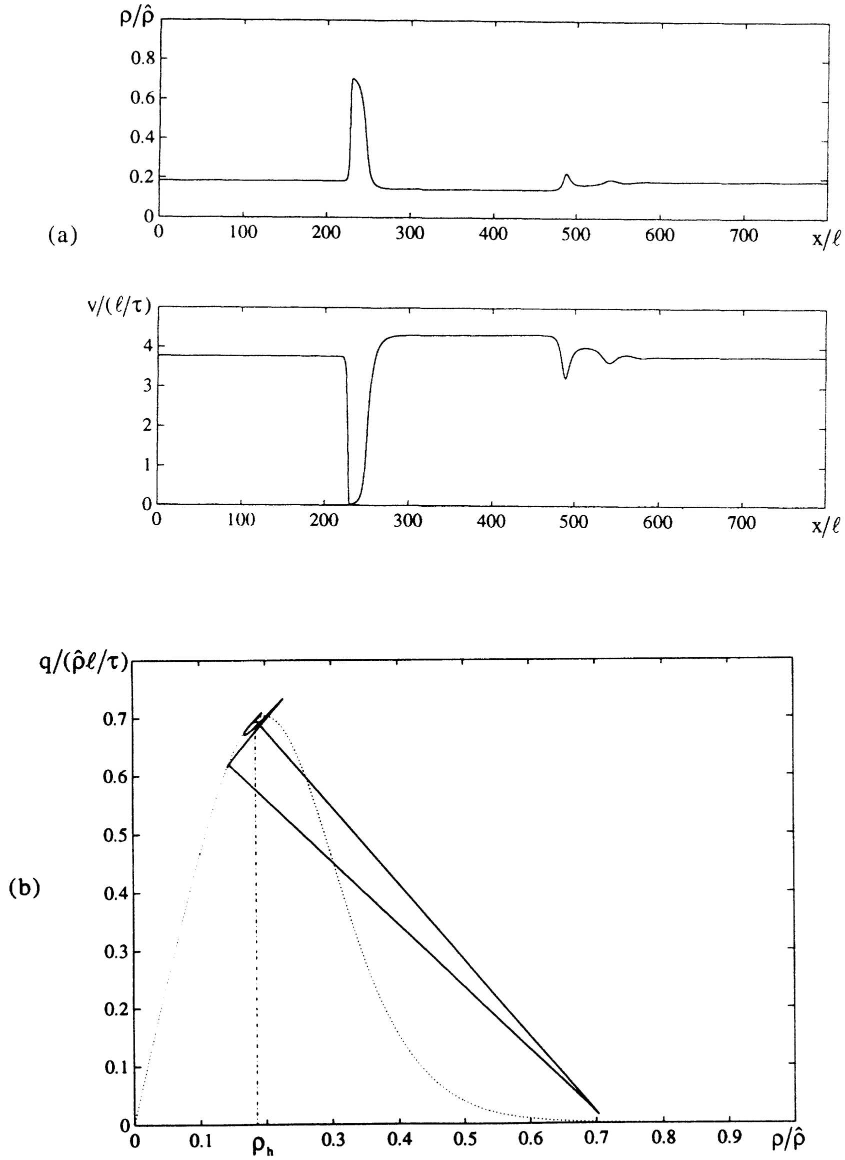

FIG. 7. The space oscillations in the transition layer of a local cluster: the distributions $\rho(x)$, $v(x)$ (a) and the corresponding curve $q(\rho)$ (b) at $t=164\tau$. The initial local perturbation $\Delta\rho(x)$ (11), $\Delta\rho_m=0.02\hat{\rho}$, $x_0=250l$; $\rho_h=0.185\hat{\rho}$. The other parameters are the same as in Fig. 1.

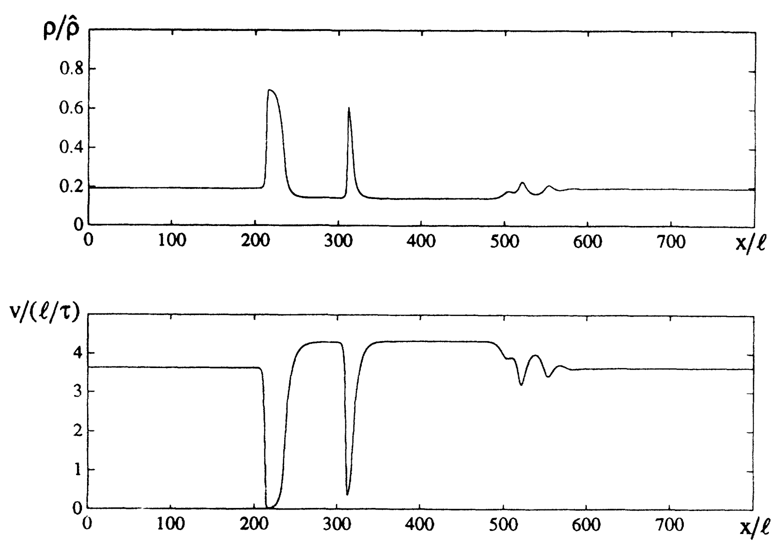

FIG. 8. The appearance of two clusters from the development of the one local perturbation (11): the distributions $\rho(x)$ and $v(x)$ at $t=176\tau$. $\rho_h=0.1935\hat{\rho}$, $\Delta\rho_m=0.01\hat{\rho}$, $x_0=250l$. The other parameters are the same as in Fig. 1.

a fixed point $x=x_0$ on the road, can be very complex functions of time. This agrees with the experimental investigations of traffic flow (e.g., [10]), where for a relative high density of vehicles the large scattering of values of the flux, which have been measured at a fixed point on a road, have been found on the "fundamental diagram." In other words, the chaotic behavior of traffic flow may be linked to the spontaneous appearance in traffic flow of a lot of interacting clusters with different parameters.

III. PARAMETERS OF STATIONARY MOVING CLUSTER

A. Equations for stationary moving cluster

In the case of clusters of vehicles moving along the circular road with a constant velocity v_g, one can introduce in (1)–(4) a self-similar variable $x \Longrightarrow x - v_g t$. In this new system of coordinates the kinetic model of traffic flow (1)–(4) reads as

$$\frac{\partial \rho}{\partial t}+\frac{\partial q^*}{\partial x}=0 , \tag{12}$$

$$\frac{\partial v}{\partial t}+(v-v_g)\frac{\partial v}{\partial x}=V(\rho)-v-\frac{c_0^2}{\rho}\frac{\partial \rho}{\partial x}+\rho^{-1}\frac{\partial^2 v}{\partial x^2} , \tag{13}$$

and the conditions (3), (4), where

$$q^*=\rho(v-v_g) . \tag{14}$$

Here, and as of now v, V, and c_0 are measured in units of l/τ; the length in units of l, the time in units of τ, the density of vehicles in units of $\hat{\rho}$.

For the time-independent functions $v(x)$ and $\rho(x)$, which describe the forms of a stationary moving cluster of vehicles in the new system coordinates, one gets from Eq. (12),

$$\frac{dq^*}{dx}=0 , \tag{15}$$

i.e., a value q^* (14) does not depend on x. It means that on the right-hand side of the formula,

$$\rho(x)=\frac{q^*}{v(x)-v_g} , \tag{16}$$

only the value v depends on x.

This allows one to derive the equations for the time-independent function $v(x)$, substituting the function $\rho(x)$ in (13) and (3) for $q^*[v(x)-v_g]^{-1}$,

$$\frac{d^2 v}{dx^2}+D_0(v(x),q^*,v_g)\frac{dv}{dx}+F(v(x),q^*,v_g)=0 \tag{17}$$

and

$$\int_0^L \frac{dx}{v(x)-v_g}=\frac{\rho_h L}{q^*} , \tag{18}$$

where

$$D_0(v(x),q^*,v_g)=\frac{q^*\{c_0^2-[v(x)-v_g]^2\}}{[v(x)-v_g]^2} , \tag{19}$$

$$F(v(x),q^*,v_g)=\left[V\left[\frac{q^*}{v(x)-v_g}\right]-v(x)\right]\times\left[\frac{q^*}{v(x)-v_g}\right] . \tag{20}$$

Equations (17) and (18) with the boundary conditions

$$v(0)=v(L),\quad dv/dx|_0=dv/dx|_L , \tag{21}$$

which follow from (4), pose an eigenvalue problem whose spectrum defines the possible values of velocities v_g and whose eigenfunctions $v(x)$ corresponding to (16) determine the form of different clusters $\rho(x)$ which can appear in the initially homogeneous flow.

The solutions $v(x)$ of Eq. (17) can be regarded as trajectories of classical "particles" which move with "time" x on the (v,w) phase plane, where

$$w=\frac{dv}{dx} . \tag{22}$$

Multiplying (17) by w (22), integrating it along the trajectory of a "particle," and taking into account the boundary conditions (21), one can derive the condition, which the solutions in the shape of a cluster of vehicles should satisfy:

$$\oint D_0(v)w(v)dv=0 .$$

In addition, the parameters of stationary moving clusters, which have been found from the numerical solution of the problem (1)–(4), have been proved by the help of the numerical solution of the problem (17)–(21). For this purpose, the unknown function $\tilde{n}(x)$, where $d\tilde{n}/dx=[q^*/(v-v_g)]-\rho_h$, was introduced and the problem (17), (18), and (21) was written as a system of three ordinary first-order differential equations for the functions $\tilde{n}(x)$, $v(x)$, $w(x)$ (22). For given ρ_h (i.e., N) and L, the corresponding boundary conditions are the following: $\tilde{n}(0)=0$, $\tilde{n}(L)=0$, $v(0)=v(L)$, $w(0)=w(L)$, and $v(0)=v_h$. The condition $v(0)=v_h=V(\rho_h)$ (8) can be claimed for the periodic case (21) because the condition $v=v_h$ is satisfied for every distribution $v(x)$ in a cluster at least at two points on the x axis, and because the position of the point $x=0$ is arbitrary. This system has been approximated on a grid $x_i=(i-1)dx$, $i=1{:}1{:}I$, $x_I=L$ by the centered box scheme [11]. A system of $3(I-1)$ nonlinear equations for the $3(I-1)$ unknown $\tilde{n}(x_i)$, $v(x_i)$, $i=2{:}1{:}(I-1)$, $w(x_i)$, $i=1{:}1{:}(I-1)$ as well as q^* and v_g have been obtained. These nonlinear equations have been linearized by employing Newton's method, and a rough approximation of their solutions has been improved iteratively until convergence was achieved.

B. Wide cluster

As follows from Fig. 2(f) and Fig. 3(e), the stationary moving cluster, which appears for the chosen parameters of the initial homogeneous flow on the road, consists of two new different stable homogeneous flows being surrounded by two fronts moving with the same velocity. Only at these fronts the density and the average velocity of vehicles on the road sharply change in space. An extent of each of these two new homogeneous states of a flow inside and outside the cluster can sufficiently exceed the characteristic value l. Such a cluster is conveniently called a *wide cluster*.

The width of a wide cluster L_s, i.e., the distance between its two fronts, as follows from the numerical calculations, depends on the value ρ_h: The higher ρ_h is, the higher L_s becomes.

On the contrary, the other parameters of a wide cluster, as there are the maximal and minimal density $(\rho_{\min}^m, \rho_{\max}^m)$, the average velocity of vehicles $(v_{\min}^m, v_{\max}^m)$, and the velocity of cluster $v_g = v_g^m$, are practically independent of the value ρ_h [Figs. 10(a)–10(c)]. The same conclusion is valid for the value $q^* = q_s^*$ [Fig. 10(d)]. Indeed, as follows from (15) and (16), one can write the expressions

$$q_s^* = \rho_{\min}^m (v_{\max}^m - v_g^m) = \rho_{\max}^m (v_{\min}^m - v_g^m) \ . \tag{23}$$

These parameters, as follows from numerical calculations, are also practically independent of the length of the road L for a wide stationary moving cluster. In other words, the values $\rho_{\min}^m$, $\rho_{\max}^m$, $v_{\min}^m$, $v_{\max}^m$, v_g^m, and q_s^* for a wide cluster sufficiently depend only on the function $V(\rho)$ and parameter c_0 in the equation of the motion (2). This means that these parameters of a wide cluster are the characteristic parameters of the traffic flow.

As follows from numerical calculations, the values $\rho_{\max}$, $\rho_{\min}$, v_{gr}, and q_{out} for the local cluster (Secs. II C 1 and II C 2) practically coincide with the characteristic values for a wide stationary moving cluster $\rho_{\max}^m$, $\rho_{\min}^m$ v_g^m and $q_{\max} = v_{\max}^m \rho_{\min}^m$, respectively, when the cluster is wide enough [$t > 180\tau$, Fig. 5(a)]. To understand this result notice that both the flux q_{out} and the flux $q_{\max}$ are determined by the same possible rate of vehicles escaping from a standstill inside the cluster. Therefore, the characteristic parameters mentioned above are practically identical for both local clusters and wide stationary moving clusters.

C. Narrow cluster

If the value ρ_h is decreased, the width of a wide cluster L_s also decreases. Therefore, by decreasing ρ_h the region of the homogeneous traffic flow inside the cluster gradually disappears. Finally, the cluster consists of two fronts and the one homogeneous flow only, which is situated outside the cluster [Fig. 10(e)]. Such a cluster is conveniently called a *narrow cluster*.

The amplitude of a narrow cluster $\rho_{\max}$ decreases with the value ρ_h, i.e., it is less than the characteristic value $\rho_{\max}^m$ for a wide cluster [Fig. 2(f)]. The amplitude of a narrow cluster $\rho_{\max}$ decreases also with the length of the road L. The other parameters of a narrow cluster $\rho_{\min}$, $v_{\min}$, $v_{\max}$, v_g, and q^*, on the contrary to those for a wide cluster, also sharply depend on ρ_h [Figs. 10(a)–10(d)] and L. The narrow cluster corresponds to the segment of a line on the (ρ, q) phase plane, which has one point ($q_{\max}$, $\rho_{\min}$, where $q_{\max} = \rho_{\min} v_{\max}$) outside the interval $[\rho_{c1}, \rho_{c2}]$ being situated on the fundamental diagram [Fig. 10(f)].

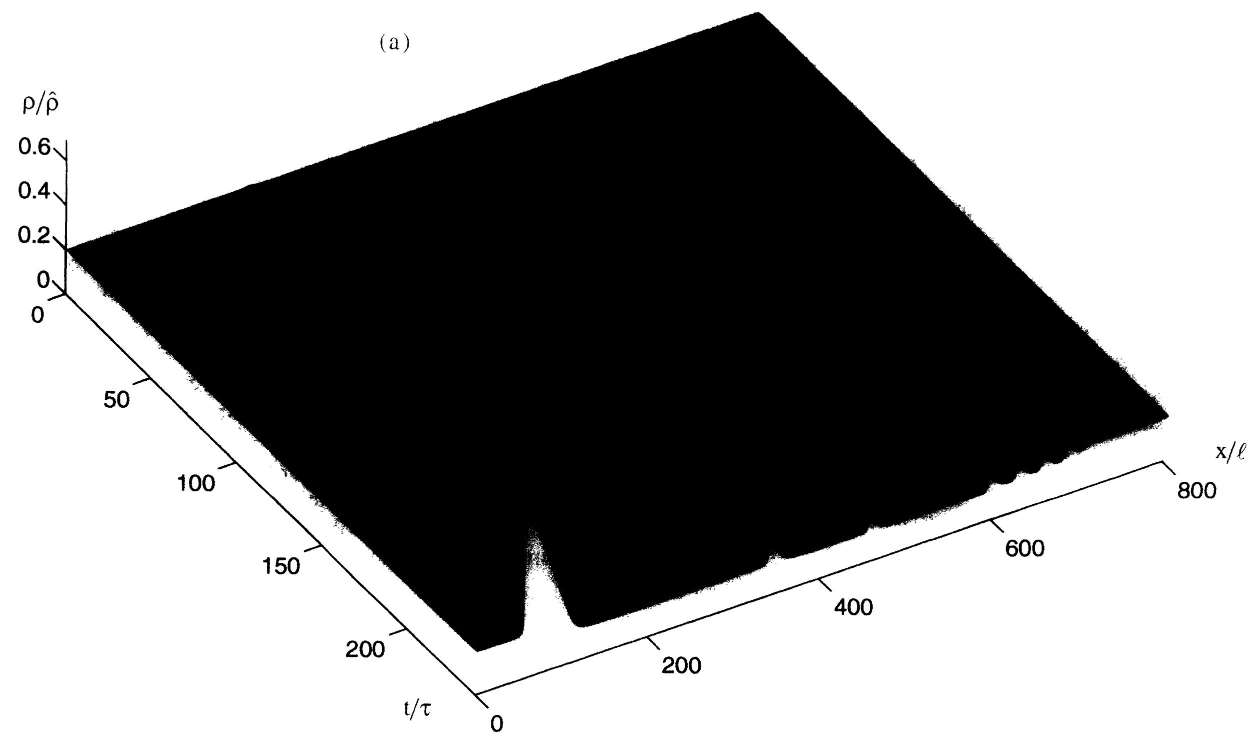

FIG. 9. The kinetics of the appearance of the sequence of clusters in traffic flow: (a) the dependence $\rho(x,t)$; (b) and (c) the distributions of the functions $\rho(x)$ and $v(x)$ at $t = 150\tau$ (b) and at $t = 180\tau$ (c); (d) the curve $q(\rho)$, which corresponds to (c). The initial local perturbation $\Delta\rho(x)$ (11), $\Delta\rho_m = 0.005\hat{\rho}$, $x_0 = 210l$; $\rho_h = 0.2\hat{\rho}$. The other parameters are the same as in Fig. 1.

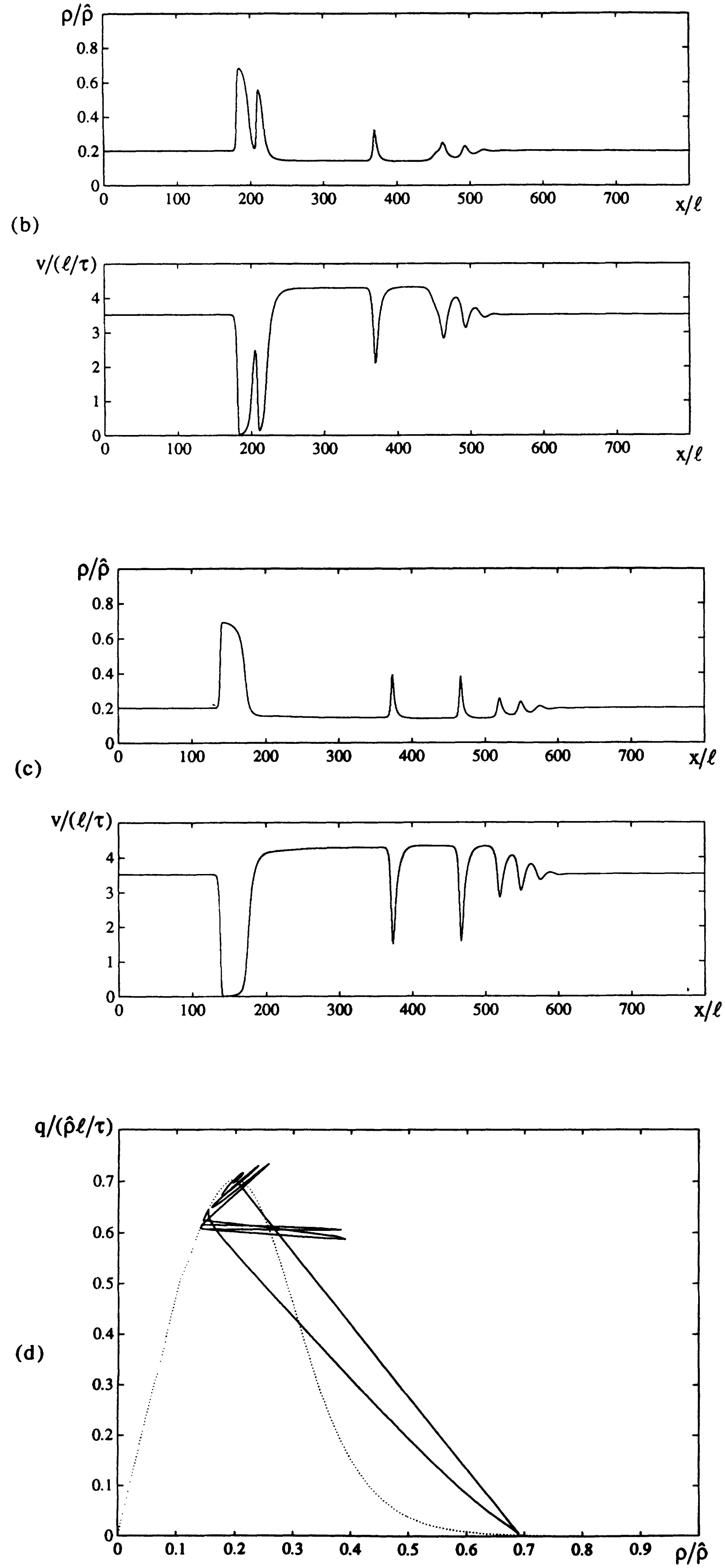

FIG. 9. (*Continued*).

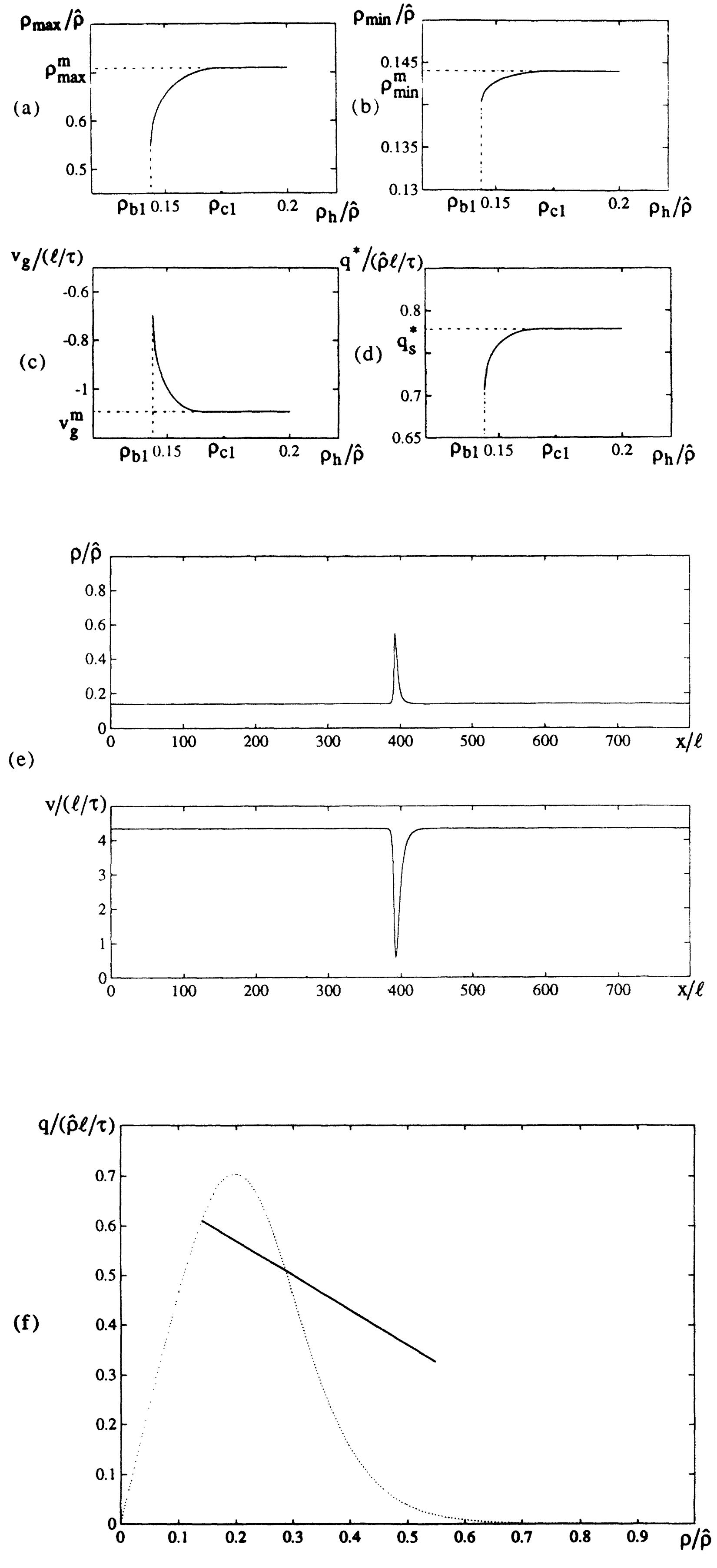

FIG. 10. The dependencies of the parameters of the stationary moving clusters on the density of vehicles in the initially homogeneous traffic flow ρ_h: (a) the maximal density of vehicles in clusters $\rho_{\max}$, (b) the minimal density $\rho_{\min}$, (c) the velocity of clusters v_g, (d) the parameter q^* (14). In (e) and (f) the distributions $\rho(x)$, $v(x)$ (e) and the curve $q(\rho)$ (f) for the narrow cluster for $\rho_h = 0.1441\hat{\rho}$ are shown. The other parameters are the same as in Fig. 1. Dotted line in (f) represents the fundamental diagram from Fig. 1(b). The found boundary density is $\rho_{b1} \cong 0.144\hat{\rho}$, the found characteristic parameters of wide clusters are $\rho_{\max}^m \cong 0.709\hat{\rho}$, $\rho_{\min}^m \cong 0.144\hat{\rho}$, $v_g^m \cong -1.09 l/\tau$, $q_s^* \cong 0.778\hat{\rho} l/\tau$.

D. Velocity of clusters. Minimum density for the existence of a cluster

From (15) and (16), for both wide and narrow clusters, one can write the expression $v_g = -(q_{\max} - q_{\min})/(\rho_{\max} - \rho_{\min})$, where the minimal flux is $q_{\min} = v_{\min}\rho_{\max}$ and the maximal flux is $q_{\max} = v_{\max}\rho_{\min}$. As for a wide cluster of large amplitude $q_{\min} \cong 0$, the latter formula has a simpler form:

$$v_g^m = -q_{\max}(\rho_{\max}^m - \rho_{\min}^m)^{-1} . \tag{24}$$

There is a boundary density of vehicles in the initially homogeneous traffic flow for the existence of the cluster ρ_{b1}: At $\rho_h < \rho_{b1}$, a cluster cannot exist. At $\rho_h = \rho_{b1}$, the cluster has the lowest amplitude $\rho_{\max}$ [Fig. 10(a)] and the smallest width $L_b = L_s(\rho_{b1})$.

As follows from the numerical calculations made, the boundary value ρ_{b1} decreases the more the length of the road L increases. It tends, for sufficiently large values of L, to the characteristic for the local cluster boundary value ρ_b [Figs. 6(a) and 6(b)], e.g., for the chosen function $V(\rho)$ and value c_0 (Fig. 1), $\rho_{b1} \cong 0.1676\hat{\rho}$ for $L = 50l$, and $\rho_{b1} \cong 0.1441\hat{\rho}$ for $L = 800l$, as for the local cluster $\rho_b \cong 0.14\hat{\rho}$. Emphasize that the critical value $\rho_{\min}^m$ for a wide cluster practically coincides with the critical boundary value ρ_b for the local cluster, i.e., also with the boundary value ρ_{b1} for a wide stationary moving cluster: $\rho_b \cong \rho_{b1} \cong \rho_{\min}^m$, if the length of the road L is large enough.

Notice that if the length of the road L is not very large, a boundary value ρ_{b1} for the stationary moving cluster is noticeably higher than a boundary value ρ_b for a local cluster. For this reason, if a local cluster in an interval of density $\rho_b < \rho_h < \rho_{b1}$ appears, this cluster on the circular road should disappear when a transition layer reaches the cluster and merges with it [see Sec. II B 1, no. vii]. This effect has been observed in the numerical simulations made.

E. "Anticluster" in traffic flow

Up to now, only the structures which can appear in the initial homogeneous flow in an interval of relative low density of vehicles, more precisely, in the interval of the values $\rho_b \le \rho_h \le \rho_{cr}$, have been considered. It has been found that a different kind of cluster, i.e., of the local region with a larger density of vehicles than in the initially homogeneous flow, can exist. It can be expected that in a homogeneous traffic flow with a very large density of vehicles, precisely, for the values ρ_h, which are in the vicinity of the second critical point ρ_{c2} (10c), an "anticluster"—a local region with a lower density of vehicles than in an initially homogeneous flow—can be formed. This supposition is confirmed by the numerical calculations (Fig. 11).

It turned out that the parameters of a wide anticluster [Figs. 11(b)–11(e)], i.e., the values $\rho_{\min}^m$, $\rho_{\max}^m$, $v_{\min}^m$, $v_{\max}^m$, v_g^m, and q_s^*, are practically the same as for a wide cluster (Sec. III C). In particular, the wide anticluster corresponds to the same segment of a line on the (ρ,q) phase plane [Fig. 3(e)].

If the value ρ_h is increased, the region of low density gets narrower, i.e., the width of the anticluster $\tilde{L}_s$ decreases. As a result, a wide anticluster [Fig. 11(a)] gradually transforms into a narrow anticluster [Fig. 11(f)]. The narrow anticluster corresponds to the segment of a line on the (ρ,q) phase plane, which outside the interval $[\rho_{c1},\rho_{c2}]$ has one point $(q_{\min}, \rho_{\max}$, where $q_{\min} = \rho_{\max} v_{\min})$ is situated on the fundamental diagram [Fig. 11(g)].

An amplitude of a narrow anticluster, i.e., a value $\rho_{\min}$, increases as the value ρ_h is increased [Fig. 11(b)]. The other parameters of the narrow anticluster also sufficiently depend on ρ_h [Figs. 11(c)–11(e)]. There is a boundary value ρ_{b2}: At $\rho_h > \rho_{b2}$, an anticluster cannot exist [Figs. 11(b)–11(e)], i.e., the value ρ_{b2} is the maximum density for the existence of a stationary moving cluster. This boundary value ρ_{b2} increases with the length of the road L, and for the values L large enough it tends to $\rho_{\max}^m$. At $\rho_h = \rho_{b2}$ the anticluster has the smallest width $\tilde{L}_b = \tilde{L}_s(\rho_{b2})$. In other words, the qualitative behavior of parameters of the anticluster, when the value ρ_h is increased, is similar as it is for the cluster discussed above when the value ρ_h is decreased.

Therefore, the stationary moving clusters can exist in the interval of the density $\rho_{b1} \le \rho_h \le \rho_{b2}$. At $\rho_h = \rho_{b1}$ the cluster has the smallest width $L_b = L_s(\rho_{b1})$. If the density is increased, the width of the cluster L_s is increased, too, and the cluster gradually transforms into an anticluster, which has the width $\tilde{L}_s = L - L_s$. If the density is further increased, the width of the anticluster is decreased, and it reaches the smallest value $\tilde{L}_b = \tilde{L}_s(\rho_{b2})$ at $\rho_h = \rho_{b2}$.

F. Average characteristics of traffic flow and comparison with other results

1. Average fundamental diagram

The functions $q(\rho)$ shown in Figs. 3, 5(c), 6(e), 6(f), 7(b), 9(d), 10(f), and 11(g) correspond to the calculated distributions $q(x)$ and $\rho(x)$ in the clusters at some fixed moment of time. However, for a comparison of these results with the results following from investigations on other models of traffic flow and with experimental results, where the average (in time) values of a flux have usually been plotted on "fundamental diagrams," a corresponding averaging of the results presented should be made.

Notice that in Sec. II C 6 it has been shown that in traffic flow the sequence of clusters, which have different amplitudes, different widths, different velocities, and are not situated periodically in space, can spontaneously appear. The faster clusters moving against the flow in time merge with the slower one upstream [1]. In the case of a circular road these faster clusters, reaching the left boundary of the road ($x=0$), appear at $x = L$, i.e., they can merge with the other slower clusters which were previously downstream of them. Because of this reason on a circular road, in time, finally often only one cluster can remain [1], although, previously, there have been many clusters in the flow. Besides this scenario, it follows from the numerical investigations made that there are a lot of others, when two, three, or more clusters with the same velocity remain on a circular road. It occurs that some of

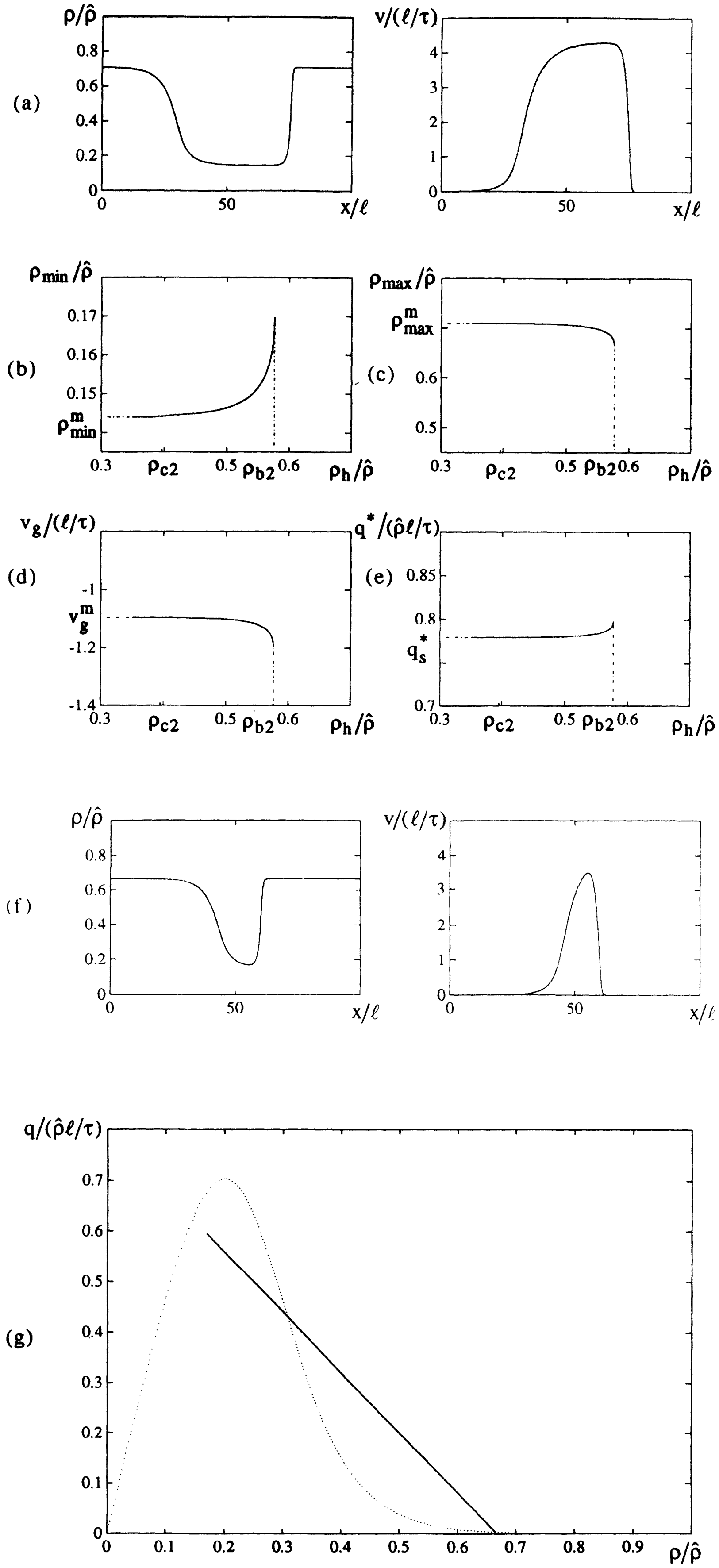

FIG. 11. The "anticlusters" in traffic flow: (a) the distributions $\rho(x)$ and $v(x)$ in a wide anticluster, (b)–(e) the dependencies of the minimal density $\rho_{\min}$ (b), the maximal density $\rho_{\max}$ (c), the velocity v_g (d), and the parameter q^* (e) for anticlusters on ρ_h. In (f) and (g) the distributions $\rho(x)$ and $v(x)$ (f) and the curve $q(\rho)$ (g) for the narrow anticluster for $\rho_h = 0.5765\hat{\rho}$ are shown. $L = 100l$. The other parameters are the same as in Fig. 1. Dotted line in (g) represents the fundamental diagram from (b). The found boundary value of density for $L = 100l$ is $\rho_{b2} \cong 0.5766\hat{\rho}$.

the different clusters show a tendency to equalize their parameters, in particular their velocities, in the process of their motion on the road. If the value L is large enough, the sequence of wide stationary moving clusters, traveling with the same velocity $v_g = v_g^m$ as the cluster shown in Fig. 2(f), can remain. Such a sequence of stationary moving wide clusters on the (ρ, q) phase plane corresponds to a segment of a line which coincides with the segment of the line for one wide cluster shown in Fig. 3(e).

For the stationary moving cluster on the circular road, the value of the average flux $\bar{q}$, averaging over a time interval large enough, practically coincides with the value $\bar{q} = L^{-1}\int_0^L q(x)dx$, where $q(x) = v(x)\rho(x)$ is the flux in the self-similar variable $x \Longrightarrow x - v_g t$ used in (12)–(14). The average density, as follows from (3), is $\bar{\rho} = \rho_h$.

Here, we restrict the consideration of an average fundamental diagram for the case of the circular road with a large enough value L, when, as follows from the results of Secs. III B–III E, that intervals of density $\bar{\rho}$, where the narrow clusters and narrow anticlusters exist, can be negligibly small. In other words, one can suppose that in the whole interval of density $\rho_{b1} \le \bar{\rho} \le \rho_{b2}$, where the clusters can exist (Secs. III D and III E), they are stationary moving wide clusters or else wide anticlusters. On the other hand, at a very large value of L, which corresponds to the mentioned condition, the boundary value ρ_{b1} for a wide cluster tends to $\rho_{\min}^m$, i.e., these parameters practically coincide: $\rho_{b1} \cong \rho_{\min}^m$. The same conclusion is valid for the wide anticluster: $\rho_{b2} \cong \rho_{\max}^m$ (Sec. III E). Therefore, in the case under consideration, the wide cluster can exist practically in the whole interval of density $\rho_{\min}^m \le \bar{\rho} \le \rho_{\max}^m$. From the results of Sec. III B it follows that: (a) the wide cluster can be approximately represented as two homogeneous states of traffic flow with the fluxes $q_{\min} = v_{\min}^m \rho_{\max}^m$ and $q_{\max} = v_{\max}^m \rho_{\min}^m$, correspondingly, inside and outside the cluster, which has the width L_s; (b) for a cluster under consideration, $q_{\min} \cong 0$. Taking this into account, one can get an approximate formula,

$$\bar{q} \cong q_{\max}\left[1 - \frac{L_s}{L}\right] \quad \text{for } \rho_{\min}^m \le \bar{\rho} \le \rho_{\max}^m .$$

On the other hand, the width of the cluster L_s is the increasing linear function of $\bar{\rho}$ [see also (58)]. As follows from Secs. III D and III E, at $\bar{\rho} \to \rho_{b1}$ the width $L_s \to L_b$ and at $\bar{\rho} \to \rho_{b2}$ the width $L_s \to L - \tilde{L}_b$, where L_b and $\tilde{L}_b$ are the smallest widths of the cluster and the anticluster, correspondingly. In other words, at very large values of L, when $L \gg L_b, \tilde{L}_b$, the line $\bar{q}(\bar{\rho})$ given by this formula (line C in Fig. 12) [12] practically coincides with the segment of a line on the (ρ, q) phase plane for wide anticlusters [Fig. 3(e)].

The homogeneous flow loses its stability with respect to fluctuations of the small amplitude in the interval of density (10a) and the critical values ρ_{c1}, ρ_{c2}, as follows from Secs. III B–III E, satisfying the conditions $\rho_{c1} > \rho_{\min}^m$ and $\rho_{c2} < \rho_{\max}^m$. Therefore, there are two intervals of density $\rho_{\min}^m < \bar{\rho} < \rho_{c1}$ and $\rho_{c2} < \bar{\rho} < \rho_{\max}^m$, where the homogeneous flow is still stable with respect to perturbations of small amplitude, and additionally the states of flow with clusters can exist, i.e., there are two intervals of "hysteresis" on the average fundamental diagram [Fig. 12(a)]. The form of the fundamental diagram realized in these intervals depends on the initial conditions.

If one starts the calculations from a distribution of the density which is disturbed by nonhomogeneous perturbations with a high enough amplitude, it will be possible, finally, to find that in the intervals of density $\rho_{\min}^m < \bar{\rho} < \rho_{c1}$ and $\rho_{c2} < \bar{\rho} < \rho_{\max}^m$ the homogeneous states cannot appear, i.e., only the states of flow with clusters can be realized. In this case, the whole average fundamental diagram consists of two parts: (i) a segment of a curve (practically a line) with a positive slope, in the interval of the density $0 < \bar{\rho} \le \rho_{\min}^m$, which corresponds to the homogeneous traffic flow; and (ii) a segment of a line in the interval $\rho_{\min}^m < \bar{\rho} < \rho_{\max}^m$ with a negative slope, which coincides with the value of the velocity of wide cluster v_g^m (24) and corresponds to the existence in the traffic flow of the wide stationary moving clusters [Fig. 12(b)]. The interval of the density $\rho_{\max}^m \le \bar{\rho} \le \hat{\rho}$, where the flux $\bar{q}$ is practically equal to zero, corresponds to a homogeneous state, which is, in fact, a standstill. Notice that by a corresponding choice of the function $V(\rho)$ and the value c_0 in (2), the width of the latter interval can be reduced practically to zero.

If, on the contrary, one starts, as has been done in Sec. II B, from the homogeneous flow in the presence of only small amplitude nonhomogeneous perturbations, the homogeneous state can occur up to the critical point $\bar{\rho} = \rho_{c1}$ (or $\bar{\rho} = \rho_{c2}$) [Fig. 12(a)]. In this case, and if only small amplitude fluctuations are present, the jump from the curves H (corresponding to the homogeneous state), to the line C (corresponding to the state with clusters), can occur on the average fundamental diagram, when $\bar{\rho}$ exceeds ρ_{c1} [jump $1 \to 2$ in Fig. 12(a)] or $\bar{\rho}$ becomes less than ρ_{c2} [jump $3 \to 4$ in Fig. 12(a)].

On the other hand, for the same reason which has been mentioned in Sec. II C 4, due to possible fluctuations of a relatively large amplitude in traffic flow, the process of random transitions between the homogeneous state and the state with clusters can occur in the intervals of density $\rho_{\min}^m < \bar{\rho} < \rho_{c1}$ and $\rho_{c2} < \bar{\rho} < \rho_{\max}^m$. In this case, one can suppose that the behavior of the average fundamental diagram in these intervals of density can strongly depend on the parameters of fluctuations in traffic flow.

Regarding the case of a not very large L (but nevertheless $L \gg l$), notice that the boundary values ρ_{b1}, ρ_{b2} and the critical values ρ_{c1}, ρ_{c2} depend on L (Sec. II A and Sec. III D). In particular, the position of the maximum point of the average fundamental diagram also depends on the length of the road L.

2. *Comparison with cellular automation model of traffic flow*

The kinetic (hydrodynamic) approach to traffic flow, which is followed in this paper, has developed since the 1950s [13]. The further advances in this approach and references can be found in [2,4–6,14,1].

Besides this kinetic approach to traffic flow, different "microscopic" models of traffic flow, i.e., models, where the behavior of each individual vehicle in traffic flow is investigated, have been developed (see, e.g., [15–17]). In the cellular automation model of traffic flow developed in [17], which belongs to the group of microscopic models, the four consecutive steps, based on fixed rules, are performed, in parallel for all vehicles: (i) acceleration, (ii) slowing down, (iii) randomization, and (iv) vehicle motion.

As follows from the investigation of a cluster formation based on the kinetic model (1)–(4) shown above, the physics of an appearance of the cluster of vehicles in an initially homogeneous traffic flow (see Sec. II C 2) is linked to the competitions between *active processes,* which try to increase the amplitude of the nonhomogeneous perturbations in traffic flow, and the *damping processes* which act in the opposite direction. The active process is linked to the falling character of the function $V(\rho)$. The damping processes are considerably connected with the diffusion (viscosity) process in traffic flow and an influence of the gradient of a pressure. If the falling character of the function $V(\rho)$ can explain why the different clusters *by principle* occur in traffic flow, the consideration of the competition between the active and the damping processes made in our paper can explain the structure and some important properties of the clusters formed.

On the other hand, it is well known (e.g., [2,5–7]) that the falling character of the function $V(\rho)$ in (2) is linked to the realization of two processes [(i) acceleration and (ii) slowing down] (see also Sec. II B), which are also very essential processes in the cellular automation model, if the density of the vehicles is large enough [17]. Apparently, because of that reason, the different clusters of vehicles, including the cluster of the large amplitude moving against the flow (see Fig. 2 in [17]), have been observed in the numerical investigations of the cellular automation model [17] and, as for a large enough density, the active process discussed above has been realized there. The analogue of the damping processes may be the limitations for both an acceleration and a slowing down for a vehicle in the algorithm of the cellular automation model.

Therefore, one can suppose that both the kinetic model

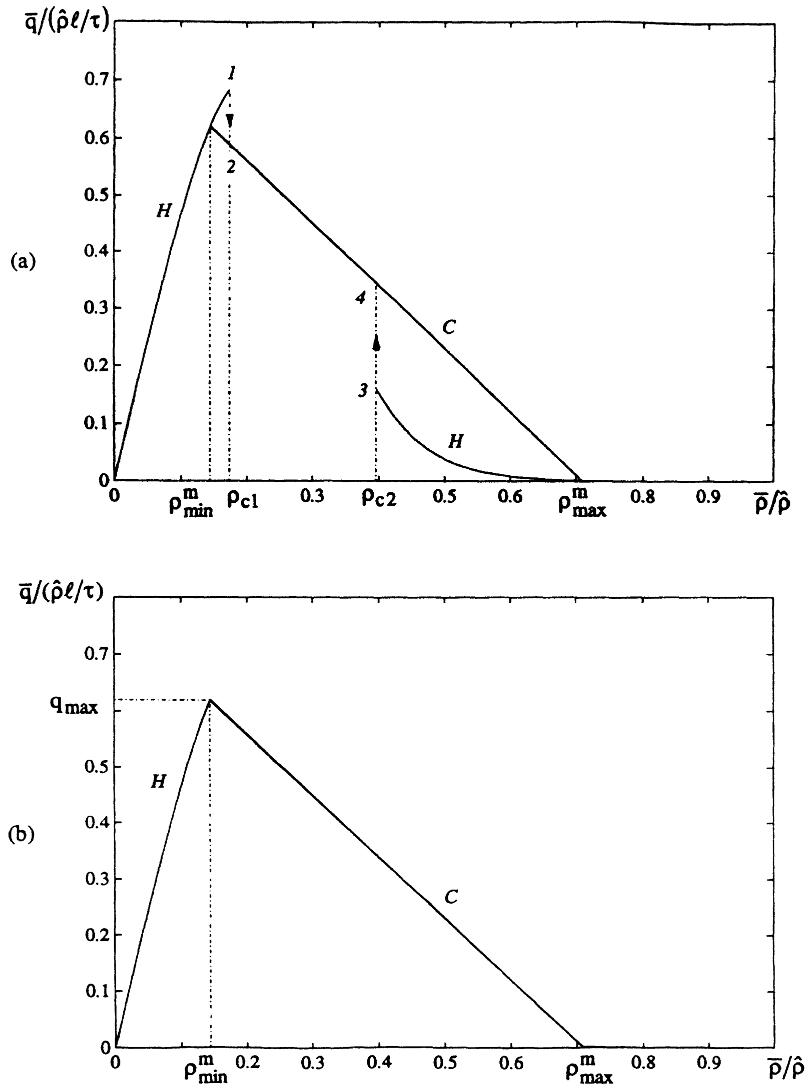

FIG. 12. The average fundamental diagram for a circular road of a large enough circumference L. The curves H correspond to the homogeneous time-independent state of traffic flow, the line C corresponds to the states of traffic flow with one or more wide stationary moving clusters (or anticlusters).

(1)–(4) [1] and the cellular automation model [17] can show qualitatively similar behavior in many aspects. These suppositions are confirmed also by (a) the qualitative similar form of the "average" fundamental diagram discussed in Sec. III F 1 and of the cellular automation model (solid line in Fig. 4 in [17]); (b) the existence of the dependence of the position of the maximum (transition) point in the fundamental diagram on the length of the road L for both models; and (c) the existence, for both models, of a very complicated scattering of points on the "fundamental diagram" for a relatively small averaging time (dots in Fig. 4 in [17]). For the kinetic model [(1)–(4)] it can appear due to the formation of the sequence of clusters with different parameters not being situated periodically in space at high density (see Sec. II C 6) and due to a possible process of random appearance and disappearance of clusters in the vicinity of critical or boundary values of density (see Sec. II C 3 and Sec. III F 1).

G. The v-q^* curve and parameters of cluster

In the (v,w) phase plane both a wide cluster [Fig. 2(f)] and a wide anticluster [Fig. 11(a)] correspond to the same closed phase trajectory [Fig. 13(a)]. The points $v^m_{\min}$ and $v^m_{\max}$ on this curve conform to the value $w=dv/dx=0$. On the other hand, for the wide cluster (and anticluster), these two points represent on the (v,w) phase plane two new homogeneous traffic flows formed by the cluster [Fig. 2(f)]. The distribution of the function $v(x)$ is described by the nonlinear ordinary differential equation of the second order (17). From the general result of the mathematical theory of such types of equations (e.g., [18]), one can suppose that the points $v^m_{\min}$ and $v^m_{\max}$ for a wide cluster, when $L \gg l$ and $L_s \gg l$, are situated on the phase space (v,w) very close to the "saddle" points of Eq. (17).

Let us consider the equation

$$F(v,q^*,v_g)=0 \,, \tag{25}$$

where, for the case of a wide cluster and the chosen parameters of traffic flow (Fig. 1), the value $v_g=v_g^m=-1.09l/\tau$ [Fig. 10(c)] can be used. The curve $q^*(v)$, which comes from Eq. (25) at this constant value v_g, has an N shape (Fig. 14): There are two characteristic points v_0 and v_0' on this curve, where the derivative

$$dq^*/dv=-(\partial F/\partial v)/(\partial F/\partial q^*) \tag{26}$$

is equal to zero. Notice that on the v axis the interval (v_0,v_0') is disposed of in the vicinity of the interval of the instability of the homogeneous flow: $v_{c2}=V(\rho_{c2})<v<v_{c1}=V(\rho_{c1})$ (Fig. 14). For the shortening one can call the curve $q^*(v)$, which follows from (25) at $v_g=\text{const}$, as a v-q^* curve.

If the value $q^*=q_s^*=0.778\hat{\rho}l/\tau$, which corresponds to a wide cluster [Fig. 10(d)], is chosen, the line $q^*=q_s^*=\text{const}$ has three points of intersection with the v-q^* curve: v_{s1}, v_{s2}, and v_{s3} (Fig. 14). The solutions of Eq. (17), linearized in the vicinity of these fixed (singular)

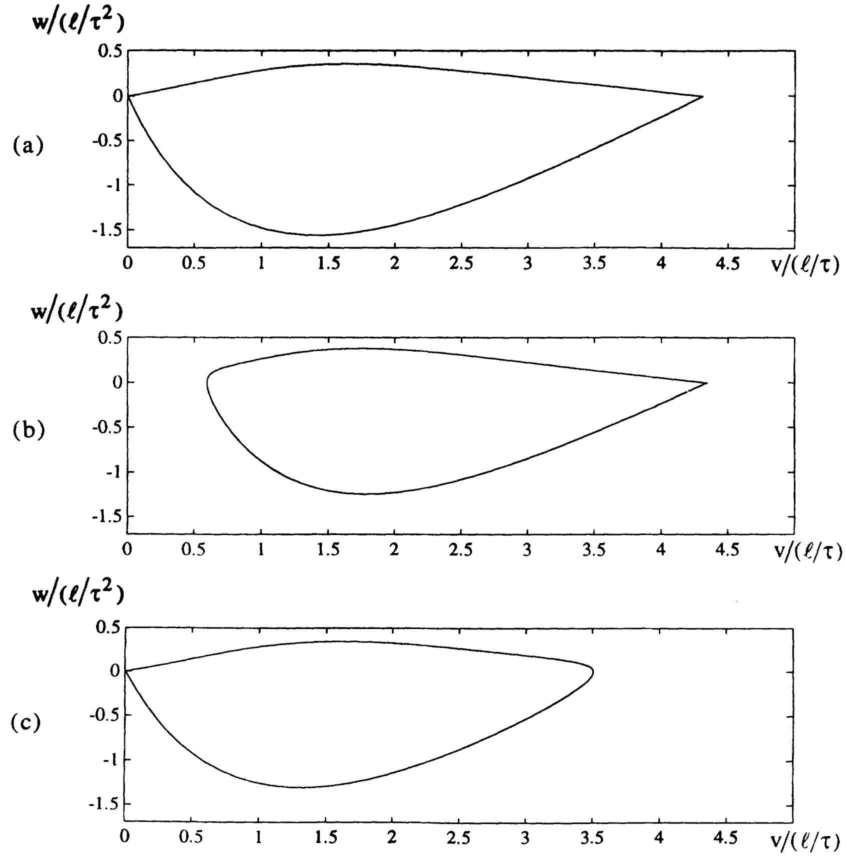

FIG. 13. The phase trajectories: (a) for the wide cluster [Fig. 2(f)] and the wide anticluster [Fig. 11(a)], (b) for the narrow cluster [Fig. 10(e)], (c) for the narrow anticluster [Fig. 11(f)].

points v_{si} ($i=1,2,3$), have the usual [18] form:

$$\delta v_i = v(x) - v_{si} \propto \exp(\lambda^{(i)} x) , \tag{27}$$

where

$$\lambda^{(i)} = \lambda_{1,2}^{(i)} = -\frac{D_0^{(i)}}{2} \pm \sqrt{(D_0^{(i)}/2)^2 - F_v^{(i)}} \; ;$$
$$F_v^{(i)} = \partial F/\partial v|_{v=v_{si}}, \quad D_0^{(i)} = D_0|_{v=v_{si}}, \quad i=1,2,3 \; . \tag{28}$$

From (25), (20), and (5) and taking into account that $\rho = q^*(v - v_g)^{-1} > 0$, one can get that on the v-q^* curve the value $\partial F/\partial q^* = q^*(v-v_g)^{-2}\xi|_{\rho=q^*/(v-v_g)} < 0$. Therefore, from (26) and from the И shape of the v-q^* curve (Fig. 14), one can see that

$$F_v^{(2)} > 0, \; F_v^{(1)} \text{ and } F_v^{(3)} < 0 \; . \tag{29}$$

It follows from (28) and (29) that, at the points $v = v_{s1}$ and $v = v_{s3}$, where $F_v^{(1,3)} < 0$, the coefficients $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ ($i=1$ and 3), independent of the values $D_0^{(i)}$ (28), are real numbers of unlike signs. At the point $v = v_{s2}$, where $F_v^{(2)} > 0$, the coefficients $\lambda_{1,2}^{(2)}$ are complex.

Hence, the singular points v_{s1} and v_{s3} are saddle points of Eq. (17). On the other hand, a wide cluster on the (v, q^*) plane corresponding to (15) and (23) represents a segment of the horizontal line $q^* = q_s^*$ (Fig. 14). The end points $v_{\min}^m$ and $v_{\max}^m$ of this segment are practically equal to the values v_{s1} and v_{s3}, correspondingly. Notice that, as one can expect from the results of Sec. II B 2, the points $v_{\min}^m$ and $v_{\max}^m$ are situated outside the interval (v_{c2}, v_{c1}) of the instability of the flow, i.e., the homogeneous traffic flows inside and outside the cluster are stable.

One can consider the peculiarities of the phase trajectories which pass near the saddle point v_{s1} (or v_{s3}). First notice that the value $D_0^{(1)} \cong 3.23 l/\tau$ is positive and the value $D_0^{(3)} \simeq -0.61 l/\tau$ is negative. This means, taking into account (26)–(28), that the values $|w|$ on these separatrices for $w > 0$ are considerably less than for $w < 0$ for the same values v. This explains the fact that the right front of the cluster [Fig. 2(f)], where $w > 0$, is relatively smooth and the left front, where $w < 0$, is sharp. So the antisymmetric form of the cluster is essentially linked to the sign of the second term in Eq. (17).

In the same way it is also possible to show that the phase trajectories near the saddle point of Eq. (17) determine the behavior of the function $v(x)$ near the value $v_{\max}$ for the narrow cluster [Figs. 10(e) and 12(b)] and near the value $v_{\min}$ for the narrow anticluster [Figs. 11(f) and 12(c)].

H. Clusters of "small" amplitude

In the numerical calculations of the cluster effect (Figs. 2–13), the function $V(\rho)$ and the value c_0 (Fig. 1) in the kinetic model of traffic flow (1)–(4) have been chosen to satisfy qualitative properties of traffic flow, especially to satisfy the conditions (5) and (10a) (Sec. II A). The additional numerical calculation, which has been made for the other examples of a function $V(\rho)$ and a value c_0, shows the same qualitative structures and parameters of clusters in a traffic flow, if the conditions (5) and (10a) are fulfilled.

As one could expect, this conclusion is also preserved for the numerical calculations of the cluster effect in a hypothetical homogeneous traffic flow, which is unstable only in the relatively small interval of the density of vehicles (10a), i.e., when $(\rho_{c2} - \rho_{c1})/\rho_{c1} \ll 1$. In this case, clusters of small amplitude are formed: $\eta = (v_{\max} - v_{\min})/v_{\max} \ll 1$. The numerical analysis shows that the properties of clusters of small amplitude and of large amplitude qualitatively coincide.

There is, however, one important quantitative peculiarity of a small amplitude cluster: The values $D_0^{(i)}$ (28) in the saddle points v_{s1}, v_{s3} [Fig. 15(c)] of Eq. (17) and also the value of the function $D_0(v(x), q^*, v_g)$ (19) in the extreme points $v_{\min}^m$, $v_{\max}^m$ due to the small amplitude of the clusters can be also small enough (of the order of $\eta \ll 1$). Therefore, the values $|\lambda^{(i)}|$ in (27) for these saddle points are almost equal to one another. As a result, the form of the stationary cluster is almost symmetric [Fig. 15(a)]. It allows for the function $D_0(v(x), q^*, v_g)$ (19) in Eq. (17),

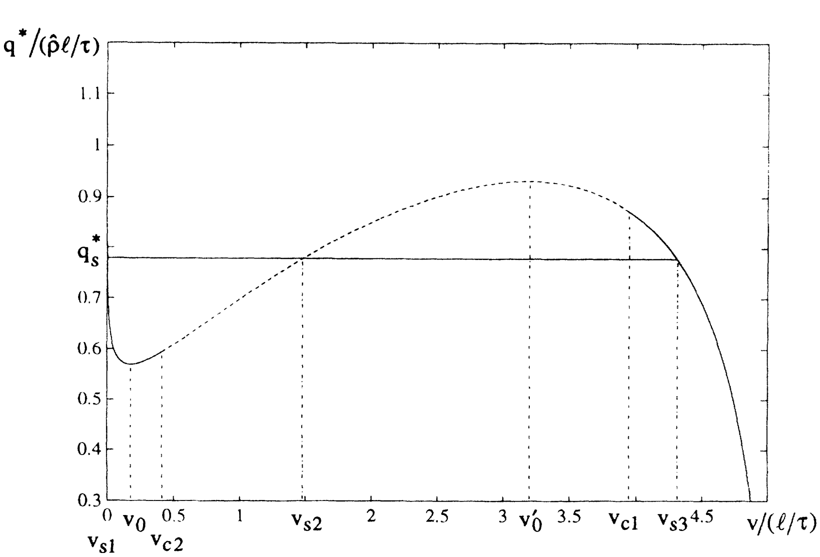

FIG. 14. The v-q^* curve for the wide cluster [Fig. 2(f)] and the wide anticluster [Fig. 11(a)]. The dashed line corresponds to the region of the average velocity of vehicles, where the homogeneous traffic flow is unstable. The found values are $v_{s1} \cong 0.002 l/\tau$, $v_{s2} \cong 1.476 l/\tau$, $v_{s3} \simeq 4.31 l/\tau$.

which corresponds to a small amplitude cluster, to write formally the following condition:

$$D_0(v,q^*,v_g)=\varepsilon D(v,q^*,v_g), \quad \text{with } \varepsilon \ll 1 \tag{30}$$

where the constant value ε should be chosen so that it is of the order of $\eta=(v_{\max}-v_{\min})/v_{\max} \ll 1$.

Clusters of large amplitude (Figs. 2–12) in comparison with the clusters of small amplitude (Fig. 15) may describe better real traffic flow. Here, traffic jams with high density inside and low density of vehicles outside are observed. Nevertheless, clusters of small amplitude, as has already been emphasized, have the same qualitative properties as clusters of large amplitude. Therefore, using (30), one can develop the qualitative nonlinear theory of a stationary moving cluster on the example of small amplitude clusters and explain the main characteristic properties of any stationary moving clusters found from the numerical analysis above.

IV. NONLINEAR THEORY OF A STATIONARY MOVING CLUSTER

A. Nonlinear equations for stationary cluster of small amplitude

Under the condition (30), one can seek the solution of the problem (17), (18), and (21) in the form of expansions

$$v(x)=v^{(0)}(x)+\varepsilon v^{(1)}(x)+\dots,$$
$$q^*=q^{*(0)}+\varepsilon q^{*(1)}+\dots, \tag{31}$$
$$v_g=v_g^{(0)}+\varepsilon v_g^{(1)}+\dots.$$

Substituting (31) into (17), (18), and (21) one can get equations for zero,

$$d^2v^{(0)}/dx^2+dU/dv^{(0)}=0, \quad U=\int_{v^{(0)}} F^{(0)}dv^{(0)}, \tag{32}$$

$$\int_0^L (v^{(0)}-v_g^{(0)})^{-1}dx=\rho_h L/q^{*(0)}, \tag{33}$$

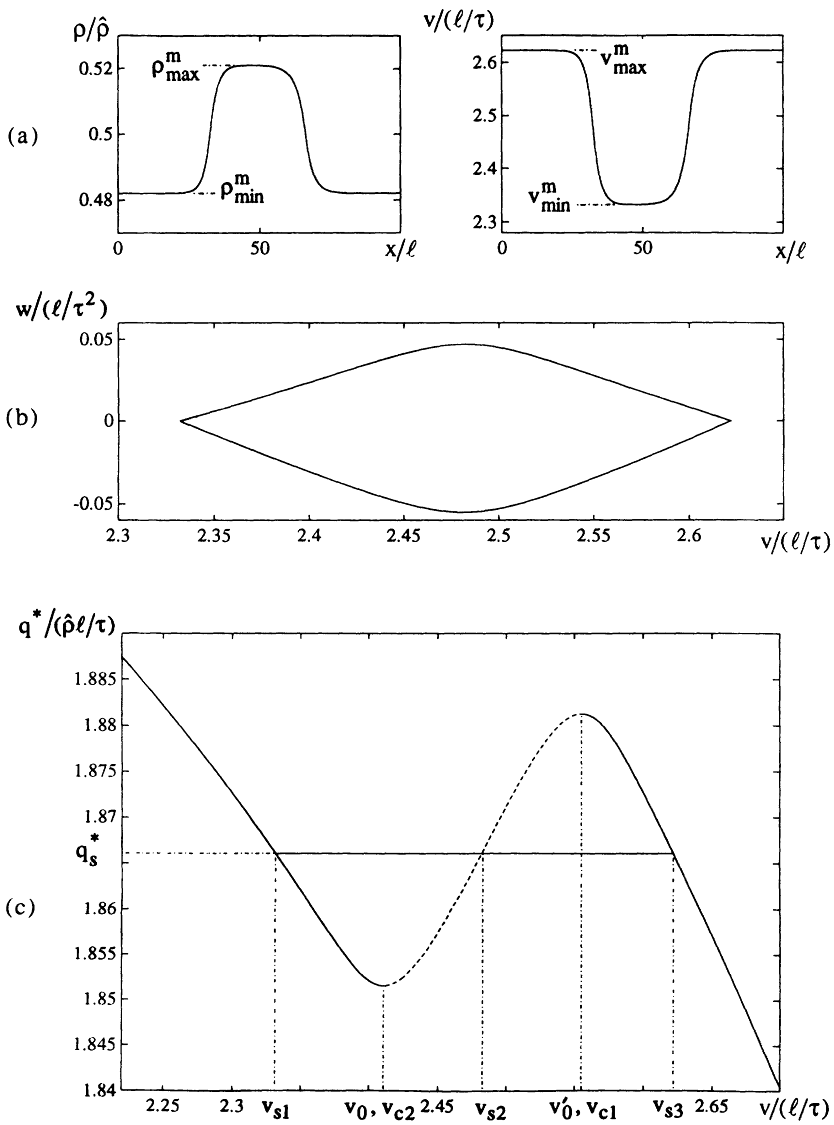

FIG. 15. The wide cluster of "small" amplitude (a), the corresponding phase trajectory (b), and the v-q^* curve (c). Results of the numerical analysis of Eqs. (1)–(4) for $V(\rho)=[0.1(1+\exp\{[(\rho/\hat{\rho})-0.5]/0.002\})^{-1}+4.8689[1-(\rho/\hat{\rho})]]l/\tau$, $c_0=3.7263l/\tau$, $L=100l$, $\rho_h=0.495\hat{\rho}$. The found values are $\rho_{c1}\cong 0.4944\hat{\rho}$, $\rho_{c2}\cong 0.5057\hat{\rho}$, $v_{s1}\cong 2.3312l/\tau$, $v_{s2}\cong 2.4827l/\tau$, $v_{s3}\cong 2.622l/\tau$, $v_0\cong v_{c2}\cong 2.411l/\tau$, $v_0'\cong v_{c1}\cong 2.555l/\tau$. The found characteristic parameters of a wide cluster are $\rho_{\max}^m\cong 0.521\hat{\rho}$, $\rho_{\min}^m\cong 0.482\hat{\rho}$, $v_g^m\cong -1.249l\tau$, $q_s^*\cong 1.866(\hat{\rho}l/\tau)$, $v_{\min}^m\cong 2.3319l/\tau$, $v_{\max}^m\cong 2.622l/\tau$.

$$v^{(0)}(0)=v^{(0)}(L), \quad dv^{(0)}/dx|_0=dv^{(0)}/dx|_L , \tag{34}$$

and the first

$$d^2v^{(1)}/dx^2+F_v^{(0)}v^{(1)}=-D^{(0)}w^{(0)}-F_{q^*}^{(0)}q^{*(1)}-F_{v_g}^{(0)}v_g^{(1)} , \tag{35}$$

$$\int_0^L \frac{v^{(1)}dx}{(v^{(0)}-v_g^{(0)})^2}-v_g^{(1)}\int_0^L \frac{dx}{(v^{(0)}-v_g^{(0)})^2}=q^{*(1)}\frac{\rho_h L}{(q^{*(0)})^2} , \tag{36}$$

$$v^{(1)}(0)=v^{(1)}(L), \quad dv^{(1)}/dx|_0=dv^{(1)}/dx|_L \tag{37}$$

approximation in ε. In (32) and (35) the functions

$$F^{(0)}=F(v^{(0)},q^{*(0)},v_g^{(0)}), \quad D^{(0)}=D(v^{(0)},q^{*(0)},v_g^{(0)}) ,$$
$$F_v^{(0)}=\partial F/\partial v|_{v^{(0)},q^{*(0)},v_g^{(0)}}, \quad F_{v_g}^{(0)}=\partial F/\partial v_g|_{v^{(0)},q^{*(0)},v_g^{(0)}} , \tag{38}$$
$$F_{q^*}^{(0)}=\partial F/\partial q^*|_{v^{(0)},q^{*(0)},v_g^{(0)}}$$

are of order of 1; $w^{(0)}=dv^{(0)}/dx$.

B. Structure of clusters

Equation (32) is formally identical to the equation describing the conservative motion of a "particle" with "coordinate" v and "time" x in a potential U. Given the values of L and N [i.e., ρ_h (8)], the form of the potential $U(v^{(0)})$ (32) and (38) depends on the two quantities, $v_g^{(0)}$ and $q^{*(0)}$. Choosing first some fixed value of $v_g^{(0)}$, one can try to find the corresponding forms of the potential U for different values of $q^{*(0)}$ and the trajectories of the "particle" in it and on the $(v^{(0)},dv^{(0)}/dx)$ phase plane. Therefore, it will be possible to find the corresponding solutions $(v^{(0)}(x),q^{*(0)})$ which satisfy the chosen value of $v_g^{(0)}$. Then, one can repeat this procedure for a new value of $v_g^{(0)}$, if the chosen value $v_g^{(0)}$ does not fulfill some requirements to be discussed below.

1. Form of potential

The extremes of $U(v^{(0)})$ correspond to the condition

$$dU/dv^{(0)}=F^{(0)}=0 , \tag{39}$$

i.e., they correspond, as follows from (38), (20), and (16), to an intersection of the curve $q^{*(0)}(v^{(0)})$, defined at the chosen value $v_g^{(0)}$ by the equation

$$V(q^{*(0)}(v^{(0)}-v_g^{(0)})^{-1})-v^{(0)}=0 , \tag{40}$$

with the straight line $q^{*(0)}=\text{const}$. In (40), the function

$$V(q^{*(0)}(v^{(0)}-v_g^{(0)})^{-1})=V(\rho)|_{\rho=q^{*(0)}(v^{(0)}-v_g^{(0)})^{-1}} . \tag{41}$$

Analogously to Sec. III G, one can call this curve $q^{*(0)}(v^{(0)})$ a v-q^* curve. A derivative $dq^{*(0)}/dv^{(0)}$ on the v-q^* curve, as follows from (40), (41), (38), and (5), is equal to

$$dq^{*(0)}/dv^{(0)}=-F_v^{(0)}/F_{q^*}^{(0)}$$
$$=[1+q^{*(0)}(v^{(0)}-v_g^{(0)})^{-2}\xi^*]\times[(v^{(0)}-v_g^{(0)})^{-1}\xi^*]^{-1} , \tag{42}$$

where a value

$$\xi^*=\xi|_{\rho=q^{*(0)}(v^{(0)}-v_g^{(0)})^{-1}}<0 \tag{43}$$

is introduced. If the shape of the v-q^* curve is known, one can reconstruct the form of the potential $U(v^{(0)})$ for different values $q^{*(0)}$ and then find the possible solutions $v^{(0)}(x)$ (Fig. 16).

For a construction of the v-q^* curve, one can use the conditions of the instability of traffic flow (10) and the condition (5). Let us reconstruct only a part of the v-q^* curve for some range of values $v^{(0)}$, $q^{*(0)}$ and also for values of the velocity $v_g^{(0)}<v^{(0)}$ (Fig. 16). Exactly, the consideration of the values $v^{(0)}$, $q^{*(0)}$ will be restricted to some vicinity of those values which satisfy the condition:

$$\rho_{c1}<q^{*(0)}/(v^{(0)}-v_g^{(0)})<\rho_{c2} . \tag{44}$$

The characteristic feature of the interval (44) follows from the above-mentioned conditions for the instability of traffic flow (10). On the one hand, to satisfy the condition (10b) in the interval (10a), it should be an interval of density ρ of traffic flow, i.e., corresponding to (14) it should fulfill the condition (44), where the value ξ (5) [and therefore the value ξ^* (43)] has to reach large enough negative values. On the other hand, outside the interval (10a) the traffic flow is stable, i.e., the value $|\xi|$ (and $|\xi^*|$) should be considerably lower than inside the interval (44). In other words, from the conditions of instability of traffic flow (10) follows that the value $|\xi^*|$ should have a relatively sharp maximum somewhere inside the interval (44).

The latter means that there are always some values of the velocity $v_g^{(0)}$ so that there is some interval

$$v_0^{(0)}<v^{(0)}<v_0'^{(0)} , \tag{45}$$

where the value $|\xi^*|$ is large enough to make on the v-q^* curve ($F^{(0)}=0$) the function

$$F_v^{(0)}=(-1+q^{*(0)}(v^{(0)}-v_g^{(0)})^{-2}|\xi^*|)q^{*(0)}(v^{(0)}-v_g^{(0)})^{-1} , \tag{46}$$

positive inside and negative outside the interval (45). Due to $v_g^{(0)}<v^{(0)}$ and $\xi^*<0$ (43), on the v-q^* curve the function $F_{q^*}^{(0)}=q^*(v^{(0)}-v_g^{(0)})^{-2}\xi^*$ is always negative. For this reason, inside the interval (45) the derivative $dq^{*(0)}/dv^{(0)}$ (42) is positive and outside this derivative it is negative. At the boundaries of (45), i.e., in the characteristic points $v^{(0)}=v_0^{(0)}$, $v'^{(0)}=v_0'^{(0)}$, the derivative is $dq^{*(0)}/dv^{(0)}=0$, as in these points the value $F_v^{(0)}=0$. Therefore, in the range of values $v^{(0)}$, $q^{*(0)}$, and $v_g^{(0)}$ under consideration, the v-q^* curve has an И shape [Fig. 16(a)] [19]. The same shape and properties of the v-q^* curve follow from the numerical calculation [Fig. 15(c)].

From the consideration made it follows that the v-q^* curve [Fig. 16(a)] consists of three branches (I, II, and III)

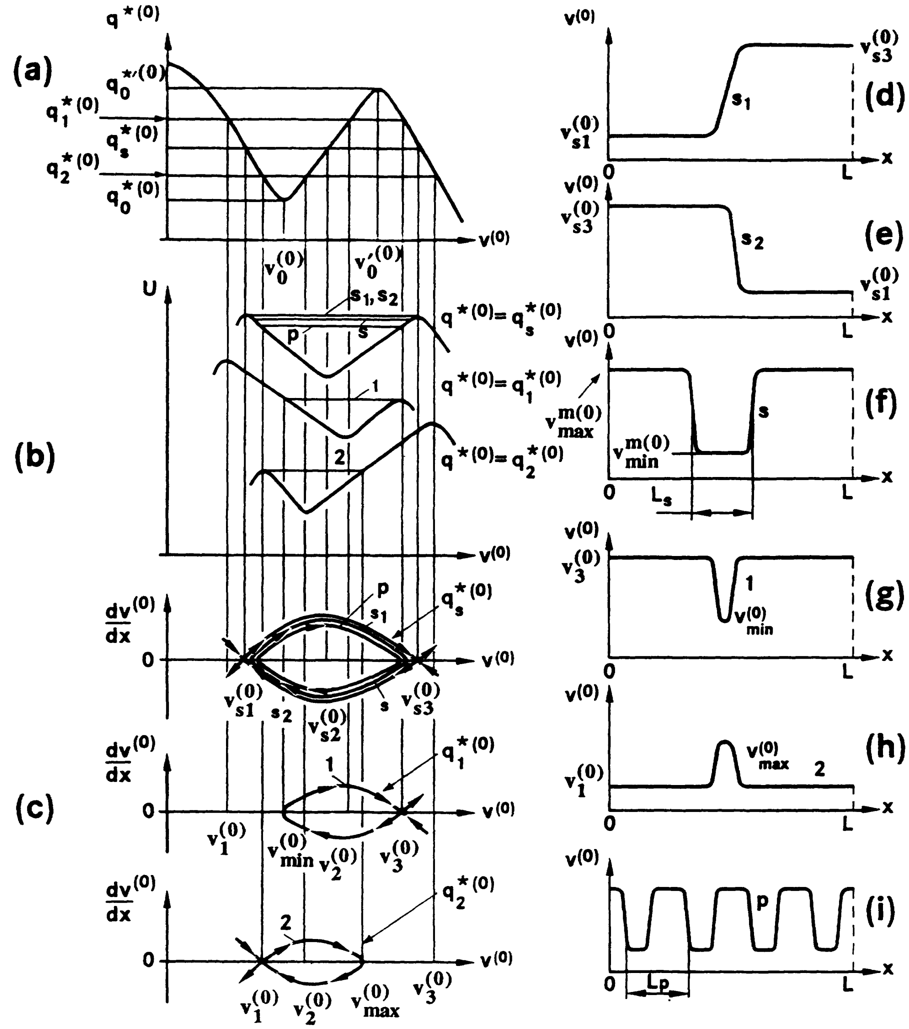

FIG. 16. Illustrating the qualitative construction of the solutions $v^{(0)}(x)$ (d)–(i): (a) the shape of a part of the v-q^* curve, (b) the form of the potential U, (c) the phase trajectories. The solutions $v^{(0)}(x)$ in (d)–(i) have been marked by the same designations $(s_1,s_2,s,1,2,p)$ as the corresponding trajectories in (b) and (c).

of the single-valued function $v^{(0)}(q^{*(0)})$. Branch I, where $v^{(0)} \le v_0^{(0)}$, corresponds to $F_v^{(0)} \le 0$. Branch II, where $v_0^{(0)} \le v^{(0)} \le v_0'^{(0)}$, corresponds to $F_v^{(0)} \ge 0$. Branch III, where $v^{(0)} \ge v_0'^{(0)}$, corresponds to $F_v^{(0)} \le 0$. On the other hand, in accordance with (32) and (39), at the extreme points of the potential, $U(v^{(0)})$ is

$$d^2U/dv^2 = F_v^{(0)} . \tag{47}$$

Therefore, the interception of some straight line $q^{*(0)}=\text{const}$ with branch II of the v-q^* curve at some point $v^{(0)}=v_2^{(0)}$, where $F_v^{(0)}>0$, corresponds to the minimum of the potential $U(v^{(0)})$, and the interception of the straight line $q^{*(0)}=\text{const}$ with branches I and III of the v-q^* curve at some points $v^{(0)}=v_1^{(0)}$ and $v^{(0)}=v_3^{(0)}$, where $F_v^{(0)}<0$ corresponds to the maximum of the potential $U(v^{(0)})$ [Fig. 16(b)]. Hence, in the range

$$q_0^{*(0)} < q^{*(0)} < q_0^{*\prime(0)} , \tag{48}$$

where the values $q_0^{*(0)} = q^{*(0)}(v_0^{(0)})$, $q_0^{*\prime(0)} = q^{*(0)}(v_0'^{(0)})$, the function $U(v^{(0)})$ has the shape of a potential pit [Fig. 16(b)].

The above-mentioned points $v^{(0)}=v_i^{(0)}$ $(i=1,2,3)$ satisfy (39), i.e., they are the singular (fixed) points of Eq. (32). It is easy to show that the fixed points $v_1^{(0)}$ and $v_3^{(0)}$, where $F_v^{(0)}<0$, are "saddle" points and the point $v_2^{(0)}$, where $F_v^{(0)}>0$, is a "center" [Fig. 16(c)]. Indeed, the solutions of Eq. (32), linearized near the extreme points $v_i^{(0)}$ $(i=1,2,3)$ of potential $U(v^{(0)})$, have the form (27), where

$$\lambda^{(i)} = \lambda_{1,2}^{(i)} = \pm(-F_v^{(0)}|_{v^{(0)}=v_i^{(0)}})^{1/2}, \quad i=1,2,3$$
$$F_v^{(0)}|_{v^{(0)}=v_2^{(0)}} > 0, \quad F_v^{(0)}|_{v^{(0)}=v_1^{(0)}}, \quad F_v^{(0)}|_{v^{(0)}=v_3^{(0)}} < 0 . \tag{49}$$

One can see from (49) that at points $v^{(0)}=v_1^{(0)}$ and $v^{(0)}=v_3^{(0)}$, where $F_v^{(0)}<0$, the coefficients $\lambda_1^{(i)}$ and $\lambda_2^{(i)}$ are real numbers of unlike signs. At a point $v^{(0)}=v_2^{(0)}$, where $F_v^{(0)}>0$, the coefficients $\lambda_1^{(2)}$ and $\lambda_2^{(2)}$ are strictly imaginary. Therefore, it follows that $v^{(0)}=v_1^{(0)}$ and $v^{(0)}=v_3^{(0)}$ are saddle points, and $v^{(0)}=v_2^{(0)}$ is a center.

2. *Form of solutions*

There is some value $q^{*(0)}=q_s^{*(0)}$ for which the potential $U(v^{(0)})$ in both saddle points $v^{(0)}=v_1^{(0)}=v_{s1}^{(0)}$ and $v^{(0)}=v_3^{(0)}=v_{s3}^{(0)}$ coincide [Fig. 16(b)]:

$$U(v_{s1}^{(0)})=U(v_{s3}^{(0)}) . \tag{50}$$

This value $q^{*(0)}=q_s^{*(0)}$ and the corresponding fixed points $v_1^{(0)}=v_{s1}^{(0)}$, $v_2^{(0)}=v_{s2}^{(0)}$, $v_3^{(0)}=v_{s3}^{(0)}$, as follows from (50) and (32), correspond to the equations

$$\begin{gathered}\int_{v_{s1}^{(0)}}^{v_{s3}^{(0)}} F^{(0)}(v^{(0)},q_s^{*(0)},v_g^{(0)})dv^{(0)}=0 \ , \\ F^{(0)}(v_{si}^{(0)},q_s^{*(0)},v_g^{(0)})=0, \quad i=1,2,3 \ .\end{gathered} \tag{51}$$

For $q^{*(0)}=q_s^{*(0)}$, the "particle" trajectories s_1 and s_2 [Fig. 16(c)] from one saddle point to another $(v_{s1}^{(0)},v_{s3}^{(0)})$ describe, with exponential accuracy, the distributions $v^{(0)}(x)$ in the form of moving steps [Figs. 16(d) and 16(e)]. But both of these solutions do not satisfy the boundary conditions (34). Contrary to it, a trajectory s, which is close to both separatrices, describes a solution $v^{(0)}(x)$ in the form of one *wide stratum* [Fig. 16(f)] which satisfies the boundary conditions (34). If $L,L_s \gg l$, then, with exponential accuracy, the extremal values $v_{\min}^{m(0)}=v_{s1}^{(0)}$ and $v_{\max}^{m(0)}=v_{s3}^{(0)}$ [Figs. 16(c) and 16(f)]. Notice that there are also a lot of other closed trajectories, which describe different periodical solutions $v^{(0)}(x)$ [Fig. 16(i)].

Taking into account the formula (16), one can see that the phase trajectory s on the (v,w) phase plane [Fig. 16(c)] and the solution $v^{(0)}(x)$ in the form of one wide stratum [Fig. 16(f)] found from the qualitative nonlinear theory approximately describe the corresponding trajectory [Fig. 15(b)] and the structure of a wide cluster of vehicles [Fig. 15(a)] found above from the numerical investigation. Therefore, for a wide cluster the values of the maximal and the minimal average velocities of vehicles $(v_{\max}^m,v_{\min}^m)$ and q_s^* (with the accuracy of ε) can be found from approximate formulas:

$$v_{\min}^m=v_{s1}^{(0)}, \quad v_{\max}^m=v_{s3}^{(0)}, \quad q_s^*=q_s^{*(0)} . \tag{52}$$

The same conclusion about qualitative agreement between the theory and the numerical investigation follows for a narrow cluster and a narrow anticluster. Indeed, one can see that for $q^{*(0)}>q_s^{*(0)}$ the trajectory 1 in Figs. 16(b) and 16(c) describes the distribution $v^{(0)}(x)$ in the form of the narrow stratum of low value $v^{(0)}$ [Fig. 16(g)]. These trajectories and solutions approximately describe the corresponding trajectory and the form of the narrow cluster, as follows from the numerical analysis. For $q^{*(0)}<q_s^{*(0)}$, the trajectory 2 [Figs. 16(b) and 16(c)] describes the distribution $v^{(0)}(x)$ in the form of a narrow stratum of high value $v^{(0)}$ [Fig. 16(h)], i.e., corresponds to a depleted local group of vehicles moving with larger velocity $v^{(0)}$. The trajectory and solution approximately describe the corresponding trajectory and the form of the narrow anticluster [20].

C. Parameters of clusters

1. *Velocity of clusters*

The expression for an evaluation of the velocity $v_g^{(0)}$ of a cluster can be found from the condition of solvability of the problem (35), (37) for the next (first) approximation in ε. This condition, known as the Fredholm alternative (e.g., [21]), is reduced to the requirement that the right-hand side of Eq. (35) has to be orthogonal to the solution $v_1^{(1)}(x)$ of the problem

$$\begin{gathered}d^2v_1^{(1)}/dx^2+F_v^{(0)}v_1^{(1)}=0 \ , \\ v_1^{(1)}(0)=v_1^{(1)}(L) \ , \\ dv_1^{(1)}/dx|_{x=0}=dv_1^{(1)}/dx|_{x=L} \ ,\end{gathered} \tag{53}$$

which is conjugate to the problem (35), (37) when the right-hand side of (35) is zero. Differentiating (32) and (34) with respect to x, one can get the problem

$$\begin{gathered}\frac{d^2}{dx^2}\frac{dv^{(0)}}{dx}+F_v^{(0)}\frac{dv^{(0)}}{dx}=0 \ , \\ \left.\frac{dv^{(0)}}{dx}\right|_{x=0}=\left.\frac{dv^{(0)}}{dx}\right|_{x=L} \ , \\ \left.\frac{d}{dx}\frac{dv^{(0)}}{dx}\right|_{x=0}=\left.\frac{d}{dx}\frac{dv^{(0)}}{dx}\right|_{x=L} \ .\end{gathered} \tag{54}$$

It follows from (53) and (54) that the function $w^{(0)}(x)\equiv dv^{(0)}/dx$ is proportional to the solution $v_1^{(1)}(x)$ of the problem (53). The functions $v^{(0)}(x)$ in Figs. 16(f)–16(h) are even functions of x. Therefore, the corresponding functions $w^{(0)}(x)$ are odd. On the other hand, the functions $F_{q^*}^{(0)}$, $F_{v_g}^{(0)}$ (38) on the right-hand side of Eq. (35) for the symmetric (with respect to point $x=L/2$) functions $v^{(0)}(x)$ [Figs. 16(f)–6(h)] are even functions of x. Consequently, the Fredholm alternative for this case, taking into account (19), (35), and (38), has the form of the expression

$$\int_0^L\{c_0^2-[v^{(0)}(x)-v_g^{(0)}]^2\} \times(v^{(0)}(x)-v_g^{(0)})^{-2}[w^{(0)}(x)]^2dx=0 \ , \tag{55}$$

which determines the value $v_g^{(0)}$ together with (32)–(34).

Outside of the fronts of a cluster, i.e., outside the regions, where an average velocity of vehicles $v^{(0)}(x)$ sharply change in space [Figs. 16(f)–16(h)], the function $w^{(0)}(x)\equiv dv^{(0)}/dx$ is close to zero. On the contrary, this function has the maximum near the point $v^{(0)}=v_{s2}^{(0)}$ [Fig. 16(c)], i.e., inside the fronts of a cluster. Taking it into account, one can get from (55) that the value $v_g^{(0)}$ for a cluster approximately is

$$v_g^{(0)}=v_{s2}^{(0)}-c_0 \ . \tag{56}$$

By comparison of this formula with formula (9b) for a phase velocity v_p of small amplitude perturbations near a threshold of an instability of a homogeneous flow one can see that $v_g-v_p\approx v_{s2}^{(0)}-v_h$. As a value $v_{s2}^{(0)}-v_h$ can be both positive and negative, the velocity of a cluster can differ from v_p not only quantitatively, but also by sign. This explains the results of the numerical investigations made in [1].

2. *Characteristic parameters of cluster*

From the numerical investigation of the parameters of a wide cluster it has been found (Sec. III B) that the values $\rho_{\min}^m$, $\rho_{\max}^m$, $v_{\min}^m$, $v_{\max}^m$, v_g^m, and q_s^* for a wide cluster do not practically depend on the density of vehicles in the initially homogeneous flow ρ_h and on the length of the road L as far as $L \gg l$. The theory presented can explain these results.

Substituting the value $v_g^{(0)} = v_{s2}^{(0)} - c_0$ from (56) into Eqs. (51) and taking into account (20), one can exclude the value $v_g^{(0)}$ and find four equations:

$$\int_{v_{s1}^{(0)}}^{v_{s3}^{(0)}} \left[V\left[\frac{q_s^{*(0)}}{v^{(0)}+c_0-v_{s2}^{(0)}} \right] - v^{(0)} \right] \left[\frac{q_s^{*(0)}}{v^{(0)}+c_0-v_{s2}^{(0)}} \right] dv^{(0)} = 0,\; V\left[\frac{q_s^{*(0)}}{v_{si}^{(0)}+c_0-v_{s2}^{(0)}} \right] - v_{si}^{(0)} = 0, \quad i=1,2,3 \tag{57}$$

for the values $v_{si}^{(0)}$ $(i=1,2,3)$, $q^{*(0)}$. The latter four values, which correspond to (23) and (52), approximately determine the values $\rho_{\min}^m$, $\rho_{\max}^m$, $v_{\min}^m$, $v_{\max}^m$ and q_s^* for a wide cluster, as one can see from (57), depend on the function $V(\rho)$ and on the value c_0, and do not depend on the values ρ_h and L. Numerical analysis of (57) shows that there is a wide range of functions $V(\rho)$ and values c_0 for which the system (57) has solutions corresponding to a wide cluster. One of the examples has been considered in Sec. III H.

Finding from (57) the value $v_{s2}^{(0)}$, one can also estimate the velocity of a wide cluster from (56). Hence, it appears that the values $\rho_{\min}^m$, $\rho_{\max}^m$, $v_{\min}^m$, $v_{\max}^m$, v_g^m, and q_s^* for a wide cluster practically do not depend on ρ_h and L. These conclusions of the nonlinear theory qualitatively explain the results of the numerical analysis conducted above (Sec. III). The formulas (56), (57) also show good presented in Sec. III H. For example, the velocity of the cluster found from the numerical calculation is $v_g \cong -1.249 l/\tau$, and from (56) and (57), $v_g^{(0)} \cong -1.244 l/\tau$. Also, for clusters of large amplitude (Sec. III B), formula (56) gives satisfactory agreement with the numerical calculations, if in (56) the value $v_{s2} \cong 1.476 l/\tau$ from the numerical calculation (Fig. 14) is used: from the numerical calculation $v_g \cong -1.09 l/\tau$, from (56) $v_g^{(0)} \simeq -1.01 l/\tau$.

From the theory presented corresponding to the numerical investigations, one can find that the width of a wide cluster L_s depends on both the density ρ_h and on the length of the road L. Indeed, the value L_s can be determined from Eq. (33). One can take into account that the regions of the fronts of a wide cluster [Fig. 16(f)] are of the order of l and for a wide cluster, $L, L_s \gg l$ (Sec. III B). So, to estimate the value L_s, one can neglect these regions of the fronts and write from Eq. (33) the approximate condition:

$$L_s = L \frac{\rho_h - \rho_{\min}^m}{\rho_{\max}^m - \rho_{\min}^m}, \tag{58}$$

where one took also into account (23) and (52). The analogue formula $L_s = L(\rho_h - \rho_{\min})/(\rho_{\max} - \rho_{\min})$ qualitatively describes the dependence of the width of the narrow cluster L_s on ρ_h and L.

From (58) it follows that corresponding to the numerical calculations (Sec. III B) the width of the wide cluster L_s monotonously increases with both the value ρ_h and L. If the value ρ_h is decreased, the value L_s also decreases and for the density ρ_h, which is still larger than $\rho_{\min}^m$, the wide cluster gradually transforms into the narrow cluster, at least when the value $L_s \to l$ [Fig. 16(g)]. The longer the length of the road L is, the less the difference between the values ρ_h and $\rho_{\min}^m$ is, which corresponds to the transformation of the wide cluster into the narrow cluster. This means that the region of density, where narrow clusters exist, is decreased with the increase in the length of the road L. These results explain the numerical investigations presented in Sec. III B.

On the other hand, one can also see from (58) that as $\rho_h \to \rho_{\min}^m$ the value L_s tends to zero even for a very large (but a finite) value L. It means that at $\rho_h < \rho_{\min}^m$ even for $L \to \infty$ a solution $v^{(0)}(x)$ [Fig. 16(f)], which describes a cluster, cannot exist, i.e., the value $\rho_h = \rho_{\min}^m$ is the lowest possible boundary density of vehicles in the initially homogeneous traffic flow for the existence of a cluster in the flow. This result of the theory can give a qualitative explanation of the numerical investigation, where it has been found that the boundary value of the density ρ_{b1} for the existence of a cluster tends to the value $\rho_{\min}^m$ if the length of the road L is increased (Sec. III D).

In the same way one can get the formula $\tilde{L}_s = L(\rho_{\max}^m - \rho_h)/(\rho_{\max}^m - \rho_{\min}^m)$ for the width of a wide anticluster $\tilde{L}_s$, where $\tilde{L}_s = L - L_s$. From this formula it follows that $\tilde{L}_s$ monotonously decreases if the value ρ_h is increased (Sec. III E) and also that the value $\rho_h = \rho_{\max}^m$ is the highest (for $L \to \infty$) possible boundary density of vehicles in the initially homogeneous flow for the existence of a wide anticluster in this flow.

V. CONCLUSIONS

A. Local cluster and other nonstationary clusters in traffic flow

(i) If a local perturbation of a finite amplitude in the initially homogeneous traffic flow occurs, a local cluster, which is surrounded both upstream and downstream by the same homogeneous flow, can spontaneously appear on a long enough road. The local cluster is a nonstationary moving localized structure. The local cluster consists of the following: (a) the proper cluster of vehicles, where the density of vehicles is considerably higher and the average velocity of vehicles is lower (up to a standstill) than in the initial flow; (b) the new almost homogeneous traffic flow formed by the cluster; and (c) a transition layer between this new flow and the initial traffic flow. The oscillations of the density and the average velocity of vehicles, which can occur in the transition layer of the local cluster at higher density of vehicles, are responsible for the spontaneous appearance of a complex localized structure which consists of a sequence of a lot of clusters.

(ii) There is a boundary density (ρ_b) for an excitation of the local cluster: if the density is less than the boundary density, the local cluster cannot be excited. There is also a critical density ($\rho_{\rm cr}$), when small-amplitude local perturbations grow in traffic flow. The boundary density is lower than the critical density: $\rho_b < \rho_{\rm cr}$. Both the boundary and the critical density are characteristic parameters of the traffic flow. In the interval of density $(\rho_b, \rho_{\rm cr})$ a local cluster can be formed, if a critical local perturbation whose amplitude exceeds a critical value, occurs. This critical local fluctuation can be considered as a "nucleation center" for the formation of local clusters. The critical amplitude of the local perturbation is a falling function of the density: it is maximal at the density ρ_b and it tends to zero, when the density approaches the value $\rho_{\rm cr}$. The parameters of the local cluster formed do not depend on the amplitude of the local perturbation but only on the parameters of the traffic flow.

(iii) An appearance of a lot of local perturbations in traffic flow can lead to the formation of many interacting moving nonstationary clusters. These clusters, especially in the vicinity of the critical density $\rho_{\rm cr}$, can have different amplitudes, different widths, different velocities and they may not be situated periodically in space. The effect of the appearance of such a complex sequence of moving nonstationary clusters, collisions between clusters which cause their merger, and also a possible process of random appearance and disappearance of clusters, may explain the chaotic behavior of traffic flow which has often been observed in experimental investigations.

B. Stationary moving clusters on circular road

(i) The critical density (ρ_{c1}) of the instability of a traffic flow on a circular road that is long enough with respect to a growth of a small-amplitude global perturbation [see (9a)] is higher than the boundary density for the excitation of the local cluster and it is lower than the critical density of the growth of small-amplitude local perturbations: $\rho_b < \rho_{c1} < \rho_{\rm cr}$.

(ii) On a circular road a stationary moving cluster of vehicles or a sequence of many stationary moving clusters can be formed. The process of a formation of a stationary moving cluster in an initially homogeneous traffic flow near the critical density ρ_{c1} is linked to the spontaneous appearance of a local cluster of vehicles on the circular road that is long enough.

(iii) There are wide or narrow stationary moving clusters in traffic flow. Besides, at a relatively large density of vehicles, there are wide or narrow stationary moving anticlusters. In the anticluster, the density is lower and the average velocity of vehicles is higher than in the initial homogeneous flow.

(iv) A wide cluster represents a local region of a large density and a low average velocity of vehicles, where the traffic flow is almost homogeneous. This region is restricted by two moving fronts, where both the density and the average velocity of vehicles sharply change in space. The homogeneous flows inside and outside the cluster are stable and differ from the initial homogeneous traffic flow. The velocity of the wide cluster, the maximal and the minimal density, and the average velocity of vehicles in it are the characteristic parameters of the traffic flow, as they do not practically depend on the density of vehicles in the initial homogeneous flow and on the length of the road, if the latter is large enough.

(v) If the density of vehicles in the initially homogeneous flow is decreased, the width of a wide cluster decreases, too, and the wide cluster gradually transforms into the narrow cluster. The amplitude of the narrow cluster decreases with the density of vehicles in the initially homogeneous flow. On the contrary, the wide anticluster gradually transforms into the narrow anticluster as the density of vehicles in the initially homogeneous flow is increased.

(vi) There is a boundary density of vehicles (ρ_{b1}) in the initially homogeneous flow, when a stationary moving cluster in traffic flow still exists. The width of a cluster is minimal at this boundary density. For the large enough length of the road this boundary density is lower than the density, which is critical for the stability of traffic flow with respect to small-amplitude global perturbations: $\rho_{b1} < \rho_{c1}$. This means that a cluster of vehicles can appear at a density of vehicles, where the homogeneous flow is stable with respect to small-amplitude fluctuations. If the length of the road is increased, the boundary density ρ_{b1} decreases and tends to the definite minimal value which is reached for the infinitely long road and practically coincides with the boundary density ρ_b of an excitation of a local cluster.

ACKNOWLEDGMENTS

We would like to thank Mr. P. Häussermann for his fruitful discussions and interest in the problem considered in the paper and also Mr. M. Schilke for help with carrying out the numerical investigations.

[1] B. S. Kerner and P. Konhäuser, Phys. Rev. E **48**, 2335 (1993).
[2] W. Leutzbach, *Introduction to the Theory of Traffic Flow* (Springer-Verlag, Berlin, 1988).
[3] J. Treiterer, Ohio State University Technical Report PB 246 094, Columbus, Ohio, 1975 (unpublished).
[4] The diffusion term in the kinetic model of traffic flow has been introduced by R. Kühne [see, e.g., R. Kühne, in *Highway Capacity and Level of Service*, edited by U. Brannolte (A. A. Balkema, Rotterdam, 1991), p. 211].

[5] I. Prigogine and R. Herman, *Kinetic Theory of Vehicular Traffic* (American Elsevier, New York, 1971).

[6] H. J. Payne, in *Mathematical Models of Public Systems,* Simulation Council Proceedings, 1971 (Simulation Councils, La Jolla, CA 1971), Vol. 1, p. 51.

[7] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).

[8] *Noise in Nonlinear Dynamical Systems, Vols. 1–3,* edited by F. Moss and P. V. E. McClintock (Cambridge University Press, New Rochelle, 1989).

[9] G. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and Natural Sciences* (Springer-Verlag, Berlin, 1983).

[10] F. L. Hall, L. A. Brian, and M. A. Gunter, Transp. Res. A **20**, 197 (1986).

[11] H. B. Keller, Soc. Ind. Appl. Math. J. Numer. Anal. **11**, 305 (1974).

[12] Notice that the line $\bar{q}(\bar{\rho})$, which corresponds to the clusters (line C in Fig. 12), does not reach exactly the points $\rho^m_{\min}$ and $\rho^m_{\max}$. Indeed, the clusters can exist in the interval $\rho_{b1} \leq \bar{\rho} \leq \rho_{b2}$, but $\rho_{b1} > \rho^m_{\min}$ and $\rho_{b2} < \rho^m_{\max}$ (Secs. III D and III E). Nevertheless, for a simplification of the consideration of the average fundamental diagram this circumstance has not been taken into account in Fig. 12, because a difference between the value $\rho^m_{\min}$ and the value ρ_{b1} (or ρ_{b2} and $\rho^m_{\max}$), when the length of the road L is large enough, can be very small.

[13] M. J. Lighthill and B. G. Whitham, Proc. R. Soc. London, Ser. A **229**, 317 (1955).

[14] A. D. May, *Traffic Flow Fundamental* (Prentice Hall, Inc., New Jersey, 1990).

[15] R. Wiedemann, *Simulation des Straßenverkehrsflusses* (Schriftenreihe des Instituts für Verkerswesens Universität Karlsruhe, Heft 8, 1974).

[16] H.-T. Fritzsche, Traffic Eng. Control **35**, 317 (1994).

[17] K. Nagel and M. Schreckenberg, J. Phys. I (France) **2**, 2221 (1992).

[18] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maier, *Qualitative Theory of Second-Order Dynamic Systems* (Wiley, New York, 1973).

[19] The velocity of a wide and of a narrow cluster does not coincide (see Sec. IV C). For this reason, although the qualitative form of the v-q^* curve for both the wide and the narrow clusters has the same И shape [Fig. 16(a)], these v-q^* curves quantitatively differ one from another. For shorting the quantity of the figures we have not taken into account this circumstance in Figs. 16(a)–16(c) which does not change the qualitative result presented.

[20] The shape of the functions $v^{(0)}(x)$ in Figs. 16(d)–16(i) is formally analogous to the shape of solutions for other physical variables found from the analysis of different nonlinear physical problems, for example, in the investigations of localized dissipative structures (autosolitons) in nonlinear active systems with diffusion. Notice also that, as follows from the properties of the local cluster found (Sec. II C), some of them are qualitatively similar to general properties of autosolitons in active systems with diffusion (for a review, see B. S. Kerner and V. V. Osipov, Usp. Fiz. Nauk **157**, 201 (1989) [Sov. Phys. Usp. **32**, 101 (1989)]; B. S. Kerner and V. V. Osipov, *Autosolitons: A New Approach to the Problems of Self-Organization and Turbulence* (Kluwer Academic, Dordrecht, 1994).

[21] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers* (McGraw Hill, New York, 1961).

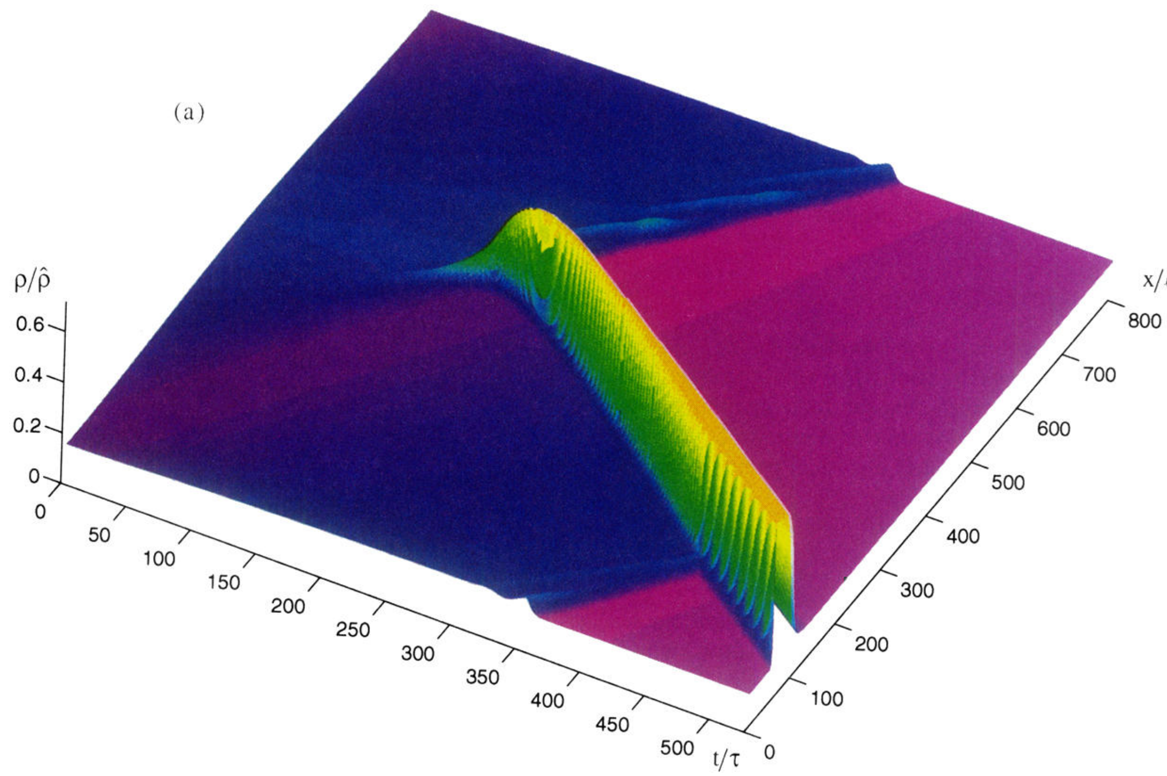

FIG. 2. The kinetics of the cluster formation: (a) the dependence $\rho(x,t)$, (b)–(e) the distributions $\rho(x)$ and $v(x)$ in the intermediate moments of time [(b) $t_1=100\tau$, (c) $t_2=150\tau$, (d) $t_3=216\tau$, (e) $t_4=300\tau$]; (f) the distributions $\rho(x)$ and $v(x)$ in the stationary cluster (at $t=490\tau$), which moves with the constant velocity $v_g^m \cong -1.09 l/\tau$ [for a visual demonstration, the functions $\rho(x)$ and $v(x)$ in (f) are shifted to the center of the road]. The initial distribution $\rho(x,0)=\rho_h+\delta\rho(x,0)$ with $\delta\rho$ (9a), $\delta\rho_0=0.02$; $\rho_h=0.174\hat{\rho}$. The other parameters are the same as in Fig. 1.

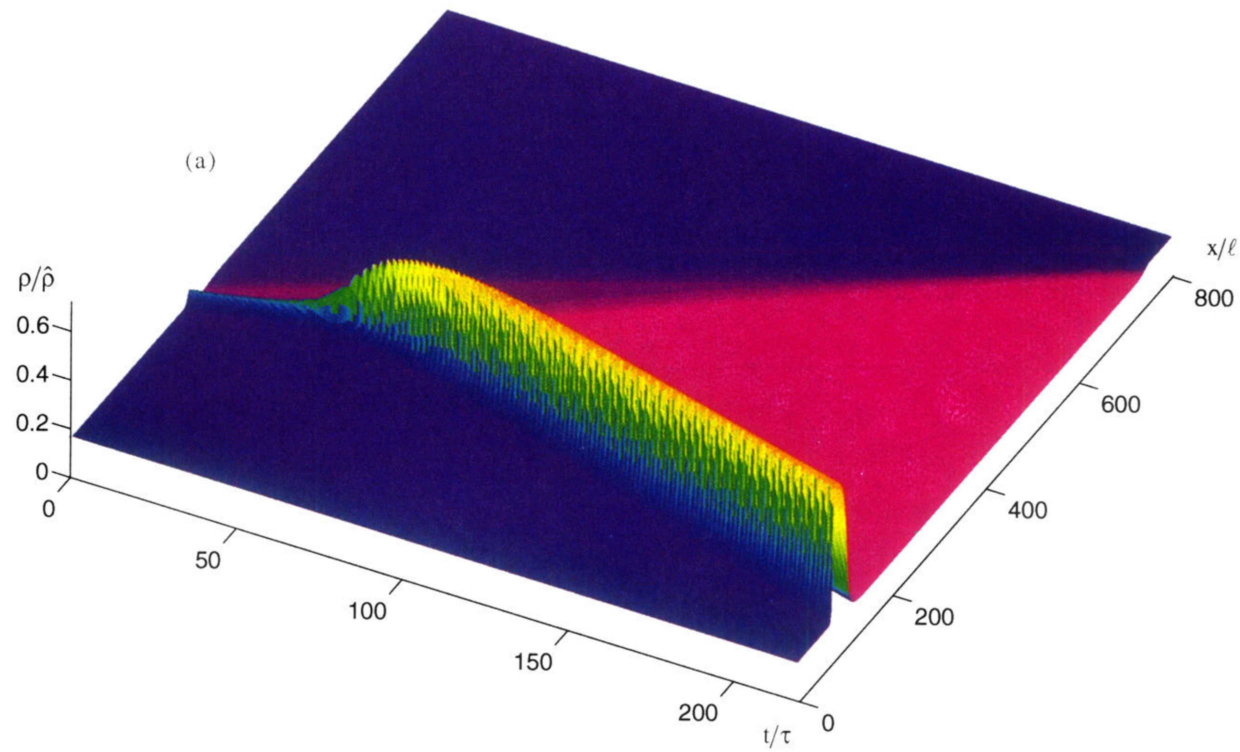

FIG. 5. The kinetics of the local cluster formation in the stable homogeneous traffic flow: (a) the dependence $\rho(x,t)$; (b) and (c) the distributions of the functions $\rho(x)$ and $v(x)$ (b) and the corresponding curve $q(\rho)$ (c) at $t=144\tau$; (d) the vehicle trajectories corresponding to (a). $\rho_h=0.17\hat{\rho}$. The initial local perturbation $\Delta\rho(x)$ (11), $\Delta\rho_m=0.06\hat{\rho}$, $x_0=250l$. The other parameters are the same as in Fig. 1.

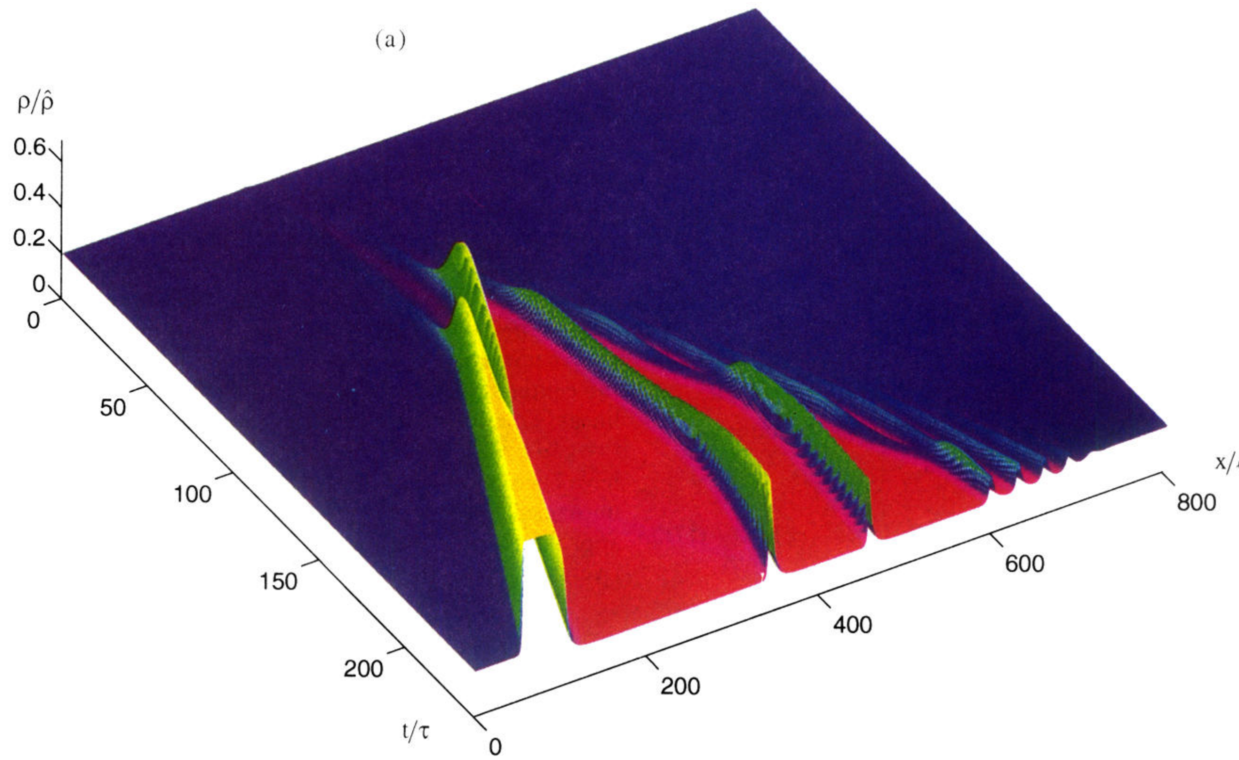

FIG. 9. The kinetics of the appearance of the sequence of clusters in traffic flow: (a) the dependence $\rho(x,t)$; (b) and (c) the distributions of the functions $\rho(x)$ and $v(x)$ at $t=150\tau$ (b) and at $t=180\tau$ (c); (d) the curve $q(\rho)$, which corresponds to (c). The initial local perturbation $\Delta\rho(x)$ (11), $\Delta\rho_m=0.005\hat{\rho}$, $x_0=210l$; $\rho_h=0.2\hat{\rho}$. The other parameters are the same as in Fig. 1.