Concise calculation of the scaling function, exponents, and probability functional of the Edwards-Wilkinson equation with correlated noise

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The linear Langevin equation proposed by Edwards and Wilkinson [Proc. R. Soc. London A **381**, 17 (1982)] is solved in closed form for noise of arbitrary space and time correlation. Furthermore, the temporal development of the full probability functional describing the height fluctuations is derived exactly, exhibiting an interesting evolution between two distinct Gaussian forms. We determine explicitly the dynamic scaling function for the interfacial width for any given initial condition, isolate the early-time behavior, and discover an invariance that was unsuspected in this problem of arbitrary spatiotemporal noise.

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I. INTRODUCTION

Following the much celebrated work of Kardar, Parisi, and Zhang (KPZ) regarding the kinetic roughening of Eden clusters, ballistic deposits, and single-step surfaces [1], there has recently been a concerted effort to tailor stochastic differential equations to actual film growth via molecular beam epitaxy (MBE), particularly in circumstances where surface diffusion of adatoms to highly coordinated sites is assumed the dominant relaxation process [2–4]. In such "ideal MBE" surface diffusion models, vacancies, overhangs, and particle desorption are strictly forbidden, apparently ensuring dynamic scaling behavior outside the KPZ universality class. For the Wolf-Villain equation (WV) [2], as well as its nonlinear variant due to Villain [3], Lai and Das Sarma [4], the relaxation is governed by a Laplacian squared term, arising from a *quasi-equilibrium* contribution to the surface diffusion current. The true asymptotic scaling properties of realistic microscopic models of MBE became a matter of some controversy, however, soon after the introduction of these conserved-particle surface diffusion equations, with some suggestions that it would be KPZ-like [5]. Even the WV model, innocuous at first glance, exhibits a rich surface roughness, as revealed by Krug, Plischke, and Siegert [6]. Interestingly, these authors show that the intrinsically *nonequilibrium* conditions of growth yield surface diffusion processes that generate a tilt-dependent mass current, ultimately responsible for the scaling properties of these models. In particular, if the nonequilibrium contribution to the net diffusion current is a decreasing function of the inclination, it stabilizes the surface, leading generically, but somewhat surprisingly, to dynamic scaling characteristic of the Edwards-Wilkinson universality class [7]. By contrast, a nonequilibrium surface diffusion current that is an increasing function of inclination incurs a growth instability, resulting in a grooved state. In fact, both types of behavior can be seen within a single model, following from a nontrivial tilt dependence of the nonequilibrium surface diffusion current, as previously observed by Siegert and Plischke [8]. Consequently, the kinetic roughening community concerned with ideal MBE surface diffusion models may, despite all their efforts, find themselves at the end of the day face-to-face with the Edwards-Wilkinson (EW) equation [9]. It is ironic, indeed, that a good decade following its introduction, the Edwards-Wilkinson equation, being the very first attempt to understand the dynamic scaling properties of stochastically roughened self-affine surfaces, is found at center stage. Because of the newly appreciated implications of EW, as well as some additional unsettled matters concerning correlated noise within the KPZ context [10–12], we have returned to this fundamental linear Langevin equation to explore some of its hitherto unaddressed statistical properties.

Rather than transforming the equation into frequency space, however, as was done by Edwards and Wilkinson in their original paper [6], and is implicit in all renormalization group schemes applied to its nonlinear generalizations [9–10], we have solved the EW equation exactly in t space, encoding all possible initial conditions in our final expression for fluctuating surface profile $h(x,t)$. Our calculation affirms explicitly the scaling behavior of the interface width $w=L^{\chi}f(t/L^{z})$ as a function of length scale L, and we obtain directly the critical exponents $\beta=\chi/z=\rho/2+\theta+\frac{1}{4}$, $\chi=\rho+2\theta+\frac{1}{2}$, and $z=2$ for the early-time, steady-state, and dynamic behaviors, respectively. Furthermore, we derive the exact expression for the scaling function f with arbitrary initial condition and noise of any given spatial and temporal correlation. These results not only show the steady-state behavior, but also capture the transient interface growth, which can be compared with numerical simulation directly, permitting determination of various phenomenological parameters. The outline of this paper is as follows. In Sec. II, we examine initially the case of spatially correlated noise and calculate physically relevant quantities, such as the interface width, the height difference correlation, etc. Generalization to temporal correlations is straightforward, but reveals a surprising spatiotemporal invariance. In Sec. III, we derive the time evolution of the full probability functional for the surface fluctuations and discuss its interesting features and consequences. Finally, we summarize the implications of our results in Sec. IV.

II. SPATIALLY AND TEMPORALLY CORRELATED NOISE

Our starting point is the EW equation, appropriate to kinetic roughening phenomena in which the competing effects are surface tension relaxation and the incessant peppering of stochastic noise. In the original EW model of sedimentation, gravity was explicitly responsible for an external stabilizing effect of the surface, while stochasticity arose from the random deposition of material. The interfacial profile, $h(x,t)$, then develops in accordance with the linear Langevin equation

$$\partial_t h(x,t) = \nu \nabla^2 h(x,t) + \eta(x,t) , \quad (2.1)$$

where $\eta(x,t)$ represents the noise, with spatially correlated variance,

$$\langle \eta(x,t)\eta(x',t') \rangle = D_0 \rho |x-x'|^{2\rho-1} \delta(t-t') .$$

After a Fourier transformation to k space, we get

$$\begin{aligned} &\partial_t h_k(t) = -\nu k^2 h_k(t) + \eta_k(t) , \\ &\langle \eta_k(t)\eta_{k'}(t') \rangle = 2D(\rho) k^{-2\rho} \delta(t-t')\delta(k+k') , \end{aligned} \quad (2.2)$$

where $D(\rho) = 2\pi\rho D_0 \int_0^\infty du\, u^{2\rho-1} \cos u = \pi D_0 \Gamma(1+2\rho) \times \cos(\pi\rho)$, which requires $0 \le \rho < \frac{1}{2}$ for convergence. The solution to this first-order linear differential equation is

$$h_k(t) = c_k e^{-\nu k^2 t} + e^{-\nu k^2 t} \int_0^t e^{\nu k^2 t'} \eta_k(t') dt' , \quad (2.3)$$

where the c_k's are Fourier coefficients specifying the initial interface profile. By adding the proper constant to $h(x,t)$, we can always set $\int_{-\infty}^{\infty} h(x,0) dx = 0$, which implies $c_{k=0} = 0$. Note that the flat initial condition [that is, $h(x,0) = 0$] corresponds to $c_k = 0$ for all k.

Let L denote part of the system; the interfacial width, $w(L,t)$, can be calculated exactly as a function of L and t. Defining, as usual, $w^2(L,t) \equiv L^{-1} \langle \int_0^L dx [h(x,t) - h_L(t)]^2 \rangle$, with $h_L(t) \equiv L^{-1} \int_0^L h(x,t) dx$, and $\langle \ \rangle$ as the statistical average, we find

$$w^2(L,t) = L^{2\rho+1} f(\nu t / L^2) + w_n^2(L,t) \quad (2.4)$$

where the scaling function $f(a = \nu t / L^2)$ has the form

$$\begin{aligned} f(a) &= \frac{D(\rho)}{4\pi^2 \nu} \int_0^\infty \frac{dq}{q^{2\rho+2}} \left[1 - \frac{2 - 2\cos q}{q^2} \right] (1 - e^{-2q^2 a}) \\ &= \frac{D_0}{4\nu(2\rho+1)(2\rho+3)} \left\{ \frac{1}{(2\rho+2)} - \frac{(8a)^{\rho+3/2}}{2\sqrt{\pi}} \Gamma(1+\rho) \left[{}_1F_1 \left[-\rho - \frac{3}{2}, \frac{1}{2}; \frac{-1}{8a} \right] - 1 - \frac{2\rho+3}{8a} \right] \right\} \end{aligned} \quad (2.5)$$

in which ${}_1F_1$ is a degenerate hypergeometric function indexed by the spatial correlation parameter ρ, while the noise-free contribution

$$w_n^2(L,t) = \int_{-\infty}^{\infty} \frac{dk\, dp}{4\pi^2} c_k c_p e^{-\nu(k^2+p^2)t} \left[\frac{e^{i(k+p)L} - 1}{i(k+p)L} - \frac{e^{ikL} - 1}{ikL} \frac{e^{ipL} - 1}{ipL} \right]$$

is transient, vanishing in the long-time limit, thanks to the exponential factor and the initial condition $c_{k=0} = 0$.

Note immediately from (2.4) that we have the *saturation-width* exponent $\chi = \rho + \frac{1}{2}$, as well as the *dynamic* exponent $z = 2$. Of course, for the case of uncorrelated noise ($\rho = 0$), these exponents had been derived by EW. In Fig. 1, we show the form of this hypergeometric scaling function for various values of ρ, which serves to indicate how the crossover to steady-state roughness is delayed for increasing noise correlation. When $t \to \infty$, we find

$$\begin{aligned} f(a)|_{a\to\infty} &= \frac{D(\rho)}{4\pi^2 \nu} \int_0^\infty \frac{dq}{q^{2\rho+2}} \left[1 - \frac{2 - 2\cos q}{q^2} \right] \\ &= \frac{D_0}{4\nu} \frac{1}{(2\rho+1)(2\rho+2)(2\rho+3)} \end{aligned} \quad (2.6)$$

which gives us the amplitude prefactor for the steady-state interfacial width. Extraction of the early-time behavior requires, by contrast, an appreciation of the asymptotics of our hypergeometric function; in the small a limit, (2.5) yields

$$\begin{aligned} f(a) &\simeq a^{\rho+1/2} \left[\frac{D(\rho)}{4\pi} \int_0^\infty \frac{du}{u^{2\rho+2}} (1 - e^{-2u^2}) \right] + O(a^1) \\ &= \frac{D_0}{4\nu} \frac{\Gamma(1+\rho)}{2(2\rho+1)} \frac{(8a)^{\rho+1/2}}{\sqrt{\pi}} + O(a^1) . \end{aligned} \quad (2.7)$$

Consequently, the initial development of the width is

$$w^2(L,t) - w_n^2(L,t) = (8\nu t)^{\rho+1/2} \frac{D_0}{4\nu} \frac{\Gamma(1+\rho)}{2(2\rho+1)\sqrt{\pi}} , \quad (2.8)$$

with the *early-time* roughness exponent $\beta = \rho/2 + \frac{1}{4}$. Other quantities of interest, such as correlation functions, are

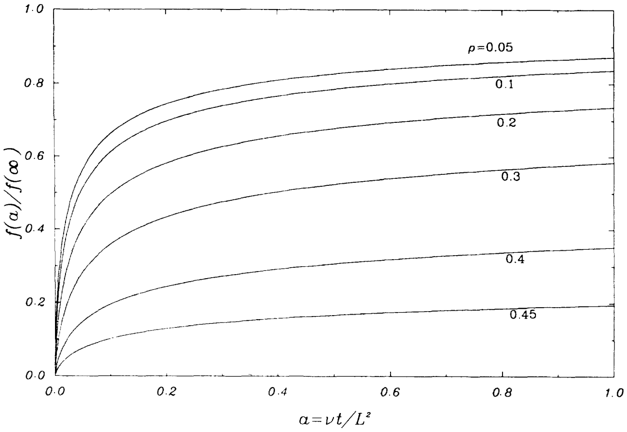

FIG. 1. The dynamic scaling function for the interfacial width, as calculated from the Edwards-Wilkinson equation under the assumption of spatially correlated stochastic noise $\langle \eta(x,t)\eta(x',t')\rangle = D_0\rho|x-x'|^{2\rho-1}\delta(t-t')$. For each value of the spatial correlation parameter ρ, the function has been normalized to approach unity for infinite argument.

readily calculated; for example,

$$\langle [h(x,t)-h(0,0)]^2\rangle - \langle [h_n(x,t)-h_n(0,0)]^2\rangle = \frac{D(\rho)}{4\pi^2\nu}(t\nu)^{\rho+(1/2)}\int_{-\infty}^{\infty} du \frac{(1-e^{-2u^2})}{(u^2)^{\rho+1}} \tag{2.9}$$

which is *independent* of x, where the subscript n is used, again, to describe the corresponding quantity in the absence of noise. In addition, the height difference correlation function in the steady state is given by

$$\begin{aligned}\lim_{t_0\to\infty} [\langle [h(x,t_0+t)-h(0,t_0)]^2\rangle & - \langle [h_n(x,t_0+t)-h_n(0,t_0)]^2\rangle] \\ &= \lim_{t_0\to\infty} \langle [h(x,t_0+t)-h(0,t_0)]^2\rangle - 0 \qquad (2.10)\\ &= \frac{D(\rho)}{2\pi^2\nu}\int_{-\infty}^{\infty} dk \frac{[1-e^{-k^2t\nu}\cos(kx)]}{(k^2)^{\rho+1}}\end{aligned}$$

so we see that the initial condition *does* change the functional form of the interfacial width, as well as the height-height correlation function. However, after subtracting the noise-free part, we regain the universal form, including both early-time scaling and steady-state behaviors, independent of the arbitrarily given initial interface profile. Note that our analysis, foregoing Fourier transformation to frequency space, permits a complete description of the surface evolution, whereas the earlier work of EW handled only the long-time limit.

Finally, we allow temporal correlation in the noise, but consider for simplicity only an initially flat profile, generalization to other starting configurations being straightforward. Assuming the noise correlation given by

$$\langle \eta(x,t)\eta(x',t')\rangle = D_0\rho\theta|x-x'|^{2\rho-1}|t-t'|^{2\theta-1} \tag{2.11}$$

a simple calculation then shows

$$w^2(L,t) = L^{2\rho+4\theta+1} f_1(\nu t/L^2) \tag{2.12}$$

where, with $a=\nu t/L^2$ as before,

$$f_1(a) = \frac{D(\rho)}{4\pi^2\nu}\int_0^\infty \frac{dq}{q^{2\rho+4\theta+2}}\left[1-\frac{2-2\cos q}{q^2}\right] g(q^2a) \tag{2.13}$$

and

$$g(y) = \int_0^y du\, u^{2\theta}e^{-u} + e^{-2y}\int_0^y du\, u^{2\theta}e^{u} . \tag{2.14}$$

Now, however, $\rho+2\theta<\frac{1}{2}$ is necessary to avoid the infrared divergence of (2.13); in addition, it might be noted that $\theta=0$ is not a well-defined limit in this continuum picture. From (2.12), it is evident that the steady-state roughness exponent $\chi=\rho+2\theta+\frac{1}{2}$, while the dynamical index $z=2$. By contrast, an examination of the asymptotics of f_1 reveals the early-time behavior

$$w^2(L,t) \simeq (\nu t)^{\rho+2\theta+(1/2)}\left[\frac{D(\rho)}{4\pi^2\nu}\int_0^\infty \frac{g(u^2)}{u^{2\rho+4\theta+2}}du\right]$$

so we have $\beta=\theta+(\rho/2)+\frac{1}{4}$.

III. PROBABILITY FUNCTIONAL

We limit ourselves, here, to the case of spatially correlated noise and derive the full probability functional characterizing the height fluctuations of the surface profile. The variance of the noise is as follows:

$$\langle \eta(x,t)\eta(x',t')\rangle = \begin{cases} D_0\rho|x-x'|^{2\rho-1}\delta(t-t') & \text{correlated ;} \\ D_0\delta(x-x')\delta(t-t') & \text{uncorrelated .}\end{cases} \tag{3.1}$$

After a Fourier transformation, we get

$$\langle \eta_k(t)\eta_{k'}(t')\rangle = 2Dk^{-2\rho}\delta(t-t')\delta(k+k') \tag{3.2}$$

where $D=D(\rho)$, as before, for the spatially correlated case, while $D=D_0\pi$ for purely white noise. Note that our $D(\rho)$ gives, as $\rho\to 0$, the same spectral density as white noise; therefore, we see that the probability functional of these two cases is the same, as expected. Since $\eta_k^*(t)=\eta_{-k}(t)$, the set of random variables $\{\eta_k(t)\}$ is reduced to $\{\eta_k(t), k>0\}$ and the variance can be rewritten as

$$\langle \eta_k(t)\eta_{k'}^*(t')\rangle = 2Dk^{-2\rho}\delta(t-t')\delta(k-k')$$

with the restriction $k>0$. Thus, $\eta_k(t)$ can be understood as uncorrelated both in k space and in t space, with the spectral density $2Dk^{-2\rho}$. That is to say, the probability density of $\{\eta_k(t)\}$ equals the product of the probability density of each element, i.e., having $k>0$ in mind,

$$P(\{\eta_k(\tau)\}|_{\tau=0}^{t}) = \prod_{k,\tau} \rho(\eta_{k,\tau})$$

To see whether or not the detailed probability distribution of the random noise can influence the macroscopic growth behavior, including particularly the probability functional of the height profile, we have used both uniform and Gaussian distributions for the noise.

(i) Uniform distribution

$$\rho(\eta_{k,\tau})=f_r(\eta_{k,\tau})f_i(\eta_{k,\tau}) ,$$

where f_r is the real part's distribution function of $\eta_{k,\tau}$ and f_i is the imaginary part's distribution function of $\eta_{k,\tau}$,

$$f_{r(i)}(x_k)=\begin{cases}0, & \text{if } |x_k|>L_k\\ \frac{1}{2}L_k, & \text{if } |x_k|<L_k\end{cases}$$

where x_k is $\eta_{k,\tau}$ and $L_k=(3Dk^{-2\rho}/\Delta t\Delta k)^{1/2}$ from (3.2).

(ii) Gaussian distribution.

$$\rho(\eta_{k,\tau})\propto\exp\left[-\Delta t\Delta k\frac{\eta_{k,\tau}\eta^*_{k,\tau}}{2Dk^{-2\rho}}\right] .$$

Recall that under the assumption of a flat initial condition, we have

$$h_k(t)=e^{-\nu k^2 t}\int_0^t e^{\nu k^2 t'}\eta_k(t')dt' .$$

By definition, the probability functional of the height profile is

$$P[\{h_k\},t]=\int D[\eta]P[\{\eta_k(\tau)\}|_{\tau=0}^{t}] \times\delta^f\left[h_k-e^{-\nu k^2 t}\int_0^t d\tau e^{\nu k^2\tau}\eta_k(\tau)\right] . \quad (3.3)$$

Substituting the expression for $P[\{\eta_k(\tau)\}|_k|_{\tau=0}^{t}]$, as well as the integral representation of the δ function into (3.3), we obtain the following result:

$$P[\{h_k\},t]=\mathcal{N}^{-1}\exp\left[-\frac{\nu}{2D}\int dk\frac{k^{2\rho+2}}{1-e^{-2\nu k^2 t}}h_k h_k^*\right] \quad (3.4)$$

with

$$\mathcal{N}=\int D[h]\exp\left[-\frac{\nu}{2D}\int dk\frac{k^{2\rho+2}}{1-e^{-2\nu k^2 t}}h_k h_k^*\right]$$

regardless of the probability distribution used for the random noise. This is, perhaps, the first analytical confirmation of the usual hypothesis about the universal behavior of the interface growth independent of the detailed distribution of the noise.

When the noise is uncorrelated in both space and time, the probability functional for the height fluctuations is given by

$$P[\{h_k\},t]\propto\exp\left[-\frac{\nu}{2\pi D_0}\int dk\frac{k^2}{1-e^{-2\nu k^2 t}}h_k h_k^*\right] \quad (3.5)$$

so that, in the small time limit, it behaves as $e^{-(1/4\pi D_0)\int dx h^2/t}$, which is Brownian in nature. When $t\to\infty$, the steady state of the probability function is $e^{-(\nu/2\pi D_0)\int dx(\nabla h)^2}$. Actually, when the noise is Gaussian, we can understand the probability functional intuitively. Because $h_k(t)$ is a linear combination of $\eta_k(\tau)$ in t space, h_k keeps the Gaussian form and remains uncorrelated in k space. To find $P[\{h_k\},t]$, we need only know its second moment $\langle h_k(t)h_k^*(t)\rangle$. A simple calculation gives the second moment and the full probability functional for the height fluctuations follows quickly.

IV. CONCLUSION

We have solved the Edwards-Wilkinson equation in closed form for noise of arbitrary spatial and temporal correlation, assuming the most general initial condition. Foregoing transformation to frequency space, we work explicitly with the time variable and are able to write down a scaling expression for the surface roughness in terms of hypergeometric functions, documenting precisely the effect of spatially correlated noise. This enables us to extract directly the early-time roughness exponent, as well as the steady-state amplitude. Direct comparison with correlated EW-type growth models [13] would be most welcome. If both space and time correlations are present, the dynamic scaling exponents depend only upon the spatiotemporal combination $\rho+2\theta$, a symmetry not shared by the KPZ equation, thanks to its characteristic nonlinearity. Finally, we have determined the complete time evolution of the probability distribution for the height fluctuations, revealing an intriguing development between distinct early-time and steady-state Gaussian forms. It is our hope that knowledge of these quantities in close form may rejuvenate similar, though admittedly more demanding, efforts in the KPZ context.

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