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Kinematics of vorticity: Vorticity-strain conjugation in incompressible fluid flows

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An exact kinematic analysis is made of the three-dimensional incompressible Euler flows. It is found that the vorticity and rate-of-strain tensors are connected with each other through an identical singular integral transform. Some formal properties of this transform are derived. In particular, there exist harmonic functions in (3+1)-dimensional space so that the boundary values (toward our three-dimensional physical space) of a pair of conjugates are simply the vorticity and rate-of-strain tensors. The generalized Cauchy-Riemann equations are explicitly written. As an application, three of Siggia's invariants are related by some integrals.

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Singular integral transforms [1] are inherent in the nonlocal nature of vortex stretching in three-dimensional incompressible flows. Nonlocality appears in the pressure term in the Euler equations and in the integral relationship between the vorticity and the strain in the vorticity equations [2-4]. The pressure Hessian [2,3,5,6] is another example, contributing to the evolution of the rate of strain. We intend to give a theoretical foundation for the vorticity-strain correlation with an explicit use of singular integral transforms.

There is a one-dimensional model for the vorticity equation, the Constantin-Lax-Majda model [7],

$$\frac{\partial \omega}{\partial t} = H(\omega)\omega, \tag{1}$$

where

$$H(\omega) = \frac{1}{\pi} \oint \frac{\omega(y)}{x-y} \mathrm{d}y$$

is the Hilbert transform and $\oint$ denotes the principal-value integral. The "vorticity" ω and "rate-of-strain" $H[\omega]$ are Hilbert conjugates and are real and imaginary parts of an analytic function in the upper-half plane (note also that $H^2 = -1$). Actually, this model could mean more than it seems.

We consider the motion of an inviscid fluid governed by the three-dimensional Euler equations,

$$\frac{Du_i}{Dt} = -\partial_i p,$$

together with the incompressibility condition $\nabla \cdot \boldsymbol{u} = 0$ ($\partial_i = \partial/\partial x_i$.) Here $D/Dt = \partial/\partial t + (\boldsymbol{u}\cdot\nabla)$ denotes the Lagrangian time derivative, $\boldsymbol{u}$ the velocity, and p the pressure. We treat the infinite space case with a fluid at rest at infinity. The velocity can be expressed as $\boldsymbol{u} = \nabla \times \boldsymbol{A}$ by the vector potential $\boldsymbol{A}$. If we take a curl under Coulomb gauge $\nabla \cdot \boldsymbol{A} = 0$, we have $\nabla^2 \boldsymbol{A} = -\boldsymbol{\omega}$, or

$$\boldsymbol{A}(\boldsymbol{x}) = \frac{-1}{4\pi} \int \frac{\boldsymbol{\omega}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} \mathrm{d}\boldsymbol{y}. \tag{2}$$

Taking the curl of (2) yields the Biot-Savart formula. In order to differentiate (2) further, a formula for the second derivative of the Newtonian potential is needed. That is, for any smooth function $g(\boldsymbol{x})$, we have [8]

$$\begin{aligned}\partial_i\partial_j g(\boldsymbol{x}) &= \frac{-1}{4\pi} \int \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} \partial_i\partial_j \Delta g(\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &= \frac{\Delta g}{3}\delta_{ij} + K_{ij}[\Delta g](\boldsymbol{x}),\end{aligned} \tag{3}$$

where

$$K_{ij}[f](\boldsymbol{x}) = \oint \frac{|\boldsymbol{x}-\boldsymbol{y}|^2 \delta_{ij} - 3(x_i - y_i)(x_j - y_j)}{4\pi |\boldsymbol{x}-\boldsymbol{y}|^5} f(\boldsymbol{y}) \mathrm{d}\boldsymbol{y}. \tag{4}$$

Here the principal-value integral means $\oint = \lim_{\epsilon\to 0} \int_{|\boldsymbol{x}-\boldsymbol{y}|\geq\epsilon}$ (similar notations will be used hereafter). The second derivative is made up of the local term due to the Dirac δ function plus the nonlocal term

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in the form of a singular integral. By using (3) and symmetrizing we find [2-4]

$$S_{ij}(\boldsymbol{x}) = \frac{3}{8\pi} \oint \frac{\epsilon_{ikl} r_k \omega_l(\boldsymbol{y}) r_j + r_i \epsilon_{jkl} r_k \omega_l(\boldsymbol{y})}{r^5} \mathrm{d}\boldsymbol{y}, \quad (5)$$

where $\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$, and ϵ_{ijk} is the fully antisymmetric tensor.

The bilateral relationship between the vorticity and the strain is best seen in terms of the vorticity tensor $\Omega_{ij} \equiv (\partial_j u_i - \partial_i u_j)/2 = -(1/2)\epsilon_{ijk}\omega_k$, which decomposes the velocity gradient as $\partial_j u_i = S_{ij} + \Omega_{ij}$. Note that $\boldsymbol{\Omega}$ and $\boldsymbol{S}$ do not commute in general. With $\boldsymbol{\Omega}$ we can write Eq. (5) as

$$S_{ij}(\boldsymbol{x}) = \frac{3}{4\pi} \oint \frac{r_k \Omega_{ki}(\boldsymbol{y}) r_j - r_i \Omega_{jk}(\boldsymbol{y}) r_k}{r^5} \mathrm{d}\boldsymbol{y} \equiv T_{ij}[\boldsymbol{\Omega}], \quad (6)$$

say.

Now we look for the inverse transform which expresses the vorticity in terms of the strain. By the definitions of $\boldsymbol{S}$ and $\boldsymbol{A}$ we have

$$S_{ij} = \frac{1}{2}(\partial_i \epsilon_{jkl} \partial_k A_l + \partial_j \epsilon_{ikl} \partial_k A_l).$$

Taking a divergence and a curl for i and j, respectively, we obtain

$$\triangle^2 A_p = -2\epsilon_{pqj} \partial_q \partial_i S_{ij}.$$

Again using (3) under $\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, we have

$$\omega_i = -\Delta A_i = -\frac{3}{2\pi} \oint \frac{\epsilon_{ijk}(x_j - y_j) S_{kl}(\boldsymbol{y})(x_l - y_l)}{|\boldsymbol{x} - \boldsymbol{y}|^5} \mathrm{d}\boldsymbol{y}.$$

In terms of $\boldsymbol{\Omega}$ this becomes

$$\Omega_{ij}(\boldsymbol{x}) = -\frac{3}{4\pi} \oint \frac{r_k S_{ki}(\boldsymbol{y}) r_j - r_i S_{jk}(\boldsymbol{y}) r_k}{r^5} \mathrm{d}\boldsymbol{y}. \quad (7)$$

A crucial observation is that $\boldsymbol{\Omega}$ and $\boldsymbol{S}$ are connected with each other through an *identical singular integral transform* (up to a minus sign), that is,

$$\boldsymbol{T}[\boldsymbol{T}[\boldsymbol{\Omega}]] = -\boldsymbol{\Omega}. \quad (8)$$

The vorticity and rate-of-strain tensors are conjugates under the transform $\boldsymbol{T}$. Because of $\mathrm{tr}(\boldsymbol{S} \cdot \boldsymbol{S}) + \mathrm{tr}(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}) = -\triangle p$ we have

$$\langle S_{ij} S_{ij} \rangle = \langle \Omega_{ij} \Omega_{ij} \rangle, \quad (9)$$

where the angular brackets denote the spatial average and tr denotes a trace. The identity (9) can be regarded as the Parseval formula for the transform $\boldsymbol{T}$. The apparently trivial shift from $\boldsymbol{\omega}$ to $\boldsymbol{\Omega}$ makes manifest the conjugate relationship between the vorticity and the strain. We note that the existence of $\boldsymbol{T}$ satisfying (8) can be attributed solely to the incompressibility condition of the velocity.

The transform $\boldsymbol{T}$ can also be considered as operating on general 3×3 matrices. Some of its properties are as follows. $\boldsymbol{T}$ is traceless; $\mathrm{tr}(\boldsymbol{T}) = 0$. Also, $\boldsymbol{T}[c(\boldsymbol{x})\boldsymbol{I}] \equiv 0$ for arbitrary scalar $c(\boldsymbol{x})$ and $\boldsymbol{I}$ is the identity matrix.

It should be noted that $\boldsymbol{T}^2 = -\boldsymbol{I}$ does not hold for general matrices. In fact, we can show by a direct computation using the Fourier transform [Eq. (11)] that $\boldsymbol{T}[\boldsymbol{T}^2[\boldsymbol{X}] + \boldsymbol{X}] = 0$ for any $\boldsymbol{X}$. It can be shown generally from this identity that

$$T_{ij}^2[X] = -X_{ij} + R_i R_j [R_k R_l [X_{kl}]] - \epsilon_{ipq} \epsilon_{jkl} R_p R_k [X_{ql}].$$

Here R_i denotes the Riesz transform defined by

$$R_i[f](\boldsymbol{x}) = c_n \oint \frac{y_i}{|\boldsymbol{y}|^{n+1}} f(\boldsymbol{x} - \boldsymbol{y}) \mathrm{d}\boldsymbol{y}$$

for $i = 1, 2, \ldots, n$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ with $c_n = \Gamma\left(\frac{n+1}{2}\right)/\pi^{(n+1)/2}$, where Γ is the gamma function and here $n = 3$. It is a generalization of the Hilbert transform into n dimensions and its Fourier transform [9] (designated by a tilde) is given by $\widetilde{R}_j = ik_j/|\boldsymbol{k}|$ (for details see Chapter III of [1]). We also have the adjoint formulas; for 3×3 matrices $\boldsymbol{f}$ and $\boldsymbol{g}$,

$$\langle \mathrm{tr}(\boldsymbol{T}[\boldsymbol{f}] \cdot \boldsymbol{g}) \rangle = \pm \langle \mathrm{tr}(\boldsymbol{f} \cdot \boldsymbol{T}[\boldsymbol{g}]) \rangle, \quad (10)$$

where + should be taken when one of $\boldsymbol{f}$ and $\boldsymbol{g}$ is symmetric and the other is antisymmetric and when both are symmetric or antisymmetric. These can be verified by writing both sides explicitly (proofs omitted).

Further properties of $\boldsymbol{T}$ can be seen through the Fourier transform,

$$\widetilde{T}_{ij}[\widetilde{\Omega}] = \widetilde{S}_{ij} = \frac{1}{|\boldsymbol{k}|^2}(k_i k_l \widetilde{\Omega}_{jl} - k_j k_l \widetilde{\Omega}_{li}), \quad (11)$$

where $\boldsymbol{k}$ is the wave number. Using the Riesz transform R_i, we can write

$$T_{ij}[\boldsymbol{\Omega}] = -R_i R_l \Omega_{jl} + R_j R_l \Omega_{li}.$$

Because of boundedness (from L^p to itself) of R_i and Eq. (8), $\boldsymbol{\Omega}$ and $T[\boldsymbol{\Omega}]$ are comparable [10] in the L^p norm [11],

$$A_p^{-1} \|\boldsymbol{\Omega}\|_p \leq \|\boldsymbol{T}[\boldsymbol{\Omega}]\|_p \leq A_p \|\boldsymbol{\Omega}\|_p,$$

with some constants A_p for $1 < p < \infty$ ($A_2 = 1$). As in the case of the Riesz transform, by analytic extension the vorticity and the rate-of-strain tensors can be regarded as the boundary values of pairs of conjugate harmonic functions in (3+1)-dimensional space: $\mathbb{R}_+^{3+1} = \{(\boldsymbol{x}, y) | \boldsymbol{x} \in \mathbb{R}^3, y > 0\}$. Let

$$\begin{aligned} u_{ij}(\boldsymbol{x}, y) &= (P_y * \Omega_{ij})(\boldsymbol{x}, y) \\ v_{ij}(\boldsymbol{x}, y) &= (P_y * S_{ij})(\boldsymbol{x}, y) \\ &= -(\Omega_{jl} * R_l R_i [P_y]) + (\Omega_{li} * R_l R_j [P_y]), \end{aligned} \quad (12)$$

where the $*$ denotes convolution and P_y is the Poisson kernel defined by

$$P_y(\boldsymbol{x}) = \int \exp(-2\pi i \boldsymbol{t} \cdot \boldsymbol{x}) \exp(-2\pi |\boldsymbol{t}| y) \mathrm{d}\boldsymbol{t} = \frac{c_n y}{(|\boldsymbol{x}|^2 + y^2)^{(n+1)/2}}.$$

The last line in (12) follows from $R_j[P_y]*f = P_y * R_j[f]$. We then have

$$\triangle u_{ij} = \left(\frac{\partial^2}{\partial y^2} + \sum_{k=1}^{3}\frac{\partial^2}{\partial x_k^2}\right) u_{ij} = 0,\ \triangle v_{ij} = 0.$$

Note that u_{ij} and v_{ij} are, respectively, antisymmetric and symmetric tensors; $u_{ij} = -u_{ji}$ $v_{ij} = v_{ji}$. As in the case of the Riesz transform (theorem 3 of Chapter III, in [1]), it can be shown through the Fourier transform that

$$S_{ij} = T_{ij}[\Omega] \Longleftrightarrow \begin{cases} \dfrac{\partial^2 u_{jl}}{\partial x_i \partial x_l} - \dfrac{\partial^2 u_{li}}{\partial x_j \partial x_l} = -\dfrac{\partial^2 v_{ij}}{\partial y^2} \\ \dfrac{\partial^2 v_{jl}}{\partial x_i \partial x_l} - \dfrac{\partial^2 v_{li}}{\partial x_j \partial x_l} = \dfrac{\partial^2 u_{ij}}{\partial y^2} \end{cases} \tag{13}$$

with $\partial_i(u_{ij} + v_{ij}) = 0$. The system of equations (13) corresponds to the generalized Cauchy-Riemann condition underlying the vorticity-strain conjugation. Moreover, $\lim_{y\to 0} u_{ij}(\boldsymbol{x}, y) = \Omega_{ij}(\boldsymbol{x})$, $\lim_{y\to 0} v_{ij}(\boldsymbol{x}, y) = S_{ij}(\boldsymbol{x})$. Therefore the vorticity and the rate-of-strain tensors can be regarded as the boundary values of conjugate harmonic functions in $\mathbb{R}_+^{3+1}$.

As an application of this transform we note the relationship between three of Siggia's invariants $I_1 = \left\langle (S_{ij}S_{ij})^2 \right\rangle$, $I_2 = \left\langle S_{ij}S_{ij}|\boldsymbol{\omega}|^2 \right\rangle$, $I_3 = \langle \omega_i S_{ij} S_{jk} \omega_k \rangle$, $I_4 = \left\langle |\boldsymbol{\omega}|^4 \right\rangle$, which describe the vorticity-strain correlation [12-15]. By subtracting the singularity it can be shown for a smooth function $\alpha(\boldsymbol{x})$ that

$$T_{ij}[\alpha \boldsymbol{X}] = \alpha T_{ij}[\boldsymbol{X}] - U_{ij}[\boldsymbol{X};\alpha], \tag{14}$$

where we have set

$$U_{ij}[\boldsymbol{X};\alpha](\boldsymbol{x}) = \frac{3}{4\pi}\oint \frac{r_k X_{ki}(\boldsymbol{y}) r_j - r_i X_{jk}(\boldsymbol{y}) r_k}{r^5}[\alpha(\boldsymbol{x}) - \alpha(\boldsymbol{y})]\mathrm{d}\boldsymbol{y}.$$

That is, a smooth function $\alpha(\boldsymbol{x})$ can be passed in and out of $\boldsymbol{T}$ by introducing a smoothing operator [16]. Letting $\alpha(\boldsymbol{x}) = \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}) = -|\boldsymbol{\omega}|^2/2$, $\beta(\boldsymbol{x}) = \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{S})$ we find

$$\begin{aligned} \langle \mathrm{tr}(\boldsymbol{T}[\boldsymbol{\Omega}]\cdot\boldsymbol{T}[\boldsymbol{\Omega}])\mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\rangle &= \langle \mathrm{tr}(\boldsymbol{T}[\boldsymbol{\Omega}]\cdot\alpha\boldsymbol{T}[\boldsymbol{\Omega}])\rangle \\ &= \langle \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{T}[\alpha\boldsymbol{T}[\boldsymbol{\Omega}]])\rangle \quad \text{by (10)} \\ &= -\langle \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\rangle - \langle \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{U}[\boldsymbol{S};\alpha])\rangle \quad \text{by (14),} \end{aligned}$$

or

$$\tfrac{1}{2}I_2 = \tfrac{1}{4}I_4 + \langle \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{U}[\boldsymbol{S};\alpha])\rangle .$$

Similarly we have

$$\langle \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{S})\mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega})\rangle = -\langle \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{S})\mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{S})\rangle + \langle \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{U}[\boldsymbol{\Omega};\beta])\rangle .$$

That is

$$\tfrac{1}{2}I_2 = I_1 - \langle \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{U}[\boldsymbol{\Omega};\beta])\rangle .$$

Note that

$$\langle \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{U}[\boldsymbol{\Omega};\beta])\rangle = -\langle \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{U}[\boldsymbol{S};\beta])\rangle ,$$

$$\langle \mathrm{tr}(\boldsymbol{\Omega}\cdot\boldsymbol{U}[\boldsymbol{S};\alpha])\rangle = -\langle \mathrm{tr}(\boldsymbol{S}\cdot\boldsymbol{U}[\boldsymbol{\Omega};\alpha])\rangle .$$

It seems worthwhile to examine further kinematic constraints imposed by the conjugate character of the vorticity-strain correlation.

An outlook for the use of the formal properties of $\boldsymbol{T}$ derived here may be in order. The model equation (1) when extended into the complex plane appears as a simple quadratic local equation and is exactly solvable [7]. It is expected that in the three-dimensional Euler equations the nonlocality may be partially reduced when seen in $\mathbb{R}_+^{3+1}$ (at least the nonlocality associated with the integral relationship between the vorticity and the strain). Therefore pursuit of the similarity with the model (1) regarding dynamics may be useful for understanding a putative singularity formation in Euler flows [17,18] and small-scale motion in Navier-Stokes turbulence. Finally, we note that a similar conjugation is also seen in two dimensions.

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