Reflectivity of cold magnetized plasmas

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The reflection of monochromatic electromagnetic radiation from a cold uniform magnetized plasma characterized by an arbitrary dielectric tensor is considered. The generalization of well-known Fresnel formulas is obtained. Some limiting cases are studied. The dielectric tensor is constructed applying a matrix generalization of the classical theory of moments.

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I. INTRODUCTION

The experimental investigation of magnetized strongly coupled plasmas poses a problem of providing a theoretical basis for consistent interpretation of upcoming results.

The diagnostics of magnetized plasmas via scheduled experiments on its reflectivity will certainly require the implementation of exact relations between optical [the dielectric tensor $\hat{\varepsilon}(\omega)=\hat{I}+4\pi i\hat{\sigma}(\omega)$, the internal dynamic conductivity tensor $\hat{\sigma}(\omega)$, etc.] and thermodynamic characteristics of magnetized dense plasmas.

The present paper contains two theoretical results aimed at establishing such relations, at least in the experimental sense.

First, a relatively methodological problem of calculation of reflection coefficients and other ellipsometry parameters of magnetized plasmas is treated. Second, the internal conductivity tensor and the dielectric tensor of strongly coupled cold plasmas are studied and constructed on the basis of exact relations and sum rules.

II. REFLECTION OF ELECTROMAGNETIC RADIATION FROM A COLD MAGNETIZED PLASMA

Consider the problem of the interaction of a monochromatic plane wave of electromagnetic (laser) radiation with a cold uniform magnetized plasma. The orientation of the magnetic field with respect to the medium surface is presumed to be arbitrary, i.e., no assumption is made about the orientation of main axes of the plasma dielectric tensor. The magnetic permeability is neglected, which is physically motivated [1,2]. Basically it is not difficult to take it into account, but that would complicate the formulas even more. Besides, since the plasma is assumed to be cold, the space dispersion is not included in our consideration.

The incident monochromatic plane electromagnetic wave of arbitrary polarization and frequency ω comes from a transparent medium with a permittivity ε_0, and falls onto a plasma magnetized by a constant and uniform magnetic field $\vec{B}_0$; the dielectric tensor of the plasma is $\hat{\varepsilon}$. The Cartesian axes x and z are shown in Fig. 1; the incidence plane coincides with the xz one. The xy plane is the interface that is assumed to be flat and uniform. The unit vectors $\vec{e}_p^{\langle i \rangle}$ and $\vec{e}_s^{\langle i \rangle}$ $(i=0,1,2,3)$ are parallel to the components of the incident wave electric vector parallel and orthogonal to the incidence plane, respectively. The superscripts 0 and 1 denote the incident and reflected waves, and 2 and 3 the refracted ones [2,3] (the plasma birefringence).

Due to the complete homogeneity along the xy plane, the dependence of the field equations' solutions on these coordinates should be universal. Thus the directions of propagation of all waves should be parallel to the xz plane, and within our system of coordinates that would mean that the wave vectors are restricted by the condition

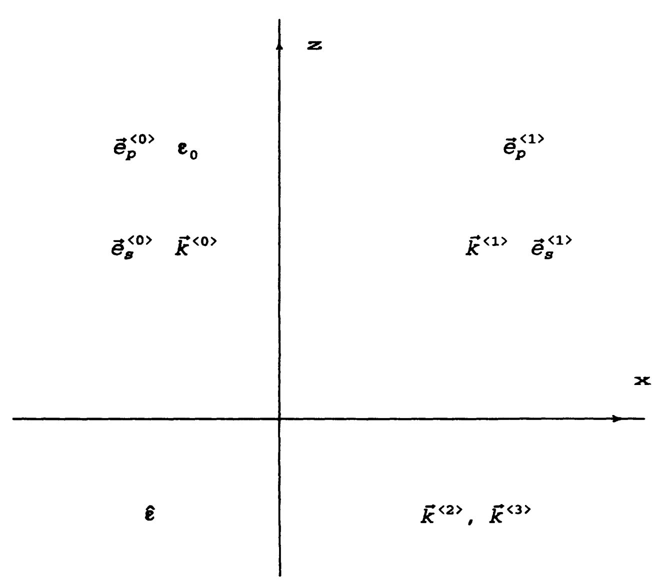

FIG. 1. Geometric parameters of the incident, reflected, and refracted waves; see the text.

$$k_{0,y}=k_{1,y}=k_{2,y}=k_{3,y}=0\ . \tag{2.1}$$

In addition,

$$k_{1,x}=k_{2,x}=k_{3,x}=\kappa_0\sqrt{\varepsilon_0}\sin\varphi_0\ , \tag{2.2}$$

where φ_0 is the angle of incidence and $\kappa_0=\omega/c$. Now the above dependence on coordinates and time can be set as

$$\exp[i(\omega t-k_x x-k_z z)]\ . \tag{2.3}$$

Some sufficiently general matrix methods are developed in [4] to solve problems about the reflection and propagation of electromagnetic waves through anisotropic optically inhomogeneous media. Following [4], consider Maxwell's equations in the medium,

$$\mathrm{rot}\vec{\mathrm{E}}=-\frac{1}{c}\frac{\partial\vec{\mathrm{H}}}{\partial t},\quad \mathrm{rot}\vec{\mathrm{H}}=\frac{1}{c}\frac{\partial \mathrm{D}}{\partial t}\ . \tag{2.4}$$

Taking into account (2.3), Eqs. (2.4) are easily simplified to

$$\Delta\vec{\mathrm{E}}-\mathrm{grad\ div}\ \vec{\mathrm{E}}+\kappa_0^2\vec{\mathrm{D}}=\mathbf{0}\ , \tag{2.5}$$

where Δ is the Laplace operator. Applying the relation $D_\mu=\varepsilon_{\mu\nu}E_\nu$ (here $\varepsilon_{\mu\nu}$ are the Cartesian components of the plasma dielectric tensor $\hat{\varepsilon}$), one can eliminate the electrical induction vector $\vec{\mathrm{D}}$ to obtain the following expression for the z component of the electric field vector $\vec{\mathrm{E}}$:

$$E_z=-f^{-1}\left\{\kappa_0^2\varepsilon_{zx}E_x+\kappa_0^2\varepsilon_{zy}E_y+ik_x\frac{\partial E_x}{\partial z}\right\}, \tag{2.6}$$

with $f=\kappa_0^2\varepsilon_{zz}-k_x^2$. In the same manner, one arrives at the system of equations

$$\varepsilon_{zz}\frac{\partial^2 E_x}{\partial z^2}-ik_x(\varepsilon_{xz}+\varepsilon_{zx})\frac{\partial E_x}{\partial z}-ik_x\varepsilon_{xy}\frac{\partial E_y}{\partial z}+(\varepsilon_{xx}f-\kappa_0^2\varepsilon_{xz}\varepsilon_{zx})E_x+(\varepsilon_{xy}f-\kappa_0^2\varepsilon_{zy}\varepsilon_{yz})E_y=0\ , \tag{2.7a}$$

$$f\frac{\partial^2 E_y}{\partial z^2}-ik_x\kappa_0^2\varepsilon_{yz}\frac{\partial E_x}{\partial z}+\kappa_0^2(\varepsilon_{yx}f-\kappa_0^2\varepsilon_{zx}\varepsilon_{yx})E_x+[(\kappa_0^2\varepsilon_{yy}-k_x^2)f-\kappa_0^4\varepsilon_{zy}\varepsilon_{yz})]E_y=0\ . \tag{2.7b}$$

The particular solutions are to be sought in the form of plane waves (2.3). Then appears the consistency condition for Eqs. (2.7a) and (2.7b), i.e., Fresnel's equation

$$k_z^4\varepsilon_{zz}+k_z^3k_x\varepsilon_{[xz]}+k_z^2\left\{k_x^2(\varepsilon_{zz}+\varepsilon_{xx})-\kappa_0^2\begin{vmatrix}\varepsilon_{yy} & \varepsilon_{yz}\\ \varepsilon_{zy} & \varepsilon_{zz}\end{vmatrix}-\kappa_0^2\begin{vmatrix}\varepsilon_{xx} & \varepsilon_{xz}\\ \varepsilon_{zx} & \varepsilon_{zz}\end{vmatrix}\right\}+k_z\{k_x^3\varepsilon_{[xz]}+\kappa_0^2k_x(\varepsilon_{yz}\varepsilon_{xy}+\varepsilon_{zy}\varepsilon_{yx}-\varepsilon_{yy}\varepsilon_{[xz]})\}$$
$$+f^{-1}[\varepsilon_{xx}f-\kappa_0^2\varepsilon_{xz}\varepsilon_{zx}][(\kappa_0^2\varepsilon_{yy}-k_x^2)f-\kappa_0^4\varepsilon_{yz}\varepsilon_{zy}]-f^{-1}\kappa_0^2[\varepsilon_{yx}f-\kappa_0^2\varepsilon_{zx}\varepsilon_{yz}][\varepsilon_{xy}f-\kappa_0^2\varepsilon_{zy}\varepsilon_{xz}]=0\ . \tag{2.8}$$

In Eq. (2.8) $\varepsilon_{[xz]}=\varepsilon_{xz}+\varepsilon_{zx}$. One should now select two of four roots of Eq. (2.8), i.e., those corresponding to the waves decaying along its way inside the plasma. The fact that such roots exist follows from the consideration that within the frequency interval for which the medium is transparent, i.e., when the tensor $\hat{\varepsilon}$ is Hermitian, the coefficients of Eq. (2.8) are real. Hence its roots are complex pair conjugate or real. The non-Hermitian contributions to $\hat{\varepsilon}$ standing for the decay would only change the decay rate (if it is not too large).

Further, using Eqs. (2.7), one can find a relation between the x and y components of the electric field vector of the wave [5]. In what follows, we will need

$$E_x^{\langle i\rangle}=(f_i^{\langle i\rangle}/f_2^{\langle i\rangle})E_y^{\langle i\rangle},\quad i=2,3\ . \tag{2.9}$$

Here the following notations are introduced:

$$f_i^{\langle i\rangle}=\varepsilon_{xy}\varepsilon_{yz}-(\varepsilon_{yy}-(k^{\langle i\rangle})^2/\kappa_0^2)(\varepsilon_{xz}+k_xk_z^{\langle i\rangle}/\kappa_0^2)\ ,$$
$$f_2^{\langle i\rangle}=\varepsilon_{yx}(\varepsilon_{xz}+k_xk_z^{\langle i\rangle}/\kappa_0^2)-\varepsilon_{yz}(\varepsilon_{xx}-(k_z^{\langle i\rangle})^2/\kappa_0^2)\ , \tag{2.10}$$

and $(k^{\langle i\rangle})^2=k_x^2+(k_z^{\langle i\rangle})^2$, $i=2,3$.

From these formulas it stems that in the case of an arbitrary direction of propagation both types of waves inside the plasma (ordinary and extraordinary) are polarized, generally speaking, elliptically. Only when the waves propagate along the magnetic field are they polarized circularly.

Now consider the boundary conditions for the electric and magnetic field vectors of the wave,

$$k_z^{\langle 0\rangle}(E_s^{\langle 0\rangle}-E_s^{\langle 1\rangle})=k_z^{\langle 2\rangle}E_y^{\langle 2\rangle}+k_z^{\langle 3\rangle}E_y^{\langle 3\rangle}\ ,$$
$$-\sqrt{\varepsilon_0}(E_p^{\langle 0\rangle}+E_p^{\langle 1\rangle})=\beta_1(E_x^{\langle 2\rangle}+E_x^{\langle 3\rangle})+\beta_2(E_y^{\langle 2\rangle}+E_y^{\langle 3\rangle})$$
$$-i\beta_3(k_z^{\langle 2\rangle}E_x^{\langle 2\rangle}+k_z^{\langle 3\rangle}E_x^{\langle 3\rangle})\ , \tag{2.11}$$

where

$$\beta_1=\kappa_0k_x\varepsilon_{zx}f^{-1},\quad \beta_2=\kappa_0k_x\varepsilon_{zy}f^{-1},\beta_3=i\kappa_0\varepsilon_{zz}f^{-1}\ .$$

Thus one arrives at a system of four equations with six unknown quantities $E_s^{\langle 1\rangle}$, $E_p^{\langle 1\rangle}$, $E_x^{\langle 2\rangle}$, $E_x^{\langle 3\rangle}$, $E_y^{\langle 2\rangle}$, and $E_y^{\langle 3\rangle}$. Having resolved this system with two more equations of relation [using Eqs. (2.9)], one can represent the answer as

$$\hat{E}^{\langle 1\rangle}=\hat{R}\hat{E}^{\langle 0\rangle}\ , \tag{2.12}$$

where $\hat{E}^{\langle 1\rangle}$ and $\hat{E}^{\langle 0\rangle}$ are matrices—columns of amplitudes of components of the electric field vectors:

$$\hat{E}^{\langle j\rangle}=\begin{bmatrix}E_p^{\langle j\rangle}\\ E_s^{\langle j\rangle}\end{bmatrix},\quad j=0,1\ , \tag{2.13}$$

and $\hat{R}$ is the reflection coefficients' matrix:

$$\hat{R}=\begin{bmatrix} R_{pp} & R_{ps} \\ R_{sp} & R_{ss} \end{bmatrix} . \qquad (2.14)$$

Its components are defined as

$$R_{pp}=R_{11}/R_0,\quad R_{ps}=R_{12}/R_0 ,$$
$$R_{sp}=R_{21}/R_0,\quad R_{ss}=R_{22}/R_0 . \qquad (2.15)$$

Here

$$R_{11}=a(\gamma_1 b_1+\gamma_2 b_2)(k_z^{\langle 3\rangle}-k_z^{\langle 2\rangle}) -(k_z^{\langle 0\rangle}+k_z^{\langle 2\rangle}b_1+k_z^{\langle 3\rangle}b_2) \times[\sqrt{\varepsilon_0}+a(\gamma_2-\gamma_1)] ,$$
$$R_{12}=-2k_z^{\langle 0\rangle}(\gamma_1 b_1+\gamma_2 b_2) ,$$
$$R_{21}=2\sqrt{\varepsilon_0}a(k_z^{\langle 2\rangle}-k_z^{\langle 3\rangle}) , \qquad (2.16)$$
$$R_{22}=a(\gamma_1 b_1+\gamma_2 b_2)(k_z^{\langle 2\rangle}-k_z^{\langle 3\rangle}) -(-k_z^{\langle 0\rangle}+k_z^{\langle 2\rangle}b_1+k_z^{\langle 3\rangle}b_2) \times[\sqrt{\varepsilon_0}+a(\gamma_1-\gamma_2)] ,$$
$$R_0=a[k_z^{\langle 0\rangle}(\gamma_1-\gamma_2)+\gamma_1 k_z^{\langle 3\rangle}-\gamma_2 k_z^{\langle 2\rangle}] +\sqrt{\varepsilon_0}(k_z^{\langle 0\rangle}+k_z^{\langle 2\rangle}b_1+k_z^{\langle 3\rangle}b_2)$$

and

$$a=\cos\varphi_0/(c_3-c_2) ,$$
$$c_i=f_i^{\langle i\rangle}/f_2^{\langle i\rangle},\quad (i=2,3) ,$$
$$\gamma_1=\beta_2+c_2(\beta_1-i\beta_3 k_z^{\langle 2\rangle}) , \qquad (2.17)$$
$$\gamma_2=\beta_2+c_3(\beta_1-i\beta_3 k_z^{\langle 3\rangle}) ,$$
$$b_1=c_3/(c_3-c_2),\quad b_2=c_2/(c_2-c_3) .$$

The wave vector components $k_z^{\langle i\rangle}$ $(i=2,3)$ are to be determined from the solution of Fresnel's Eq. (2.8), and the z components of the vector $\vec{k}^{\langle 0\rangle}$ of the incident wave, $k_z^{\langle 0\rangle}=-\kappa_0\sqrt{\varepsilon_0}\cos\varphi_0$. Formulas (2.12)–(2.17) together with Eq. (2.8) provide the general solution of the problem on reflection of a monochromatic plane electromagnetic wave from an anisotropic medium characterized by the dielectric tensor $\hat{\varepsilon}(\omega)$. Along with magnetized plasmas one can also treat anisotropic metals, dielectrics, etc., always when the spatial dispersion is insignificant.

As seen from Eqs. (2.15) and (2.16), $(R_{ps}, R_{sp}\neq 0)$, the reflection coefficient is generally a nondiagonal matrix. In a system of coordinates with the z axis parallel to the external magnetic field, the $\hat{\varepsilon}(\omega)$ matrix simplifies to

$$\hat{\varepsilon}=\begin{bmatrix} \varepsilon_\perp & ig & 0 \\ -ig & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_\parallel \end{bmatrix} , \qquad (2.18)$$

where $\varepsilon_\perp$ and $\varepsilon_\parallel$ are the transverse and longitudinal (with respect to the external magnetic field) components of the dielectric tensor, with g being the component of $\hat{\varepsilon}(\omega)$ due to the plasma gyrotropy.

In the general case under consideration this is not true, and the $\hat{\varepsilon}(\omega)$ tensor should be obtained from Eq. (2.18) by an orthogonal transformation to the initial coordinate system (see, e.g., [5]). Thus the components of this tensor will be expressed in terms of $\varepsilon_\perp$, $\varepsilon_\parallel$, g, and three angles of Euler (in the case of plasmas) which define the magnetic field orientation with respect to the selected system of coordinates.

The transition $\vec{B}_0\to\vec{0}$ to the case when the plasma becomes nongyrotropic is treated in Appendix A.

For the case of oblique incidence onto the magnetized plasma, the magnetic field $\vec{B}_0$ being orthogonal to its surface, the matrix of reflection coefficients is determined by Eqs. (2.15) and (A3). Still these expressions are too general.

Now consider the case of normal incidence. Then $k_x=0$, and from Eqs. (A1) and (A2) one obtains

$$k_z^{\langle i\rangle}=\pm\kappa_0(\varepsilon_\perp\pm g)^{1/2}\quad (i=2,3),\quad c_2c_3=1 . \qquad (2.19)$$

Upon quite cumbersome but simple transformations, one derives corresponding reflection coefficients:

$$R_{pp}=-R_{ss}=\frac{n_1n_2-\varepsilon_0}{\varepsilon_0+n_1n_2+\sqrt{\varepsilon_0}(n_1+n_2)} ,$$
$$R_{ps}=-R_{sp}=i\sqrt{\varepsilon_0}\frac{n_2-n_1}{\varepsilon_0+n_1n_2+\sqrt{\varepsilon_0}(n_1+n_2)} , \qquad (2.20)$$

where $n_{1,2}\equiv(\varepsilon_\perp\pm g)^{1/2}$ are the refraction coefficients for ordinary and extraordinary waves propagating along the external magnetic field. The transition $B_0\to 0$ in this case is trivial, since in this limiting case $n_1=n_2=\sqrt{\varepsilon}$.

Notice that the components of the dielectric tensor enter into Eq. (2.20) only through combinations $\varepsilon_\perp\pm g$. This is caused by the fact that in this particular case the problem can be resolved in a different, simpler way.

Observe that the Hermitian matrix $\hat{\varepsilon}$ in Eq. (2.18) is reducible by simple unitary transformation to a diagonal form [5] with eigenvalues $n_{1,2}^2$, $\varepsilon_\parallel$. Since in a collisional plasma the dielectric tensor $\hat{\varepsilon}$ of Eq. (2.18) is, generally, non-Hermitian, it can always be represented as

$$\hat{\varepsilon}=\hat{\varepsilon}_1+i\hat{\varepsilon}_2 ,$$

where $\hat{\varepsilon}_1$ and $\hat{\varepsilon}_2$ are the Hermitian matrices of the same type. It can easily be seen that they commute, which is the necessary and sufficient condition of their simultaneous (in the same orthogonal basis and by the same unitary transformation) diagonalization. Then one arrives at Eq. (2.20) upon using the boundary conditions in the coordinate system corresponding to the new basis, and Eq. (2.9) for the x and y components of the electric field vector, which is very simple in this particular case;

$$E_x=\pm iE_y .$$

Consider now [4] the relative reflection coefficients; the latter characterize the variation of the state of polarization of the incident wave as a result of reflection,

$$\rho_{11}=R_{pp}/R_{ss},\quad \rho_{12}=R_{ps}/R_{ss},\quad \rho_{21}=R_{sp}/R_{ss} . \qquad (2.21)$$

One also introduces the angles of polarization, Ψ_{ij} and Δ_{ij}, of the reflecting system:

$$\tan(\Psi_{11}\exp(i\Delta_{11}))=\rho_{11} ,\quad \tan(\Psi_{12}\exp(i\Delta_{12}))=\rho_{12} ,\quad \tan(\Psi_{21}\exp(i\Delta_{21}))=\rho_{21} . \tag{2.22}$$

The latter definitions form the system of basic equations of ellipsometry. In addition,

$$E_p^{\langle 0\rangle}/E_s^{\langle 0\rangle}=\exp(i\Delta_0)/\tan\Psi_0 , \tag{2.23a}$$

where

$$\tan\Psi_0=|E_s^{\langle 0\rangle}|/|E_p^{\langle 0\rangle}|,\quad \Delta_0=\arg(E_p^{\langle 0\rangle}/E_s^{\langle 0\rangle}) \tag{2.23b}$$

and

$$E_p^{\langle 1\rangle}/E_s^{\langle 1\rangle}=\exp(i\Delta_1)/\tan\Psi_1 , \tag{2.24a}$$

where

$$\tan\Psi_1=|E_s^{\langle 1\rangle}|/|E_p^{\langle 1\rangle}|,\quad \Delta_1=\arg(E_p^{\langle 1\rangle}/E_s^{\langle 1\rangle}) . \tag{2.24b}$$

The angles Ψ_0, Δ_0, Ψ_1, and Δ_1 determine the states of polarization of both incident and reflected waves completely. That is, the ratios of Eqs. (2.23) or (2.24) are real if both waves are linearly polarized Then Δ_0 and Δ_1 are equal to 0 or π. The final relation between the magnitudes introduced beforehand,

$$\frac{\exp(i\Delta_1)}{\tan\Psi_1}=\frac{\tan(\Psi_{11}\exp(i\Delta_{11}))(\exp(i\Delta_0)/\tan\Psi_0)}{1+\tan(\Psi_{21}\exp(i\Delta_{21}))(\exp(i\Delta_0)/\tan\Psi_0)}+\frac{\tan(\Psi_{12}\exp(i\Delta_{12}))}{1+\tan(\Psi_{21}\exp(i\Delta_{21}))(\exp(i\Delta_0)/\tan\Psi_0)} , \tag{2.25}$$

is the theoretical basis of "zero" methods of experimental determination of angles of polarization of reflecting systems. The angles of polarization Ψ_{11}, Δ_{11}, Ψ_{12}, Δ_{12}, Ψ_{21}, and Δ_{21} can be measured, and one can utilize Eq. (2.21) along with Eqs. (2.15)–(2.17) to determine optical and structural characteristics of a strongly coupled magnetized plasma with the dielectric tensor $\hat{\varepsilon}$ ([6], see Sec. III). This approach is thus suggested as an effective method of diagnostics of such plasmas.

In conclusion, the total (i.e., when $|E_p^{\langle 0\rangle}|\neq 0$, $|E_s^{\langle 0\rangle}|\neq 0$ longitudinal ($|E_p^{\langle 0\rangle}|\neq 0$, $|E_s^{\langle 0\rangle}|=0$) and transverse ($|E_p^{\langle 0\rangle}|=0$, $|E_s^{\langle 0\rangle}|\neq 0$ reflection coefficients are determined. These traditional characteristics are to be measured as the ratios of time averaged intensities of reflected and incident waves;

$$R=|E^{\langle 1\rangle}|^2/|E^{\langle 0\rangle}|^2,\quad R_{\parallel}=|E^{\langle 1\rangle}|^2/|E_p^{\langle 0\rangle}|^2,\quad R_{\perp}=|E^{\langle 1\rangle}|^2/|E_s^{\langle 0\rangle}|^2 . \tag{2.26}$$

Formulas (2.13) and (2.14) for R, $R_{\parallel}$, and $R_{\perp}$ can be used to obtain

$$R=\frac{|R_{pp}+\gamma R_{ps}|^2+|R_{sp}+\gamma R_{ss}|^2}{1+|\gamma|^2},\quad R_{\parallel}=|R_{pp}|^2+|R_{sp}|^2,\quad R_{\perp}=|R_{ps}|^2+|R_{ss}|^2 , \tag{2.27}$$

where $\gamma=E_s^{\langle 0\rangle}/E_p^{\langle 0\rangle}$. In particular, when the magnetic field is orthogonal to the plasma surface, and in the case of normal incidence, upon using Eq. (2.20), one obtains

$$R=\frac{|n_1n_2-\varepsilon_0-\sqrt{\varepsilon_0}(n_2-n_1)|^2+|n_1n_2-\varepsilon_0+\sqrt{\varepsilon_0}(n_2-n_1)|^2}{2|n_1n_2+\varepsilon_0+\sqrt{\varepsilon_0}(n_2+n_1)|^2} ,$$
$$R_{\parallel}=R_{\perp}=\frac{|n_1n_2-\varepsilon_0|^2+\varepsilon_0|n_1-n_2|^2}{|n_1n_2+\varepsilon_0+\sqrt{\varepsilon_0}(n_2+n_1)|^2} . \tag{2.28}$$

In the first case the incident wave was presumed to be circularly polarized ($|E_p^{\langle 0\rangle}|=|E_s^{\langle 0\rangle}|$), and in both longitudinal and transverse cases the polarization was arbitrary. If $\vec{B}_0=0$, the usual Fresnel's reflection coefficients follow from Eqs. (2.28).

In Sec. III the dielectric tensor $\varepsilon_{\mu\nu}(\omega)$ of cold magnetized plasmas is constructed on the basis of exact relations and sum rules.

III. CONDUCTIVITY AND DIELECTRIC TENSOR OF STRONGLY COUPLED MAGNETIZED PLASMAS

Consider the reflection of electromagnetic radiation of frequency ω by a strongly coupled magnetized plasma. Let us suppose that the wavelength of both refracted waves is much longer than the characteristic length of the magnetized plasma system, i.e., [7],

$$v_{T,A}|k_{\parallel}^{\langle i\rangle}|/\omega \ll 1,\quad v_{T,A}|k_{\perp}^{\langle i\rangle}|/|\omega_{H,A}| \ll 1 , \tag{3.1}$$

where $k_{\parallel}^{\langle i\rangle}$ and $k_{\perp}^{\langle i\rangle}$ ($i=2,3$) are the projections of the wave vector of the two refracted waves parallel and perpendicular to the external magnetic field direction, $v_{T,A}$ is the average velocity of particles of species A, and $\omega_{H,A}=Z_AeB_0/m_Ac$ is the cyclotron frequency of species A with charges Z_Ae and masses m_A, B_0 being the field strength of the external magnetic field, and c the vacuum light velocity.

Inequalities (3.1) are to be satisfied for all plasma species. In this case one can neglect the spatial dispersion in the plasma system, and the internal conductivity tensor $\sigma_{\mu\nu}(\omega)$ describing the response of the system to the Maxwellian field perturbation is given by the Kubo linear response formula [8,9]:

$$\sigma_{\mu\nu}(z)=\sum_{A,B}\sigma_{\mu\nu}^{AB}(z)=\sum_{A,B}\frac{i}{z}\left\{\frac{\omega_{p,A}^2}{4\pi}\delta_{\mu\nu}\delta^{AB}+G_{\mu\nu}^{AB}(z)\right\}, \tag{3.2}$$

where $\omega_{p,A}=(4\pi Z_A^2 e^2 n_A/m_A)^{1/2}$ is the plasma frequency of particles of species A, n_A being the number density of species A. δ^{AB} like $\delta_{\mu\nu}$ is the Kronecker delta, $z=\omega+i\eta$, $\eta>0$,

$$G_{\mu\nu}^{AB}=-\frac{i}{\hbar}\int_0^\infty e^{izt}\langle[I_\mu^A(t),I_\nu^B(0)]\rangle dt \tag{3.3}$$

is the retarded current-current Green function, and $I_\mu^{A(t)}$ is the current operator of species A in the Heisenberg representation,

$$I_\mu^A(t)=\int d\vec{r}\left\{i\frac{Z_A e\hbar}{2m_A}\left[\Psi_A^\dagger(\vec{r},t)\left[\nabla_\mu+\frac{iZ_A e}{c}A_\mu(\vec{r})\right]\Psi_A(\vec{r},t)-\text{H.c.}\right]-[\text{rot}\Psi_A^\dagger(\vec{r},t)\vec{\mu}_A\Psi_A(\vec{r},t)]_\mu\right\}. \tag{3.4}$$

Here $\Psi_A^\dagger(\vec{r},t)$ and $\Psi_A(\vec{r},t)$ are the creation and annihilation operators of a particle of species A, A_μ is the vector potential of the external magnetic field, $A_x=-B_0 y$, $A_y=A_z=0$, and $\vec{\mu}_A$ is the magnetic momentum operator of species A.

Hereafter our notations are as follows: $A,B,C,\ldots$ designate different plasma species; $\mu,\nu=x,y,z$ are the Cartesian indices. The matrix in the vector space $\hat{\sigma}(\omega)$ [with the elements $\sigma_{\mu\nu}(\omega)$ from Eq. (3.2) where the summation over the particle species is to be carried out] is called the conductivity tensor; the matrix $\overleftrightarrow{\sigma}(\omega)$ in the species and vector space [with the elements $\sigma_{\mu\nu}^{AB}(\omega)$] is termed the conductivity matrix.

In Eq. (3.3), $[\hat{R},\hat{S}]=\hat{R}\hat{S}-\hat{S}\hat{R}$ is the commutator of the Heisenberg operators $\hat{R}$ and $\hat{S}$, the angular brackets $\langle\ \rangle$ denote averaging over the Hamiltonian

$$\hat{H}=-\sum_A\frac{\hbar^2}{2m_A}\int d\vec{r}\left[\nabla+\frac{iZ_A e}{c}\vec{A}(\vec{r})\right]\Psi_A^\dagger(\vec{r})\left[\nabla-\frac{iZ_A e}{c}\vec{A}(\vec{r})\right]\Psi_A(\vec{r})$$
$$+\sum_{A,B}\int d\vec{r}d\vec{r}'\Psi_A^\dagger(\vec{r})\Psi_B^\dagger(\vec{r}')\frac{Z_A Z_B e^2}{|\vec{r}-\vec{r}'|}\Psi_A(\vec{r})\Psi_B(\vec{r}')-\sum_A\int d\vec{r}\Psi_A^\dagger(\vec{r})\vec{\mu}_A\cdot\vec{B}_0\Psi_A(\vec{r}). \tag{3.5}$$

The dielectric tensor employed in Fresnel's equations of Sec. II is connected with the conductivity tensor by the well-known relation

$$\varepsilon_{\mu\nu}(z)=\delta_{\mu\nu}+\frac{4\pi i}{z}\sigma_{\mu\nu}(z). \tag{3.6}$$

The Green function $G_{\mu\nu}^{AB}(z)$ from Eq. (3.3) is analytic in the upper complex half-plane. On the imaginary axis at frequencies $z=\Omega_n=2\pi n i k_B T$ ($n=0,1,\ldots$; T being the temperature) it coincides with the Matsubara version of Green's temperature function. There exists a well elaborated field perturbation theory for the temperature Green's function [10] bounded, by its very nature, to weakly coupled plasmas.

In this paper we suggest an alternative approach to the determination of the Green's function $G_{\mu\nu}^{AB}(z)$ applicable to strongly coupled plasmas. It is based on exact relations and sum rules, and uses the matrix version of the Nevanlinna formula from the classical theory of moments.

Earlier we applied the approach based on exact relations and sum rules to the investigation of the conductivity tensor only. The longitudinal and transverse conductivity of strongly coupled unmagnetized plasmas were studied in [11] and [12], respectively. The results of approach [11] were used to calculate the dynamical characteristics of three-dimensional [13] and two-dimensional [14] one-component plasmas (OCP's), binary ionic mixtures (BIM's) [15], and two-component plasmas (TCP's) [16].

The dynamical properties of strongly coupled plasmas have also been investigated within the quasilocalized charges model [17] and the mean field theory (dynamical [18] and static [19]), and the approach based on the representation of the Green's function as continued fractions [20], but the implementation of the classical method of moments [13] proved to produce the best overall agreement with the OCP molecular dynamics (MD) data [21]. In the case of multicomponent plasmas, however, the agreement of results based on the application of the method of moments to the conductivity tensor [15,16] with the corresponding MD data [22] is less satisfactory.

A first step in the investigation of the dielectric tensor of strongly coupled magnetized plasmas was made elsewhere [6].

The aim of this section is to extend the results of Refs. [11] and [6] to the investigation of the conductivity matrix of magnetized multicomponent plasmas. The matrix generalization of the classical theory of moments will be applied to express the conductivity matrix in terms of static correlations.

In contrast to the earlier approach [11–16], where only the frequency moments of the conductivity tensor were considered, here the contributions of each of the plasma species are taken into account. This should result in a better description of multicomponent plasmas. In the case of OCP's both approaches coincide.

One of the authors (J.O.) demonstrated recently [23] that the inclusion of transverse photons into the system Hamiltonian (first suggested by Kalman and Genga [24]) needs a careful application of the theory of moments in order to avoid physically incorrect conclusions. Therefore in Eq. (3.5) we have omitted contributions due to the transverse photons, which were employed in our recent papers [12,14].

Consider the frequency moments of the non-negative Hermitian part of the conductivity matrix $\overleftrightarrow{\sigma}_H(\omega)=\sigma_{\mu\nu}^{AB}(\omega)+(\overline{\sigma_{\nu\mu}^{BA}(\omega)})$ ($\overline{\lambda}$ being the complex conjugate of the complex number λ),

$$\overleftrightarrow{M}_n=\int_{-\infty}^{\infty}\omega^n\overleftrightarrow{\sigma}_H(\omega)d\omega,\quad n=0,1,2,3\ . \tag{3.7}$$

On the basis of Kubo linear response theory, one obtains

$$\begin{aligned}&(M_0)_{\mu\nu}^{AB}=\tfrac{1}{4}\omega_{p,A}^2\delta^{AB}\delta_{\mu\nu}\ ,\\&\overleftrightarrow{M}_1=\overleftrightarrow{M}_0^{1/2}\overleftrightarrow{\omega}_1\overleftrightarrow{M}_0^{1/2}\ ,\\&\overleftrightarrow{M}_2=\overleftrightarrow{M}_0^{1/2}(\overleftrightarrow{\omega}_1^2+\overleftrightarrow{\Omega}^2)\overleftrightarrow{M}_0^{1/2}\ ,\\&\overleftrightarrow{M}_3=\overleftrightarrow{M}_0^{1/2}(\overleftrightarrow{\omega}_1^3+\overleftrightarrow{\omega}_1\overleftrightarrow{\Omega}^2+\overleftrightarrow{\Omega}^2\overleftrightarrow{\omega}_1)\overleftrightarrow{M}_0^{1/2}\ ,\end{aligned} \tag{3.8}$$

where

$$(\omega_1)_{\mu\nu}^{AB}=i\omega_{H,A}\delta^{AB}D_{\mu\nu},\quad D_{\mu\nu}=\frac{1}{B_0}[[\vec{B}_0\times\vec{e}_\nu]\times\vec{e}_\mu]\ ,$$

$\vec{e}_\nu$ being the unit vector in direction ν, and

$$(\Omega^2)_{\mu\nu}^{AB}=\omega_{p,A}\omega_{p,B}\sum_{q,c}{}'\frac{q_\mu q_\nu}{q^2}\left[S_{AB}(\vec{q})\delta^{BC}-\frac{Z_c n_c}{Z_A n_A}S_{AC}(\vec{q})\delta^{AB}\right]\ .$$

Here the prime at the summation sign indicates that $\vec{q}\neq\vec{0}$, $S_{AB}(\vec{q})=(n_A n_B V^2)^{-1}\langle n_{\vec{q}}^A n_{-\vec{q}}^B\rangle$ is the partial static structure factor of particles A and B, V is the plasma volume, and $n_{\vec{q}}^A$ is the Fourier transform of the operator of number density of particles A.

Note that $\det\overleftrightarrow{\Omega}^2=0$.

It can be shown that higher order frequency moments M_n and $n\geq 4$ diverge [11,25].

The matrix form of the Nevanlinna formula from the classical theory of moments [26–28] expresses the conductivity matrix $\overleftrightarrow{\sigma}(z)$ [satisfying the sum rules Eqs. (3.8)] via the matrix function $\overleftrightarrow{q}(z)=\overleftrightarrow{\tau}(z)\overleftrightarrow{\Omega}^2$, $\overleftrightarrow{\tau}(z)$ being analytical in the upper complex half-plane and there having a positive definite anti-Hermitian part. In addition, the function $\overleftrightarrow{\tau}(z)$ should satisfy the limiting condition $\lim_{z\to\infty}(\overleftrightarrow{\tau}(z)/z)=\overleftrightarrow{0}$ with the Hermitian part $\lim_{z\to\infty}\overleftrightarrow{\tau}_H=\overleftrightarrow{0}$ ($\overleftrightarrow{0}$ being the zero matrix).

This matrix version of the Nevanlinna formula is given by (see Appendix B)

$$\overleftrightarrow{\sigma}(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\overleftrightarrow{\sigma}_H(\omega)}{\omega-z}d\omega=\frac{i}{\pi}\overleftrightarrow{M}_0^{1/2}\overleftrightarrow{A}(z)[\overleftrightarrow{B}(z)]^{-1}\overleftrightarrow{M}_0^{1/2}\ , \tag{3.9}$$

where

$$\begin{aligned}&\overleftrightarrow{A}(z)=z\overleftrightarrow{I}+\overleftrightarrow{q}(z)\ ,\\&\overleftrightarrow{B}(z)=z^2\overleftrightarrow{I}-z\overleftrightarrow{\omega}_1-\overleftrightarrow{\Omega}^2+(z\overleftrightarrow{I}-\overleftrightarrow{\omega}_1)\overleftrightarrow{q}(z)\ ,\end{aligned} \tag{3.10}$$

$\overleftrightarrow{I}$ being the unit matrix.

Equation (3.9) is the most general expression for the analytical in the upper complex half-plane matrix function $\overleftrightarrow{\sigma}(z)$ satisfying the sum rules Eqs. (3.8).

Due to the Onsager principle, within the system of coordinates with the z axis parallel to the external magnetic field, the dielectric tensor (and the conductivity tensor) has the form (2.18). This is possible only if in this system of coordinates the function $\overleftrightarrow{q}(z)$ verifies

$$\begin{aligned}&(q')_{xx}^{AB}=(q')_{yy}^{AB},\quad (q')_{xy}^{AB}=-(q')_{yx}^{AB}\ ,\\&(q')_{xz}^{AB}=(q')_{zx}^{AB}=(q')_{yz}^{AB}=(q')_{zy}^{AB}=0\ .\end{aligned} \tag{3.11}$$

The prime at the matrix component designates that of the matrix transformed in the new system of coordinates.

Keeping this in mind, one now regards the matrices $\tilde{\sigma}_\pm(\omega)$ and $\tilde{\sigma}_\parallel(\omega)$ in the particle space with the elements

$$\begin{aligned}&(\tilde{\sigma}_\pm)^{AB}=(\sigma_{xx}^{AB}\pm i\sigma_{xy}^{AB})\ ,\\&(\tilde{\sigma}_\parallel)^{AB}=\sigma_{zz}^{AB}\ .\end{aligned} \tag{3.12}$$

With the analogous definitions for the matrices $\tilde{A}_\alpha$, $\tilde{B}_\alpha$, $\tilde{M}_{0,\alpha}$, $\tilde{q}_\alpha$, $\tilde{\omega}_{1,\alpha}$, and $\tilde{\Omega}_\alpha^2$ ($\alpha=\pm,\|$), one finds from Eq. (3.9) ($\tilde{I}^{AB}=\delta^{AB}$ being the unit matrix in the particle space)

$$\begin{aligned}(\tilde{\sigma}_\pm)^{AB}&=\frac{i}{4\pi}\sum_{C,D,E}(\tilde{M}_{0,\pm}^{1/2})^{AC}(\tilde{A}_\pm)^{CD}((\tilde{B}_\pm)^{-1})^{DE}(\tilde{M}_{0,\pm}^{1/2})^{EB}\\&=\frac{i\omega_{p,A}\omega_{p,B}}{4\pi}\sum_c(z\delta^{AC}+\tilde{q}_\pm^{AC})([z^2\tilde{I}-z\tilde{\omega}_{1,\pm}-\tilde{\Omega}_\pm^2+z\tilde{q}_\pm-\tilde{\omega}_{1,\pm}\tilde{q}_\pm]^{-1})^{CB}\ ,\end{aligned} \tag{3.13a}$$

$$\begin{aligned}(\tilde{\sigma}_\parallel)^{AB}&=\frac{i}{4\pi}\sum C,D,E\,(\tilde{M}_{0,\parallel}^{1/2})^{AC}(\tilde{A}_\parallel)^{CD}((\tilde{B}_\parallel)^{-1})^{DE}(\tilde{M}_{0,\parallel}^{1/2})^{EB}\\&=\frac{i\omega_{p,A}\omega_{p,B}}{4\pi}\sum_c(z\delta^{AC}+\tilde{q}_\parallel^{AC})([z^2\tilde{I}-\tilde{\Omega}_\parallel^2+z\tilde{q}_\parallel]^{-1})^{CB}\ .\end{aligned} \tag{3.13b}$$

For the sake of simplicity we now examine a fully ionized two-component plasma (TCP) consisting of electrons with number density n_e, and ions with charges Ze and number density n_i. Due to the neutrality condition, $Zn_i=n_e$. In the case of the TCP the matrices $\tilde{\Omega}^2_\alpha$ $(\alpha=\pm,\|)$ simplify, and one has

$$(\tilde{\Omega}^2_\pm)^{AB}=\omega_{p,A}\omega_{p,B}L_\perp,\quad (\tilde{\Omega}^2_\|)^{AB}=\omega_{p,A}\omega_{p,B}L_\| \ , \tag{3.14}$$

with

$$L_\perp=\frac{1}{2}\sum_{\vec{k}\neq 0}S_{ei}(\vec{k})\frac{k_\perp^2}{k^2},\quad L_\|=\sum_{k\neq 0}S_{ei}(\vec{k})\frac{k_\|^2}{k^2} \ , \tag{3.15}$$

$S_{ei}(\vec{k})$ being the partial electron-ion static structure factor, and $k_\perp$ (and $k_\|$) the projection of the vector $\vec{k}$ on the direction perpendicular (parallel) to the external magnetic field.

From Eq. (3.13a), for the transverse (with respect to the external magnetic field) part of the conductivity matrix (taking into account that $\det\tilde{q}_{\pm(\|)}=0$), one now obtains

$$(\tilde{\sigma}_\pm)^{ee}=\frac{i\omega^2_{p,e}(z\pm\omega_{H,i})}{4\pi B_\pm}[z+\mathrm{Sp}\tilde{q}_\pm]-(\tilde{\sigma}_\pm)^{ei} \ ,$$

$$(\tilde{\sigma}_\pm)^{ei}=(\tilde{\sigma}_\pm)^{ie}=\frac{i}{4\pi z B_\pm}L_\perp\omega^2_{p,e}\omega^2_{p,i}(z+Q_\pm) \ , \tag{3.16a}$$

$$(\tilde{\sigma}_\pm)^{ii}=\frac{i\omega^2_{p,i}(z\pm\omega_{H,e})}{4\pi B_\pm}[z+\mathrm{Sp}\tilde{q}_\pm]-(\tilde{\sigma}_\pm)^{ei} \ .$$

For the longitudinal part of the conductivity, from Eq. (3.13b) we obtain

$$(\tilde{\sigma}_\|)^{ee}=\frac{i\omega^2_{p,e}z}{4\pi B_\|}[z+\mathrm{Sp}\tilde{q}_\|]-(\tilde{\sigma}_\|)^{ei} \ ,$$

$$(\tilde{\sigma}_\|)^{ei}=(\tilde{\sigma}_\|)^{ie}=\frac{i}{4\pi z B_\|}L_\|\omega^2_{p,e}\omega^2_{p,i}(z+Q_\|) \ , \tag{3.16b}$$

$$(\tilde{\sigma}_\|)^{ii}=\frac{i\omega^2_{p,i}z}{4\pi B_\|}[z+\mathrm{Sp}\tilde{q}_\|]-(\tilde{\sigma}_\|)^{ei} \ .$$

Here the following notations are used:

$$\mathrm{Sp}\tilde{q}_\alpha=(\tilde{q}_\alpha)^{ee}+(\tilde{q}_\alpha)^{ii},\quad (\alpha=\pm,\|) \ ,$$
$$Q_\alpha=(\tilde{q}_\alpha)^{ee}-\frac{\omega_{p,e}}{\omega_{p,i}}(\tilde{q}_\alpha)^{ei}=(\tilde{q}_\alpha)^{ii}-\frac{\omega_{p,i}}{\omega_{p,e}}(\tilde{q}_\alpha)^{ie} \ , \tag{3.17}$$

$$B_\pm=(z\pm\omega_{H,e})(z\pm\omega_{H,i})(z+\mathrm{Sp}\tilde{q}_\pm)-L_\perp\omega_p^2(z+Q_\pm) \ , \tag{3.18a}$$

$$B_\|=z^2(z+\mathrm{Sp}\tilde{q}_\|)-L_\|\omega_p^2(z+Q_\|) \ , \tag{3.18b}$$

$\omega_p^2=\omega^2_{p,e}+\omega^2_{p,i}$. The latter equation in (3.17) results from the symmetry property of the conductivity matrix $(\tilde{\sigma}_\alpha)^{ei}=(\tilde{\sigma}_\alpha)^{ie}(\alpha=\pm,\|)$.

From Eqs. (3.16a), (3.16b), and (3.2), for the three independent components of the conductivity tensor $\sigma_\pm(z)=\sigma'_{xx}(z)\pm i\sigma'_{xy}(z)$ and $\sigma_\|(z)=\sigma'_{zz}(z)$ [$\sigma'_{\mu\nu}(z)$ are the components of the conductivity tensor in the system of coordinates with the z axis parallel to the external magnetic field], one obtains

$$\sigma_\pm(z)=(\tilde{\sigma}_\pm)^{ee}+2(\tilde{\sigma}_\pm)^{ei}+(\tilde{\sigma}_\pm)^{ii}$$
$$=\frac{i\omega_p^2}{4\pi B_\pm}z[z+\mathrm{Sp}\tilde{q}_\pm(z)] \ , \tag{3.19a}$$

$$\sigma_\|(z)=(\tilde{\sigma}_\|)^{ee}+2(\tilde{\sigma}_\|)^{ei}+(\tilde{\sigma}_\|)^{ii}$$
$$=\frac{i\omega_p^2}{4\pi B_\|}z[z+\mathrm{Sp}\tilde{q}_\|(z)] \ . \tag{3.19b}$$

There is no phenomenological choice for the matrix function $\tilde{q}(z)$ which would give the unique conductivity tensor $\hat{\sigma}(z)$. As a first approximation, however, one can set $\tilde{q}(z)$ equal to its static value $\tilde{q}(z)\equiv\tilde{q}(0)$. The constant matrix $\tilde{q}(0)$ can be obtained from the components of the static conductivity tensor $\sigma_{\pm,0}=\sigma_\pm(\omega=0)$ and $\sigma_{\|,0}=\sigma_\|(\omega=0)$ only, since $z=0$ is a singular point for the conductivity matrix, but a regular one for the conductivity tensor of a multicomponent plasma.

From Eqs. (3.18a) and (3.19a) there follows the existence of nonvanishing static values $\sigma_{\pm,0}$ only if the following conditions are satisfied:

$$\omega_{H,e}\omega_{H,i}\mathrm{Sp}\tilde{q}_\pm(\omega=0)-L_\perp\omega_p^2\tilde{Q}_\pm(\omega=0)=0 \ . \tag{3.20}$$

Further introducing the notations $ih_\pm=\mathrm{Sp}\tilde{q}_\pm(\omega=0)$ we obtain the following expressions for the conductivity tensor components $\sigma_\pm(z)$:

$$\sigma_\pm(z)=\frac{i}{4\pi}\omega_p^2(z+ih_\pm)$$
$$\times\{(z\pm\omega_{H,e})(z\pm\omega_{H,i})$$
$$-L_\perp\omega_p^2+ih_\pm[z\pm(\omega_{H,e}+\omega_{H,i})]\}^{-1} \ . \tag{3.21a}$$

In a similar way, for the longitudinal (with respect to the external magnetic field) component of the conductivity tensor $[\tilde{Q}_\|(\omega=0)=0, ih_\|=\mathrm{Sp}\tilde{q}_\|(\omega=0)]$, one obtains

$$\sigma_\|(z)=\frac{i}{4\pi}\frac{\omega_p^2(z+ih_\|)}{z^2-L_\|\omega_p^2+ih_\|z} \ . \tag{3.21b}$$

The "constants" $h_\pm$ and $h_\|$ are defined by the static values of the conductivity components $\sigma_{\pm,0}$ and $\sigma_{\|,0}$. In the simple impact approximation with the free path time $\tau_A(B_0)$ for a particle of species A (depending generally on the external magnetic field strength), the static values of the conductivity components are given by

$$\sigma_{\pm,0}=\sum_{A=e,i}\frac{\omega^2_{p,A}\tau_A(B_0)}{4\pi}\frac{1}{1\pm\omega_{H,A}\tau_A(B_0)} \ ,$$
$$\sigma_{\|,0}=\sum_{A=e,i}\frac{\omega^2_{p,A}\tau_A(B_0)}{4\pi} \ . \tag{3.22}$$

In the first order of the ratio $\sqrt{m_e/m_i}$ the static conductivity is given by the electron conductivity only. Omitting the ion terms in Eqs. (3.22) and comparing with Eqs. (3.21a) and (3.21b) (with $z=0$), within the same accuracy one obtains

$$h_{\pm}=h_{\perp}=(L_{\perp}\omega_p^2-\omega_{H,e}\omega_{H,i})\tau_e(B_0)\approx L_{\perp}\omega_p^2\tau_e(B_0)\ ,$$
$$h_{\parallel}=L_{\parallel}\omega_p^2\tau_e(B_0)\ , \tag{3.23}$$

where we took into account that $\omega_{H,i}\ll\omega_p$ in strongly coupled plasmas for all available experimental values of the magnetic field strength.

Equations (3.21) together with Eqs. (3.23) are the simplest approximations giving an analytical expression in the upper half-plane for the conductivity tensor $\sigma_{\mu\nu}(z)$, and interpolating between the static value of the conductivity tensor $q_{\mu\nu}(z=0)$ and the high-frequency sum rules of the conductivity matrix $\sigma_{\mu\nu}^{AB}(z)$ [Eqs. (3.8)].

For $B_0=0$, $\omega_{H,e}=\omega_{H,i}=0$, and one obtains that the conductivity tensor is described by the scalar function $\sigma(z)=\sigma_{\pm}=\sigma_{\parallel}(z)$ and that $\sigma(z)$ coincides with the corresponding result in [11].

Equations (3.21) and (3.5) ensure that for the components of the dielectric tensor in the system of coordinates with the z axis parallel to the external magnetic field [see Eq. (2.18)], and within the first order of the ratio $\sqrt{m_e/m_i}$, we have

$$\varepsilon_{\perp}(\omega)=1-\frac{\omega_p^2}{\omega}\frac{(\omega+ih_{\perp})[\omega^2+h_{\perp}(i\omega-\tau_e^{-1})]}{\{[\omega^2+h_{\perp}(i\omega-\tau_e^{-1})]^2-\omega_{H,e}^2(\omega+ih_{\perp})^2\}}\ ,$$
$$g(\omega)=\frac{\omega_p^2}{\omega}\frac{\omega_{H,e}(\omega+ih_{\perp})^2}{\{[\omega^2+h_{\perp}(i\omega-\tau_e^{-1})]^2-\omega_{H,e}^2(\omega+ih_{\perp})^2\}}\ ,$$
$$\varepsilon_{\perp}(\omega)=1-\frac{\omega_p^2}{\omega}\frac{\omega+ih_{\parallel}}{\omega^2+h_{\parallel}(i\omega-\tau_e^{-1})}\ . \tag{3.24}$$

Equation (3.24) together with Eqs. (2.28) are the main results of our paper. They describe the reflectivity coefficient of a magnetized strongly coupled plasma with a magnetic field perpendicular to the plasma surface and in the case of normal incidence of the electromagnetic radiation.

In the case of inclined incidence (the magnetic field being perpendicular to the surface of the plasma) one uses Eqs. (2.15) and (A3) instead of Eqs. (2.28).

If, in addition, the magnetic field is not normal to the plasma surface, the dielectric tensor should be obtained from (2.18) and (3.24) by an orthogonal transformation to the initial coordinate system (with the z axis normal to the plasma surface). After that one uses Eqs. (2.15) and (2.16) to obtain the reflectivity coefficients.

The quality of our approximation [Eqs. (3.24)] for the dielectric tensor is limited by our knowledge of the quantities $L_{\perp}$, $L_{\parallel}$, and τ_e.

Parameters $L_{\perp}$ and $L_{\parallel}$ can be expressed by standard methods of quantum field theory within the random phase approximation (RPA) in terms of the polarization operators Π_e and Π_i of electrons and ions. The electron-ion structure factor $S_{ei}(\vec{k})$ is given by [10]

$$S_{ei}(\vec{k})=\frac{4\pi e^2k_BT}{Vn_e^2}\frac{\Pi_e(\vec{k})\Pi_i(\vec{k})}{k^2+4\pi e^2[\Pi_e(\vec{k})+Z\Pi_i(\vec{k})]}\ . \tag{3.25}$$

Equation (3.25) together with (3.15) define the parameters $L_{\perp}$ and $L_{\parallel}$ in terms of the electron and ion polarization operators. The expression of the polarization operators within the RPA is well known [29].

The parameters $L_{\perp}$ and $L_{\parallel}$ can also be investigated experimentally using the laser or the electron beam scattering experiments [30], or directly from the analysis of the asymptotic behavior of the Hall coefficient.

Neglecting the ion cyclotron frequency, it follows from Eqs. (3.21) and (3.23) that in the system of coordinates with the z axis parallel to the external magnetic field the parameter

$$\sigma^*(\omega)=\frac{\sigma'_{zz}(\omega)\sigma'_{xx}(\omega)}{\sigma'_{xy}(\omega)}=\frac{\omega_p^2}{4\pi\omega_{H,e}}\frac{\omega+i\tau_eL_{\parallel}\omega_p^2}{\omega+i\tau_eL_{\perp}\omega_p^2}\frac{\omega^2-L_{\perp}\omega_p^2+i\omega\tau_eL_{\perp}\omega_p^2}{\omega^2-L_{\parallel}\omega_p^2+i\omega\tau_eL_{\parallel}\omega_p^2}\ . \tag{3.26}$$

In particular,

$$\sigma^*(\omega\ll\tau_e\omega_p^2)\simeq\frac{\omega_p^2}{4\pi\omega_{H,e}}\frac{L_{\perp}}{L_{\parallel}}\left\{1-\frac{i\omega}{\tau_e\omega_p^2}(L_{\parallel}^{-1}-L_{\perp}^{-1})+\cdots\right\}\ , \tag{3.27}$$
$$\sigma^*(\omega\gg\tau_e\omega_p^2)\simeq\frac{\omega_p^2}{4\pi\omega_{H,e}}\left\{1+\frac{(L_{\parallel}-L_{\perp})\omega_p^2}{\omega^2}+\cdots\right\}\ .$$

Knowing the relaxation time τ_e, the relations in Eqs. (3.27) define the parameters $L_{\perp}$ and $L_{\parallel}$ measuring the Hall and longitudinal conductivities at high and low frequencies of the perturbing electric field.

The relaxation time $\tau_e(B_0)$ (or the static conductivity tensor) is obtained by direct measurement. They can also be calculated by a method described in [31].

IV. CONCLUSION

Two closely related problems are resolved in this paper. The reflectivity and other ellipsometric characteristics of cold magnetized plasmas (including the case of magnetic field inclined with respect to the plasma surface) are expressed in terms of the dielectric tensor. Various limiting cases are considered.

The dielectric tensor is further constructed within the method of moments to satisfy all known exact relations and sum rules. The matrix generalization of the Nevanlinna formula is employed to include into consideration multicomponent plasmas. The interactions between different plasma species (like electron-electron, ion-ion, and electron-ion interactions in the case of a two-component plasma) are included on an equal basis.

Notice that in the case of a one-component plasma the application of the Nevanlinna formula is virtually equivalent to the introduction of the dynamic, or static local field correction (picked up to satisfy the sum rules) within the mean field theory.

To our knowledge in the two-component plasma there is no easy option for inclusion of electron-ion field corrections to the RPA polarization operators within the mean field approach. On the other hand, the method of moments permits us to include electron-ion correlations as well as electron-electron and ion-ion correlations.

The solution of the two problems considered in the paper constitutes a basis for future experiments on reflection of laser radiation by magnetized plasmas.

APPENDIX A

To consider the $\vec{B}_0 \to \vec{0}$ transition one should know the behavior of solutions of the Fresnel's equations (2.8) in this limiting case. Though the latter are quite complicated, one can notice that, since generally the matrix $\hat{\varepsilon}(\omega)$ is determined by the orthogonal transformation (which certainly is independent of the field), it suffices to carry out the proof of existence of the $\vec{B}_0 \to 0$ limit for a matrix of the type given by Eq. (2.18). Thus it will be valid in a general case as well.

Hence in Eq. (2.8) one can set $\varepsilon_{xz}=\varepsilon_{zx}=\varepsilon_{zy}=-\varepsilon_{yz}=0$, then the solution $k_z^{\langle i \rangle}$ $(i=2,3)$ is found immediately:

$$(k_z^{\langle i \rangle})^2=\varepsilon_\perp\kappa_0^2-\frac{k_x^2}{2}\frac{\varepsilon_\parallel+\varepsilon_\perp}{\varepsilon_\parallel} \pm\frac{1}{2}\left[\left[k_x^2\frac{\varepsilon_\parallel+\varepsilon_\perp}{\varepsilon_\parallel}-2\kappa_0^2\varepsilon_\perp\right]^2 -\frac{4f}{\varepsilon_\parallel}(\varepsilon_\perp(\kappa_0^2\varepsilon_\perp-k_x^2)-\kappa_0^2g^2)\right]^{1/2} . \quad \text{(A1)}$$

In this special case the coefficients c_i $(i=2,3)$ of Eq. (2.17) are

$$c_i=[\varepsilon_\perp-(k^{\langle i \rangle})^2/\kappa_0^2]/ig . \quad \text{(A2)}$$

Further, upon carrying out all necessary transformations for the quantities determining the reflection coefficient, one finds

$$R_{11}=\frac{g^2\varepsilon_\parallel\kappa_0^5c_2c_3\cos\varphi_0}{f(k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle})^2}-\left\{k_z^{\langle 0\rangle}+\frac{k_z^{\langle 2\rangle}k_z^{\langle 3\rangle}+\kappa_0^2\varepsilon_\perp-k_x^2}{k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle}}\right\}\left\{\sqrt{\varepsilon_0}-\cos\varphi_0\frac{\kappa_0\varepsilon_\parallel}{f}\left[\frac{(k_z^{\langle 2\rangle})^3-(k_z^{\langle 3\rangle})^3}{(k_z^{\langle 3\rangle})^2-(k_z^{\langle 2\rangle})^2}+\frac{\kappa_0^2\varepsilon_\perp-k_x^2}{k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle}}\right]\right\},$$

$$R_{22}=-\frac{g^2\varepsilon_\pm\kappa_0^5c_2c_3\cos\varphi_0}{f(k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle})^2}-\left\{-k_z^{\langle 0\rangle}+\frac{k_z^{\langle 2\rangle}k_z^{\langle 3\rangle}+\kappa_0^2\varepsilon_\perp-k_x^2}{k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle}}\right\}\left\{\sqrt{\varepsilon_0}+\cos\varphi_0\frac{\kappa_0\varepsilon_\parallel}{f}\left[\frac{(k_z^{\langle 2\rangle})^3-(k_z^{\langle 3\rangle})^3}{(k_z^{\langle 3\rangle})^2-(k_z^{\langle 2\rangle})^2}+\frac{\kappa_0^2\varepsilon_\perp-k_x^2}{k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle}}\right]\right\},$$

$$R_{12}=2\frac{ig}{f}k_z^{\langle 0\rangle}\varepsilon_\parallel\kappa_0^3\frac{c_2c_3}{k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle}}, \quad \text{(A3)}$$

$$R_{21}=-2ig\kappa_0\frac{k_z^{\langle 0\rangle}}{k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle}},$$

$$R_0=\sqrt{\varepsilon_0}\left\{k_z^{\langle 0\rangle}+\frac{k_z^{\langle 2\rangle}k_z^{\langle 3\rangle}+\kappa_0^2\varepsilon_\perp-k_x^2}{k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle}}\right\}-\cos\varphi_0\frac{\kappa_0\varepsilon_\parallel}{f}\left[-\frac{(k_z^{\langle 2\rangle})^3-(k_z^{\langle 3\rangle})^3}{(k_z^{\langle 3\rangle}-(k_z^{\langle 2\rangle})^2}k_z^{\langle 0\rangle}-\frac{\kappa_0^2\varepsilon_\perp-k_x^2}{k_z^{\langle 2\rangle}+k_z^{\langle 3\rangle}}k_z^{\langle 0\rangle}+k_z^{\langle 2\rangle}k_z^{\langle 3\rangle}\right],$$

where $k_z^{\langle i \rangle}$ and c_i $(i=2,3)$ are obtained from Eqs. (A1) and (A2).

Assume now that the following limiting relations are valid for the dielectric tensor components (at least they are applicable to ideal plasmas):

$$\varepsilon_\parallel(\vec{B}_0\to\vec{0})\simeq\varepsilon,\quad \varepsilon_\perp(\vec{B}_0\to\vec{0})\simeq\varepsilon+\Delta , \quad \text{(A4)}$$

where $\Delta(\vec{B}_0\to\vec{0})$ tends to zero as B_0^2, and $g(\vec{B}_0\to 0)$ as B_0. In Eq. (A4) ε is the isotropic plasma permeability, and only leading terms of the $\vec{B}_0\to\vec{0}$ expansion are pointed out. Then from Eqs. (A1) and (A2) it follows that, when $\vec{B}_0\to\vec{0}$,

$$(k^{\langle i \rangle})^2\simeq\varepsilon\kappa_0^2-k_x^2\Delta/2\varepsilon\pm\frac{\kappa_0 g}{\sqrt{\varepsilon_0}}(\kappa_0^2\varepsilon-k_x^2)^{1/2} , \quad \text{(A5)}$$

$$c_i\simeq\frac{ik_x^2}{2\varepsilon\kappa_0^2 g}\Delta\pm i(\kappa_0^2\varepsilon)^{-1/2}(\kappa_0^2\varepsilon-k_x^2)^{1/2},\quad i=2,3 . \quad \text{(A6)}$$

Hence the product c_2c_3 for small values of the magnetic field behaves like

$$(c_2c_3)\simeq 1-\kappa_x^2/(\kappa_0^2\varepsilon)+O(B_0^2) . \quad \text{(A7)}$$

Now, utilizing Eqs. (A5) and (A7), it is easy to carry out the transition $B_0\to\vec{0}$ in formulas (A3), also taking into account that

$$\lim_{\vec{B}\to\vec{0}} k_z^{\langle 2\rangle}=\lim_{\vec{B}\to\vec{0}} k_z^{\langle 3\rangle}=k_z=-(\kappa_0^2\varepsilon-k_x^2)^{1/2} .$$

In this manner one derives well-known Fresnel's formulas [32,2].

APPENDIX B

Suppose that the non-negative Hermitian matrix function $\overleftrightarrow{\sigma}_H(\omega)$ is uniformly continuous and possesses sufficient smoothness (e.g., that it satisfies the Lipshitz condition), and that additionally

$$\overleftrightarrow{M}_r=\int_{-\infty}^{\infty}\omega^r\overleftrightarrow{\sigma}_H(\omega)d\omega<\infty,\quad r=0,1,2,3 . \quad \text{(B1)}$$

Inequalities (B1) are substantiated by Eqs. (3.8).

Due to the Riesz-Herglotz theorem [26] the matrix function $\overleftrightarrow{\sigma}(z)$, being analytical in the upper complex half-plane $\mathrm{Im} z>0$ and there having a non-negative definite Hermitian part $\overleftrightarrow{\sigma}_H(z)$ is representable as

$$\overleftrightarrow{\sigma}(z)=\overleftrightarrow{\alpha}_0-i\overleftrightarrow{\alpha}_1 z+\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\overleftrightarrow{\sigma}_H(\omega)}{\omega-z}d\omega \ , \tag{B2}$$

where $\overleftrightarrow{\alpha}_0$ has a non-negative Hermitian part, and $\overleftrightarrow{\alpha}_1$ is a non-negative Hermitian matrix.

The constant matrices $\overleftrightarrow{\alpha}_0$ and $\overleftrightarrow{\alpha}_1$ are to be determined from the asymptotic behavior of the matrix function $\overleftrightarrow{\sigma}(\omega)$ at $|\omega|\to\infty$. They describe a possible polarization of atoms and ions in nonhydrogen plasmas at $|\omega|\to\infty$. The "infinite" value of the frequency must be understood as a value much greater than the plasma frequency but much less than the excitation frequency of atoms and ions. Since only the long-wavelength limiting case is investigated, the plasma species can be regarded as pointlike particles, and hence the polarizability at high frequencies decreases as ω^{-2} and both $\overleftrightarrow{\alpha}_0=\overleftrightarrow{\alpha}_1=\overleftrightarrow{0}$. Thus the conductivity matrix satisfies the Kramers-Kronig relation

$$\overleftrightarrow{\sigma}(z)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\overleftrightarrow{\sigma}_H(\omega)}{\omega-z}d\omega \ . \tag{B3}$$

The problem is to establish the matrix function $\overleftrightarrow{\sigma}(z)$ satisfying the sum rules (3.8), and specifying the representation (B3). The explicit expression can be obtained by the Nevanlinna-type formula of the matrix problem of moments [28].

The latter one makes it possible to construct the matrix function $\overleftrightarrow{\sigma}(z)$ (being analytical in the upper complex half-plane and there having a non-negative definite Hermitian part) in terms of its first $2\nu+1$ ($\nu=0,1,\ldots$) frequency moments.

Denote by $\{\overleftrightarrow{P}_k(\omega)\}$ and $\{\overleftrightarrow{Q}_k(\omega)\}$, $k=0,1,\ldots,\nu$, the system of orthogonal matrix polynomials with the weight function $\overleftrightarrow{\sigma}_H(\omega)$. The polynomials $\{\overleftrightarrow{P}_k(\omega)\}$ (and $\{\overleftrightarrow{Q}_k(\omega)\}$) are defined by the first $2\nu+1$ moments as follows [28]:

$$\overleftrightarrow{P}_0(z)=\overleftrightarrow{M}_0^{-1/2} \ ,$$

$$\overleftrightarrow{P}_k(z)=(\overleftrightarrow{M}_{2k}-\check{B}_k^*\check{S}_{k-1}^{-1}\check{B}_k)^{-1/2}(-\check{B}_k^*\check{S}_{k-1}^{-1},\overleftrightarrow{I})\times\begin{bmatrix}\check{b}_{k-1}\\ z^k\overleftrightarrow{I}\end{bmatrix} \ , \tag{B4}$$

$$\overleftrightarrow{Q}_0(z)=\overleftrightarrow{0} \ ,$$

$$\overleftrightarrow{Q}_k(z)=-(\overleftrightarrow{M}_{2k}-\check{B}_k^*\check{S}_{k-1}^{-1}\check{B}_k)^{-1/2}(-\check{B}_k^*\check{S}_{k-1}^{-1},\overleftrightarrow{I})\times\begin{bmatrix}\check{c}_{k-1}\\ \overleftrightarrow{y}_k\end{bmatrix} \tag{B5}$$

and

$$\check{S}_{k-1}=\begin{bmatrix}\overleftrightarrow{M}_0 & \overleftrightarrow{M}_1 & \cdots & \overleftrightarrow{M}_{k-1}\\ \overleftrightarrow{M}_1 & \overleftrightarrow{M}_2 & \cdots & \overleftrightarrow{M}_k\\ \vdots & \vdots & \vdots & \vdots\\ \overleftrightarrow{M}_{k-1} & \overleftrightarrow{M}_k & \cdots & \overleftrightarrow{M}_{2k-2}\end{bmatrix} \ ,$$

$$\check{b}_{k-1}=\begin{bmatrix}\overleftrightarrow{I}\\ z\overleftrightarrow{I}\\ \vdots\\ z^{k-1}\overleftrightarrow{I}\end{bmatrix} \ ,$$

$$\check{c}_{k-1}=-\begin{bmatrix}\overleftrightarrow{0} & \overleftrightarrow{0} & \cdots & \overleftrightarrow{0} & \overleftrightarrow{0}\\ \overleftrightarrow{M}_0 & \overleftrightarrow{0} & \cdots & \overleftrightarrow{0} & \overleftrightarrow{0}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \overleftrightarrow{M}_{k-2} & \overleftrightarrow{M}_{k-3} & \cdots & \overleftrightarrow{M}_0 & \overleftrightarrow{0}\end{bmatrix}\check{b}_{k-1} \ , \tag{B6}$$

$$\overleftrightarrow{y}_k=-(\overleftrightarrow{M}_{k-1}+z\overleftrightarrow{M}_{k-2}+\cdots+z^{k-1}\overleftrightarrow{M}_0) \ ,$$

$$\check{B}_k=\begin{bmatrix}\overleftrightarrow{M}_k\\ \overleftrightarrow{M}_{k+1}\\ \vdots\\ \overleftrightarrow{M}_{2k-1}\end{bmatrix} \ .$$

In formulas (B4)–(B6), $(\check{A},\check{B})$ is the block-matrix line and $\binom{\check{A}}{\check{B}}$ the block-matrix column, where $\check{A}$ and $\check{B}$ are, in general, block matrices as well; $\check{A}^*$ is the Hermitian conjugate of $\check{A}$. Note that the matrices in (B4)–(B6) are generally not commutable.

Let R_ν be the set of all non-negative definite matrix functions of limited variation $\overleftrightarrow{s}(\omega)$ such that

$$\int_{-\infty}^{\infty}\omega^r d\overleftrightarrow{s}(\omega)=\int_{-\infty}^{\infty}\omega^r\overleftrightarrow{\sigma}_H(\omega)d\omega=\overleftrightarrow{M}_r \ , \quad r=0,1,\ldots,2\nu \ . \tag{B7}$$

Kovalishina proved [28] that there is a univalent correspondence between the R_ν matrix functions and the matrix functions $\overleftrightarrow{t}_\nu(z)$ being analytical in the upper complex half-plane, and there having a positive anti-Hermitian part such that the matrix $z^{-1}\overleftrightarrow{t}_\nu(z)$ for $z^{\to\infty}$ converges to zero.

This correspondence is set up by the generalization of the Nevanlinna formula [26–28]:

$$\int_{-\infty}^{\infty}\frac{d\overleftrightarrow{s}(\omega)}{\omega-z}=[\overleftrightarrow{\alpha}_\nu(z)-\overleftrightarrow{\beta}_\nu(z)\overleftrightarrow{t}_\nu(z)]\times[\overleftrightarrow{\gamma}_\nu(z)-\overleftrightarrow{\delta}_\nu(z)\overleftrightarrow{t}_\nu(z)]^{-1} \ . \tag{B8}$$

In particular, among the functions $\overleftrightarrow{t}_\nu(z)$ there (for a given ν) is only one matrix function $\overleftrightarrow{T}_\nu(z)$ satisfying the equality

$$\int_{-\infty}^{\infty}\frac{d\omega\overleftrightarrow{\sigma}_H(\omega)}{\omega-z}=[\overleftrightarrow{\alpha}_\nu(z)-\overleftrightarrow{\beta}_\nu(z)\overleftrightarrow{T}_\nu(z)]\times[\overleftrightarrow{\gamma}_\nu(z)-\overleftrightarrow{\delta}_\nu(z)\overleftrightarrow{T}_\nu(z)]^{-1} \ . \tag{B9}$$

The matrices $\overleftrightarrow{\alpha}_\nu$, $\overleftrightarrow{\beta}_\nu$, $\overleftrightarrow{\gamma}_\nu$, and $\overleftrightarrow{\delta}_\nu$ are defined by the polynomials $\overleftrightarrow{P}_k(z)$ and $\overleftrightarrow{Q}_k(z)$ as follows:

$$\begin{bmatrix}\overleftrightarrow{\alpha}_\nu(z) & \overleftrightarrow{\beta}_\nu(z)\\ \overleftrightarrow{\gamma}_\nu(z) & \overleftrightarrow{\delta}_\nu(z)\end{bmatrix}=\begin{bmatrix}\overleftrightarrow{I} & \overleftrightarrow{0}\\ \overleftrightarrow{0} & \overleftrightarrow{I}\end{bmatrix}+z\sum_{k=0}^{\nu}\begin{bmatrix}\overleftrightarrow{Q}_k^*(\bar{z})\overleftrightarrow{P}_k(0) & \overleftrightarrow{Q}_k^*(\bar{z})\overleftrightarrow{Q}_k(0)\\ -\overleftrightarrow{P}_k^*(\bar{z})\overleftrightarrow{P}_k(0) & -\overleftrightarrow{P}_k^*(\bar{z})\overleftrightarrow{Q}_k(0)\end{bmatrix} \ . \tag{B10}$$

Here $\bar{z}$ is the complex conjugate of z.

In our case $\nu=1$; and from Eqs. (3.8), (B4)–(B6), and (B10) one obtains

$$
\begin{aligned}
\overleftrightarrow{\alpha}_1(z)&=\overleftrightarrow{I}-z\overleftrightarrow{M}_0^{1/2}\overleftrightarrow{\Omega}^{-2}\overleftrightarrow{M}_0^{-1/2}\overleftrightarrow{\omega}_1 ,\\
\overleftrightarrow{\beta}_1(z)&=z\overleftrightarrow{M}_0^{1/2}\overleftrightarrow{\Omega}^{-2}\overleftrightarrow{M}_0^{1/2} ,\\
\overleftrightarrow{\gamma}_1(z)&=-\overleftrightarrow{M}_0^{-1}z+z(z\overleftrightarrow{I}-\overleftrightarrow{\omega}_1)\overleftrightarrow{M}_0^{-1/2}\Omega^{-2}\overleftrightarrow{M}_0^{1/2}\overleftrightarrow{\omega}_1 ,\\
\overleftrightarrow{\delta}_1(z)&=\overleftrightarrow{I}-z(z\overleftrightarrow{I}-\overleftrightarrow{\omega}_1)\overleftrightarrow{M}_0^{-1/2}\Omega^{-2}\overleftrightarrow{M}_0^{-1/2} .
\end{aligned}
\tag{B11}
$$

Using Eq. (B9) for $\nu=1$ and taking into account the Kramers-Kronig relation Eq. (B3), one arrives at the following representation of the conductivity matrix:

$$
\overleftrightarrow{\sigma}(z)=-\frac{i}{\pi}[\overleftrightarrow{\alpha}_1(z)-\overleftrightarrow{\beta}_1(z)\overleftrightarrow{T}_1(z)]\times[\overleftrightarrow{\gamma}_1(z)-\overleftrightarrow{\delta}_1(z)\overleftrightarrow{T}_1(z)]^{-1} . \tag{B12}
$$

Note, however, that in our case the matrix $\overleftrightarrow{\Omega}^{-2}=(\overleftrightarrow{\Omega}^2)^{-1}$ does not exist, since $\det\overleftrightarrow{\Omega}^2=\overleftrightarrow{0}$. This is due to the fact that representation (B9) [with the particular case (B12)] is given for strongly positive definite sequences of frequency moments $\{\overleftrightarrow{M}_r\}$, $r=0,1,\ldots,2\nu$ only, i.e., the sequences for which for all $k\le\nu$ the matrices $(\overleftrightarrow{M}_{2k}-\breve{B}_k^*\breve{S}_{k-1}^{-1}\breve{B}_k)$ are strongly positive definite. In our case for $k=1$ we define $(\overleftrightarrow{M}_2-\breve{B}_1^*\breve{S}_0^{-1}\breve{B}_1=\overleftrightarrow{M}_0^{1/2}\overleftrightarrow{\Omega}^2\overleftrightarrow{M}_0^{1/2})$ a non-negative matrix only. In order to obtain a representation for our case as well, one rewrites Eq. (B12) in the following manner:

$$
\overleftrightarrow{\sigma}(z)=\frac{i}{\pi}\overleftrightarrow{M}_0^{1/2}[z\overleftrightarrow{I}+\overleftrightarrow{\tau}(z)\overleftrightarrow{\Omega}^2]\times[z^2\overleftrightarrow{I}-z\overleftrightarrow{\omega}_1-\overleftrightarrow{\Omega}^2+(z\overleftrightarrow{I}-\overleftrightarrow{\omega}_1)\overleftrightarrow{\tau}(z)\overleftrightarrow{\Omega}^2]^{-1}\overleftrightarrow{M}_0^{1/2} , \tag{B13}
$$

where we introduced the matrix function $\overleftrightarrow{\tau}(z)=-[\overleftrightarrow{M}_0^{1/2}\overleftrightarrow{T}_1(z)\overleftrightarrow{M}_0^{1/2}-\overleftrightarrow{\omega}_1]^{-1}$, which is analytical in the upper complex half-plane and there having a positive definite anti-Hermitian part such that in the upper half-plane the matrix $z^{-1}\overleftrightarrow{\tau}(z)$ for $z\to\infty$ converges to zero.

Taking into account that the matrices $\overleftrightarrow{M}_0$ and $\overleftrightarrow{\omega}_1$ are commutable, one easily proves that for all positive definite matrices $\overleftrightarrow{\Omega}^2$ the right-hand side of Eqs. (B12) and (B13) coincide. The right-hand side of Eq. (B13), however, also exists for non-negative matrices with $\det\overleftrightarrow{\Omega}^2=\overleftrightarrow{0}$. Therefore, Eq. (B13) is the general solution for the conductivity matrix satisfying the frequency moments $\{\overleftrightarrow{M}_r\}$, $r=0,1,2$. If, in addition, the Hermitian part of the matrix $\overleftrightarrow{\tau}(z)$ for $z\to\infty$ converges to zero, the conductivity matrix defined by Eq. (B13) also satisfies the frequency moment $\overleftrightarrow{M}_3$.

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