Global stability and local stability of Hopfield neural networks with delays

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It is well known that Hopfield neural networks without delays exhibit no oscillations and possess global stability (i.e., all trajectories tend to some equilibrium). In the present paper we show that if the bound $\tau\beta\|T_2\|<1$ is satisfied, then a corresponding Hopfield neural network with delays $\tau>0$, interconnection matrix T_2 associated with delays, and gain of the neurons given by β, will exhibit similar qualitative properties as the original Hopfield neural network without delays ($\|T_2\|$ denotes the matrix norm induced by the Euclidean vector norm). Specifically, we show that if the above bound is satisfied, then a Hopfield neural network without delays and a corresponding Hopfield neural network with delays will have identical asymptotically stable equilibria, and both networks are globally stable. In addition to the above, we provide in the present paper an effective method of determining the asymptotic stability of an equilibrium of a Hopfield neural network with delays, assuming that the above bound is satisfied. Our results are consistent with the results reported by Marcus and Westervelt [Phys. Rev. A **39**, 347 (1989)]. Specifically, the present results, all of which are obtained by rigorous proof, give support to these results, which are based on linearization arguments, numerical simulations, and experimental results.

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I. INTRODUCTION

Hopfield neural networks, which constitute a class of artificial neural networks, are of great common interest (see, e.g., [1–8]) and can be described by a system of differential equations of the form

$$\dot{x}=-Cx+TS(x)+b\ , \tag{1.1}$$

where $x\in\mathbb{R}^n$ denotes the state variables associated with the neurons, b is a real n-vector representing bias terms, C is a real $n\times n$ diagonal matrix representing self-feedback terms, $T=T^T$ is a real symmetric $n\times n$ matrix representing neuron interconnections, and $S(x)=[s_1(x_1),\dots,s_n(x_n)]^T$ is a real n-vector valued function whose components are sigmoidal nonlinearities representing the neurons. One of the reasons why Hopfield neural networks have received a great deal of attention is because these networks possess global stability (see [3]), i.e., the outputs of the neurons will always converge to some equilibrium, and no oscillations will be present. Since Hopfield neural networks have the potential of performing parallel computation, some electronic implementations of Hopfield neural networks in VLSI technology have already been realized (see, e.g., [9–11]). However, in the implementation of artificial neural networks, time delays are unavoidably encountered, and it is known that time delays can cause systems to oscillate (see, e.g., [12,13] and [14]). Therefore, it is crucial to take time delays into consideration and to investigate the qualitative properties of Hopfield neural networks with delays. A class of such networks can be described by systems of equations of the form

$$\dot{x}(t)=-Cx(t)+T_1S(x(t))+T_2S(x(t-\tau))+b\ , \tag{1.2}$$

where x,b,C,S,T are the same as in (1.1), and $T_1+T_2=T$ (i.e., T_1 denotes the part of the interconnections associated with no delays while T_2 denotes the part of the interconnections associated with delays). We note that T_1,T_2 need not be symmetric; however, we will assume that T is symmetric.

Hopfield neural networks with delays (1.2) have also received attention (see, e.g., [15–17]). Most of these works consider only the local stability of system (1.2). Existing results concerning global stability are applicable to a special class of system (1.2) [for example, the components of $S(x)$ are piecewise linear functions instead of general sigmoidal functions (see, e.g., [16])]. For the general Hopfield neural networks with delays given by Eq. (1.2), there are no existing results which provide explicit answers to the question whether system (1.2) is globally stable when the delay is sufficiently small. The above question was studied extensively in [14], and it is suggested in [14] that the answer to this question is affirmative. The neural network model considered in [14] is a special case of system (1.2), where C is the identity matrix, T_1 is the null matrix, and all the components of $S(x)$ are identical functions. It is asserted in [14] that this special class of Hopfield neural networks with delays has the global stability property when the delays are smaller than some bound which can be determined explicitly. However, this assertion lacks proof. (In the words of the authors, "our results are based on local rather than global stability analysis and therefore do not provide a rigorous guarantee of stability. Rather, we support our results with numerical and experimental evidence suggesting that stability criteria presented here are valid under the conditions investigated.")

In the present paper, we provide a bound for the delay τ of system (1.2), which depends on the interconnection matrix and on the neuron gains. We prove rigorously that when the delay is smaller than this bound, system

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(1.2) will have the same global stability and also the same local stability properties as system (1.1). As a consequence of this, it can be concluded that the system (1.2) will be globally stable for sufficiently small delays when the corresponding system (1.1) is globally stable. In our proof, we make use of an energy functional for system (1.2) and we show that this energy functional decreases along the solutions of (1.2), ultimately converging to some equilibrium of system (1.2). Some of our results are consistent with existing results (small gain results) reported in [14] and therefore offer theoretical support of these results. We also show that any (asymptotically) stable equilibrium of (1.2) corresponds to a local minimum of the energy functional. When delays are smaller than the bound mentioned above, we prove that the set of all (asymptotically) stable equilibria of (1.2) is identical to the set of all the (asymptotically) stable equilibria of (1.1). In other words, not only the global stability of system (1.2), but also the local (asymptotic) stability of each equilibrium of (1.2) will be unaffected by small delays. Moreover, we establish an effective criterion for the (asymptotic) stability of each single equilibrium of (1.2).

Before proceeding further, it is important to point out that system (1.2) constitutes a special case of neural networks with delays described by equations of the form

$$\dot{x}_i(t) = -c_i x_i(t) + \sum_{j=1}^{n} t_{ij} s_j(x_j(t-\tau_{ij})) + b_i \ , \qquad i=1,\dots,n \qquad (1.2')$$

where $C=\text{diag}\{c_1,\dots,c_n\}$, $T=[t_{ij}]_{n\times n}$, $S(x)=[s_1(x_1),\dots,s_n(x_n)]^T$, $b=[b_1,\dots,b_n]^T$, and where C, T, $S(x)$, and b are defined as in (1.2). Equation (1.2′) provides a generalization which admits different time delays $\tau_{ij}\geq 0$ for different interconnections. Work by the present authors is in progress to extend the present results to neural networks described by (1.2′).

In the next section, we provide the necessary notation used throughout this paper. In Sec. III, we establish our main result for the global stability of Hopfield neural networks with delays. In Sec. IV we investigate the local stability of equilibria of neural networks (1.2). A specific example is given in Sec. V to demonstrate the applicability of some of our results. Concluding remarks are given in the final section (Sec. VI).

II. NOTATION

Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^n$ denote real n space. If $x\in\mathbb{R}^n$, then $x^T=(x_1,\dots,x_n)$ denotes the transpose of x. Let $\mathbb{R}^{n\times m}$ denote the set of $n\times m$ real matrices. If $B=[b_{ij}]_{n\times m}\in\mathbb{R}^{n\times m}$, then B^T denotes the transpose of B, and $\det(B)$ denotes the determinant of B. For $x\in\mathbb{R}^n$, let $\|x\|$ denote the Euclidean vector norm, $\|x\|=(x^Tx)^{1/2}$, and for $A\in\mathbb{R}^{n\times n}$, let $\|A\|$ denote the norm of A induced by the Euclidean vector norm, i.e., $\|A\|=[\lambda_{\max}(A^TA)]^{1/2}$. I denotes the identity $n\times n$ matrix.

Let $\mathbb{R}^+$ denote the set of non-negative real numbers, i.e., $\mathbb{R}^+=[0,+\infty)$. Let X be a subset of $\mathbb{R}^n$ and let Y be a subset of $\mathbb{R}^m$. We denote by $C[X,Y]$ the set of all continuous functions from X to Y, and we denote by $C^k[X,Y]$ the set of all functions from X to Y which have continuous derivatives up to order k. Let $\tau>0$, $x\in C[[-\tau,+\infty),\mathbb{R}^n]$, and $t>0$. We define $x_t\in C[[-\tau,0],\mathbb{R}^n]$ as $x_t(s)=x(t+s)$ for $s\in[-\tau,0]$. For any $\phi\in C[[-\tau,0],\mathbb{R}^n]$, the norm of ϕ, denoted by $|\phi|$, is defined as $|\phi|=\max\{\|\phi(t)\|:t\in[-\tau,0]\}$.

The system (1.2) is said to be globally stable if for any solution $x(t)$, $\lim_{t\to\infty}x(t)$ exists. The definitions of stability and asymptotic stability of any equilibrium of (1.2) are contained in standard texts (see, e.g., [18]).

III. GLOBAL STABILITY OF HOPFIELD NEURAL NETWORKS WITH DELAYS

In the present section, we consider the Hopfield neural networks with delays described by the retarded type differential-difference equation (1.2), given by $\dot{x}(t)=-Cx(t)+T_1S(x(t))+T_2S(x(t-\tau))+b$, where $C=\text{diag}\{c_1,\dots,c_n\}$ with $c_i>0$ for $i=1,\dots,n$, $T=T_1+T_2$ is a symmetric matrix, $b\in\mathbb{R}^n$, $\tau>0$, and $S(x)=[s_1(x_1),\dots,s_n(x_n)]^T$ is a sigmoidal vector function such that $s_i(\cdot)\in C^1(\mathbb{R},\mathbb{R})$, $s_i'(\rho):=ds_i/d\rho(\rho)>0$, $\lim_{\rho\to\infty}s_i(\rho)=1$, $\lim_{\rho\to-\infty}s_i(\rho)=-1$, and $\lim_{|\rho|\to\infty}s_i'(\rho)=0$ for $i=1,\dots,n$. In order to establish our main results, we need to present some existing properties for system (1.2).

Lemma 1. Any solution of system (1.2) is bounded.

Remark 1. The proof of Lemma 1 is identical to the proof when $\tau=0$ (see, e.g., [2]), noticing that the c_i's are positive and the $s_i(x_i)$'s are bounded, $i=1,\dots,n$.

Assumption A. For any equilibrium x_e of system (1.2) [i.e., $-Cx_e+TS(x_e)+b=0$ with $T=T_1+T_2$], the matrix $J(x_e)$ is a nonsingular matrix, where

$$J(x)=-T+\text{diag}\{c_i(s_1'(x_1))^{-1},\dots,c_n(s_n'(x_n))^{-1}\} \ . \qquad (3.1)$$

Lemma 2. For almost all $b\in\mathbb{R}^n$ (except a set with Lebesgue measure 0), system (1.2) satisfies Assumption A.

Remark 2. Lemma 2 can easily be proved by using Sard's Theorem (see Lemma 3.3 of [6,19]).

Lemma 3. When system (1.2) satisfies Assumption A, the set of equilibria of system (1.2) is a discrete set.

Remark 3. Lemma 3 can be proved by the inverse function theorem (see Remark 3.4 of [6]).

Remark 4. Throughout this paper, we will assume that system (1.2) satisfies Assumption A. We note that this assumption is also required in global stability studies of artificial neural networks without delays (see, e.g., [6,2]). In fact, Assumption A is very mild, since by Lemma 2 this assumption is satisfied for almost all $b\in\mathbb{R}^n$.

We are now in a position to establish the following result.

Theorem 1. Suppose that Assumption A is satisfied for system (1.2), and suppose that

$$\tau\beta\|T_2\|<1 , \tag{3.2}$$

where $\beta=\max_{x\in\mathbb{R}^n}\|D(x)\|$, and $D(x):=\mathrm{diag}\{s_1'(x_1),\ldots,s_n'(x_n)\}$. Then, system (1.2) is globally stable.

Proof. Let $y=S(x)$, and suppose x_t is a function in $C([-\tau,0],\mathbb{R}^n)$. Then $y_t=S(x_t)$ is also in $C([-\tau,0],\mathbb{R}^n)$. We define an energy functional $E(x_t)$ associated with (1.2) by

$$E(x_t)=-y_t^T(0)Ty_t(0)+2\sum_{i=1}^{n}\int_0^{(y_t(0))_i}c_is_i^{-1}(\sigma)d\sigma-2y_t^T(0)b+\int_{-\tau}^{0}[y_t(\theta)-y_t(0)]^TT_2^Tf(\theta)T_2[y_t(\theta)-y_t(0)]d\theta , \tag{3.3}$$

where $f(\theta)\in C^1([-\tau,0],\mathbb{R}^+)$ which will be specified later. After changing integration variables, (3.3) can be represented by

$$E(x_t)=-y^T(t)Ty(t)+2\sum_{i=1}^{n}\int_0^{y_i(t)}c_is_i^{-1}(\sigma)d\sigma-2y^T(t)b+\int_{t-\tau}^{t}[y(w)-y(t)]^TT_2^Tf(w-t)T_2[y(w)-y(t)]dw . \tag{3.4}$$

The derivative of $E(x_t)$ with respect to t along any solution of (1.2) can be calculated as

$$\begin{aligned}
\frac{dE(x_t)}{dt}&=-2y^T(t)TD(x(t))[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]\\
&\quad+2x^T(t)CD(x(t))[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]\\
&\quad-2[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]^TD(x(t))^b\\
&\quad-[y(t-\tau)-y(t)]^TT_2^Tf(-\tau)T_2[y(t-\tau)-y(t)]\\
&\quad-\int_{t-\tau}^{t}[y(w)-y(t)]^TT_2^Tf'(w-t)T_2[y(w)-y(t)]dw\\
&\quad-\int_{t-\tau}^{t}[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]^TD(x(t))T_2^Tf(w-t)T_2[y(w)-y(t)]dw\\
&\quad-\int_{t-\tau}^{t}[y(w)-y(t)]^TT_2^Tf(w-t)T_2D(x(t))[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]dw\\
&=-2[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]^TD(x(t))[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]\\
&\quad+2[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]^TD(x(t))T_2[y(t-\tau)-y(t)]\\
&\quad-[y(t-\tau)-y(t)]^TT_2^Tf(-\tau)T_2[y(t-\tau)-y(t)]\\
&\quad-\int_{t-\tau}^{t}[y(w)-y(t)]^TT_2^Tf'(w-t)T_2[y(w)-y(t)]dw\\
&\quad-\int_{t-\tau}^{t}[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]^TD(x(t))T_2^Tf(w-t)T_2[y(w)-y(t)]dw\\
&\quad-\int_{t-\tau}^{t}[y(w)-y(t)]^TT_2^Tf(w-t)T_2D(x(t))[-Cx(t)+T_1y(t)+T_2y(t-\tau)+b]dw\\
&=-\int_{-\tau}^{0}\alpha(x_t,\theta)^TM(x_t,\theta)\alpha(x_t,\theta)d\theta ,
\end{aligned} \tag{3.5}$$

where $f'(\theta)=df/d\theta(\theta)$ and $\alpha(x_t,\theta)=[\alpha_1^T,\alpha_2^T,\alpha_3^T]$ such that

$$\alpha_1=-Cx(t)+T_1y(t)+T_2y(t-\tau)+b , \tag{3.6}$$

$$\alpha_2=T_2[y(t-\tau)-y(t)] , \tag{3.7}$$

$$\alpha_3=T_2[y(t+\theta)-y(t)] , \tag{3.8}$$

and

$$M(x_t,\theta)=\begin{bmatrix}\dfrac{2D(x(t))}{\tau} & -\dfrac{D(x(t))}{\tau} & D(x(t))T_2^Tf(\theta)\\ -\dfrac{D(x(t))}{\tau} & \dfrac{f(-\tau)}{\tau}I & 0\\ f(\theta)T_2D(x(t)) & 0 & f'(\theta)I\end{bmatrix} , \tag{3.9}$$

where I denotes the $n\times n$ identity matrix. To obtain the last equation in (3.5), we changed the integration variables from w back to θ. We will show that if the hypotheses of Theorem 1 are satisfied, then $M(x_t,\theta)$ is positive definite for all

$\theta\in[-\tau,0]$ and all x_t which satisfy Eq. (1.2). In doing so, we let $U=U_3U_2U_1$, where

$$U_1=\begin{bmatrix} I & 0 & 0\\ -\frac{I}{2} & I & 0\\ 0 & 0 & I\end{bmatrix},\quad U_2=\begin{bmatrix} I & 0 & 0\\ 0 & I & 0\\ -\frac{\tau}{2}f(\theta)T_2 & 0 & I\end{bmatrix},$$

and

$$U_3=\begin{bmatrix} I & 0 & 0\\ 0 & I & 0\\ 0 & -\frac{1}{2}f(\theta)T_2D(x(t))\left[\frac{f(-\tau)}{\tau}I-\frac{D(x(t))}{2\tau}\right]^{-1} & I\end{bmatrix}.$$

It is not difficult to verify that $\tilde{M}=UM(x_t,\theta)U^T$ is a diagonal matrix. In fact,

$$\tilde{M}=\mathrm{diag}\{M_1,M_2,M_3\}\ , \tag{3.10}$$

where

$$M_1=\frac{2D(x(t))}{\tau}\ , \tag{3.11}$$

$$M_2=\frac{f(-\tau)}{\tau}I-\frac{D(x(t))}{2\tau}\ , \tag{3.12}$$

and

$$M_3=f'(\theta)I-\frac{f(\theta)T_2D(x(t))}{2}\left[\left[\frac{f(-\tau)}{\tau}I-\frac{D(x(t))}{2\tau}\right]^{-1}+2\tau D^{-1}(x(t))\right]\frac{D(x(t))T_2^Tf(\theta)}{2}\ . \tag{3.13}$$

It follows that $M(x_t,\theta)$ is positive definite if and only if $\tilde{M}$ is positive definite, and if and only if M_1, M_2, and M_3 are all positive definite.

We now show that if the conditions $\tau\beta\|T_2\|<1$ is satisfied, where $\beta=\max_{x\in\mathbf{R}}\|D(x)\|$, then we can always find a suitable $f(\theta)\in C([-\tau,0],\mathbf{R}^+)$ such that M_1, M_2, and M_3 are positive definite for all x_t which satisfy Eq. (1.2) and for all θ. From this it follows that $M(x_t,\theta)$ is positive definite and, therefore, $dE(x_t)/dt\le 0$ along any solution x_t of (1.2).

By the assumption that $s_i'(\rho)>0$ for all $\rho\in\mathbf{R}$, the matrix M_1 is automatically positive definite. The matrix M_2 will always be positive definite if condition

$$2f(-\tau)-\beta>0 \tag{3.14}$$

is satisfied. For M_3, it is easily shown that if

$$f'(\theta)>\tfrac{1}{4}f^2(\theta)\|T_2\|^2\left\|\left\{D(x(t))\left[\left[\frac{f(-\tau)}{\tau}I-\frac{D(x(t))}{2\tau}\right]^{-1}+2\tau D^{-1}(x(t))\right]D(x(t))\right\}\right\| \tag{3.15}$$

is true, then M_3 is also positive definite. Notice that the matrix

$$H:=\left\{D(x(t))\left[\left[\frac{f(-\tau)}{\tau}I-\frac{D(x(t))}{2\tau}\right]^{-1}+2\tau D^{-1}(x(t))\right]D(x(t))\right\}$$

is a diagonal matrix, i.e., $H=\mathrm{diag}\{h_1,\dots,h_n\}$. It is easy to show that

$$h_i=\frac{4f(-\tau)s_i'(x_i(t))\tau}{2f(-\tau)-s_i'(x_i(t))}$$

for $i=1,\dots,n$. Since $s_i'(x_i(t))<\beta$ by the definition of β, we have, in view of (3.14), that

$$h_i\le\frac{4f(-\tau)\beta\tau}{2f(-\tau)-\beta}\ .$$

Therefore, we obtain

$$\|H\|\le\frac{4f(-\tau)\beta\tau}{2f(-\tau)-\beta}\ ,$$

and furthermore, condition (3.15) will be satisfied if (3.14) is satisfied and

$$f'(\theta)>\tfrac{1}{4}f^2(\theta)\|T_2\|^2\frac{4f(-\tau)\beta\tau}{2f(-\tau)-\beta} \tag{3.16}$$

is satisfied.

Next, we need to show that there is an $f \in C^1([-\tau,0],\mathbb{R})$ such that conditions (3.14) and (3.16) are satisfied. We choose

$$f(-\tau)=(\beta\tau^2\|T_2\|^2)^{-1} . \quad (3.17)$$

Condition (3.14) is satisfied by the choice (3.17). Furthermore,

$$\left[f(-\tau)\tau\|T_2\|-\frac{1}{\beta\tau\|T_2\|}\right]^2+1-\frac{1}{\beta^2\tau^2\|T_2\|^2} =1-\frac{1}{\beta^2\tau^2\|T_2\|^2}<0$$

is true because $\beta\tau\|T_2\|<1$. It follows that

$$kf(-\tau)\tau<1 , \quad (3.18)$$

where

$$k=\frac{\|T_2\|^2 f(-\tau)\beta\tau}{2f(-\tau)-\beta} . \quad (3.19)$$

Since $kf(-\tau)\tau<1$, we can always find an l, such that $0<l<1$ and $kf(-\tau)\tau<l$. Therefore, we will always have $\gamma>0$ where γ is given by

$$\gamma=\frac{l}{kf(-\tau)}-\tau . \quad (3.20)$$

We now choose $f(\theta)$ on $[-\tau,0]$ as

$$f(\theta)=\frac{l}{k(\gamma-\theta)} . \quad (3.21)$$

It is easily verified that this choice is consistent with (3.17). Clearly, $f \in C([-\tau,0],\mathbb{R}^+)$ since $\gamma>0$. The derivative of $f(\theta)$ is given by

$$f'(\theta)=\frac{1}{k(\gamma-\theta)^2}=\frac{k}{l}f^2(\theta)>kf^2(\theta) , \quad (3.22)$$

since $l<1$. Combining (3.19) and (3.22), we can verify that $f(\theta)$ satisfies condition (3.16).

Therefore, we have shown that if $\beta\tau\|T_2\|<1$, then there exists an $f(\theta)$ [given by (3.21), where γ, k, and $f(-\tau)$ are given by (3.20), (3.19), and (3.17), respectively] such that conditions (3.14) and (3.16) are satisfied. Thus $M(x_t,\theta)$ is positive definite for all x_t satisfying Eq. (1.2) and all $\theta\in[-\tau,0]$. Therefore, we have in fact shown that

$$\frac{dE(x_t)}{dt}\le 0 \quad (3.23)$$

along any solution x_t of Eq. (1.2), where $E(x_t)$ is the energy functional given by (3.3).

We know that if $dE(x_t)/dt=0$ for some x_t satisfying Eq. (1.2), then $\alpha_1=0$, $\alpha_2=0$, and $\alpha_3=0$ for all $\theta\in[-\tau,0]$, where the α_i's are given by (3.6)–(3.8).

Since for any x_t satisfying Eq. (1.2), x_t is bounded (see Lemma 1), and since $dE(x_t)/dt\le 0$, it follows from the invariance theory (see Chap. 4, Lemmas 1.4 and 2.1 of [18]), that the limit set of x_t as $t\to\infty$ is a connected subset of the set of all equilibria of system (1.2). By Assumption A, this set is a discrete set (Lemma 3). Thus we have proved that x_t approaches some equilibrium of system (1.2) as $t\to\infty$. Specifically, we have shown that $\lim_{t\to\infty}x(t)$ exists.

Remarks 5. (i) A global stability condition for a special case of system (1.2) appeared in [16] where all components of $S(x)$ are assumed to be saturation functions instead of general sigmoidal functions. The results in [16] are more restrictive than Theorem 1. (ii) In [14] a special case of system (1.2) is considered, given by the equation

$$\dot{x}(t)=-x(t)+TS(x(t-\tau)) , \quad (3.24)$$

where x and T are the same as in (1.2), and the components of $S(x)$ are identical sigmoidal functions. By linearization of (3.24) about an equilibrium, the authors of [14] obtain the bound $\tau\beta\lambda_{\min}(T)<\pi/2$ for the asymptotic stability of the equilibrium of (3.24). This local asymptotic stability criterion for an equilibrium is conjectured to be also a bound for the global stability of (3.24) in [14].

When applying Theorem 1 to (3.24), we obtain the bound $\tau\beta\|T\|<1$ for the global stability of system (3.24). Although this bound is more conservative than the bound given in the preceding paragraph, the present result is obtained by proof and involves no conjectures. Experimental results obtained in [14] suggest that both of the bounds given above may be conservative.

IV. LOCAL STABILITY OF HOPFIELD NEURAL NETWORKS WITH DELAYS

In the preceding section we showed that when $\tau\beta\|T_2\|<1$, Hopfield neural networks with delays described by Eq. (1.2) are globally stable, i.e., any solution of (1.2) will converge to *some* equilibrium of (1.2). Since in the implementation of Hopfield neural networks as associative memories, information is stored in specific asymptotically stable equilibria (called stable memories), good criteria which ensure the asymptotic stability of an equilibrium of (1.2) are of great interest. We address this issue in the present section.

At the present time, there are no known general results which provide necessary and sufficient conditions for the asymptotic stability of an equilibrium for Hopfield neural networks with delays [given by (1.2)]. However, several results have been reported which provide sufficient conditions for the asymptotic stability of an equilibrium for (1.2). These results are frequently obtained by linearizing (1.2) about an equilibrium of interest (see, e.g., [14]). Other results, which make use of sector conditions for nonlinearities, have been obtained by Lyapunov's Second Method (see, e.g., [17]). It should be emphasized that in the case of delay equations, even for linear systems given by

$$\dot{x}(t)=Ax(t)+Bx(t-\tau) , \quad (4.1)$$

there are no known general results which constitute necessary and sufficient conditions for the asymptotic stability of the equilibrium $x=0$. [In (4.1), $A\in\mathbb{R}^{n\times n}$,

$B \in \mathbf{R}^{n\times n}$, $\tau > 0$, and $x \in \mathbf{R}^n$.] However, many sufficient conditions for the asymptotic stability of the equilibrium $x=0$ of (4.1) have been established (see, e.g., [20–23]). In the present section we will show that if the conditions of Theorem 1 are satisfied, then the asymptotic stability of any equilibrium of system (1.2) can be deduced from the asymptotic stability of the same equilibrium of system (1.1). In other words, if $\tau\beta\|T_2\| < 1$ then (as shown in the preceding section), Hopfield neural networks (1.1) and Hopfield neural networks with delays (1.2) are both globally stable, and furthermore (as will be shown in the present section), both have the same local stability properties at any equilibrium. This enables us to verify the asymptotic stability of the equilibria of system (1.2) by ascertaining the asymptotic stability of corresponding equilibria of system (1.1).

In order to proceed further, we require the following.

Definition. An element $\phi \in C([-\tau,0],\mathbf{R}^n)$ is called a *local minimum* of the energy functional E defined by (3.3) if there exists a $\delta > 0$, such that for any $\tilde{\phi} \in C([-\tau,0],\mathbf{R}^n)$, $E(\phi) \le E(\tilde{\phi})$ whenever $|\phi - \tilde{\phi}| < \delta$.

We are now able to establish the following results.

Theorem 2. Suppose that the conditions of Theorem 1 are satisfied. If x_e is an equilibrium of (1.2), then the following statements are equivalent: (i) x_e is a stable equilibrium of (1.2); (ii) x_e is an asymptotically stable equilibrium of (1.2); (iii) ϕ_{x_e} is a local minimum of the energy functional E given by (3.3), where $\phi_{x_e} \in C([-\tau,0],\mathbf{R}^n)$ such that $\phi_{x_e} \equiv x_e$; (iv) $J(x_e)$ is positive definite, where $J(x)$ is given in Eq. (3.1).

Proof.

(a) (i) $\Longrightarrow$ (ii). Since Assumption A is satisfied, the set of equilibria of system (1.2) is a discrete set by Lemma 3. Therefore, when $\epsilon > 0$ is sufficiently small, there is no other equilibrium in $U(x_e,\epsilon)$, a neighborhood of x_e, given by

$$U(x_e,\epsilon) := \{x \in \mathbf{R}^n : \|x - x_e\| < \epsilon\} . \tag{4.2}$$

Since x_e is a stable equilibrium of (1.2), there exists an $\eta > 0$ such that for any $\phi \in C([-\tau,0],\mathbf{R}^n)$ satisfying $|\phi - x_e| < \eta$, $|x_t - x_e| < \epsilon$ for all $t > 0$, where x_t is the solution of (1.2) with initial condition ϕ. Thus $x_t \in C([-\tau,0],U(x_e,\epsilon))$ for all t. In view of Theorem 1, x_t will converge to some equilibrium of system (1.2). Since x_e is the only equilibrium of (1.2) in $U(x_e,\epsilon)$, it follows that x_t converges to x_e. Thus, we have shown that x_e is an attractive equilibrium of system (1.2). Therefore the stable equilibrium x_e of (1.2) is an asymptotically stable equilibrium of system (1.2).

(b) (ii) $\Longrightarrow$ (iii). Since x_e is an asymptotically stable equilibrium of system (1.2), there exists an $\eta > 0$ such that for any $\phi \in C([-\tau,0],\mathbf{R}^n)$ satisfying $|\phi - x_e| < \eta$, x_t converges to x_e, where x_t is the solution of (1.2) with initial condition ϕ. Therefore $E(\phi_{x_e}) \le E(x_t) \le E(\phi)$ for any $\phi \in C([-\tau,0],\mathbf{R}^n)$ satisfying $|\phi - x_e| < \eta$. Therefore ϕ_{x_e} is a local minimum of the energy functional E.

(c) (iii) $\Longrightarrow$ (iv). Let $\tilde{E}$ be a function from $\mathbf{R}^n$ to $\mathbf{R}$ defined by

$$\tilde{E}(x) := -y^T T y + 2\sum_{i=1}^{n} \int_0^{y_i} c_i s_i^{-1}(\sigma)\,d\sigma - 2y^T b , \tag{4.3}$$

where $y = S(x)$. Comparing E with $\tilde{E}$, we note that $\tilde{E}$ is a function defined on $\mathbf{R}^n$ while E is a functional defined on $C([-\tau,0],\mathbf{R}^n)$. Since ϕ_{x_e} is a local minimum of E, x_e must be a local minimum of $\tilde{E}$. For otherwise, there would exist a sequence $\{x_n\} \subset \mathbf{R}^n$ such that $x_n \to x_e$ as $n \to \infty$ and $\tilde{E}(x_n) < \tilde{E}(x_e)$. Let ϕ_{x_n} denote the constant function $\phi_{x_n} \equiv x_n$ in $C([-\tau,0],\mathbf{R}^n)$. Then $|\phi_{x_n} - \phi_{x_e}| \to 0$ as $n \to \infty$, and

$$E(\phi_{x_n}) = \tilde{E}(x_n) < \tilde{E}(x_e) = E(\phi_{x_e}) .$$

This contradicts the fact that ϕ_{x_e} is a local minimum of E. Therefore, x_e is a local minimum of $\tilde{E}$. Hence, $\tilde{J}(x_e)$ is positive semidefinite (see, e.g., Theorem 3.6 of [24]), where $\tilde{J}(x)$ is the Hessian matrix of $\tilde{E}$ given by

$$\tilde{J}(x) = \left[\frac{\partial^2 \tilde{E}}{\partial x_i \partial x_j} \right]_{n\times n} . \tag{4.4}$$

It can be shown that

$$\tilde{J}(x) = 2D(x)J(x)D(x) ,$$

where $D(x) = \mathrm{diag}\{s_1'(x_1), \ldots, s_n'(x_n)\}$, and $J(x)$ is given by Eq. (3.1). Therefore, $J(x_e)$ is also positive semidefinite. By Assumption A, $J(x_e)$ is a nonsingular matrix. Thus we have shown that $J(x_e)$ is positive definite.

(d) (iv) $\Longrightarrow$ (i). We need to prove that x_e is a stable equilibrium of system (1.2), i.e., for any $\epsilon > 0$, there exists a $\delta > 0$, such that for any $\phi \in C([-\tau,0],\mathbf{R}^n)$, if $|\phi - x_e| < \delta$, then $|x_t - x_e| < \epsilon$ where x_t is the solution of (1.2) with initial condition ϕ.

Since $J(x_e)$ is positive definite, then $\tilde{J}(x_e)$ must also be positive definite where $\tilde{J}(x)$ is the Hessian matrix of $\tilde{E}$ given by (4.4). Furthermore,

$$\nabla_x \tilde{E}(x) = 2(-Ty + Cx - b)D(x) ,$$

where $D(x)$ is given in part (b). Therefore, $\nabla_x \tilde{E}(x_e) = 0$ since x_e is an equilibrium of (1.2). It follows (by Theorem 3.6 of [24]) that x_e is a local minimum of $\tilde{E}$, i.e., there exists a $\delta_1 > 0, \delta_1 < \epsilon$, such that whenever $0 < \|x - x_e\| \le \delta_1$, $\tilde{E}(x_e) < \tilde{E}(x)$. Let $r = \min\{\tilde{E}(x) : \|x - x_e\| = \delta_1\}$. Then it is true that $r > \tilde{E}(x_e)$. Since $E(\phi_{x_e}) = \tilde{E}(x_e)$, it follows that $r > E(\phi_{x_e})$. Note that E is a continuous functional. Therefore, there exists a $\delta > 0$, $\delta < \delta_1$ such that whenever $|\phi - x_e| < \delta$, where $\phi \in C([-\tau,0],\mathbf{R}^n)$, we have $E(\phi) < r$. Suppose x_t is any solution of (1.2) with the initial condition ϕ such that $|\phi - x_e| < \delta$. We will show that $|x_t - x_e| < \delta_1 < \epsilon$. Otherwise, there would exist a $t_0 > 0$ such that $\|x_{t_0}(0) - x_e\| = \delta_1$, i.e., $\|x(t_0) - x_e\| = \delta_1$. By the definition of E and $\tilde{E}$, we have $E(x_{t_0}) \ge \tilde{E}(x(t_0)) \ge r$. Therefore, we obtain $E(x_{t_0}) > E(\phi)$ which contradicts the fact that E is monotonically decreasing along any solution of (1.2). Thus, we have shown that x_e is an asymptotically stable equilibrium of system (1.2). □

Remark 6. We note that statement (iv) in Theorem 2 is independent of the delay τ. Therefore, if system (1.2) satisfies Assumption A, and if the condition $\tau\beta\|T_2\|<1$ is satisfied, then the locations of the (asymptotically) stable equilibria of system (1.2) will not depend on the delay τ. This is true if in particular $\tau=0$. Therefore, if $\tau\beta\|T_2\|<1$, then system (1.2) and system (1.1) [obtained by letting $\tau=0$ in (1.2)] will have identical (asymptotically) stable equilibria. We state this in the form of a corollary.

Corollary 1. Under the conditions of Theorem 1, x_e is an (asymptotically) stable equilibrium of system (1.2) if and only if x_e is an (asymptotically) stable equilibrium of system (1.1). This is true if and only if $J(x_e)$ is positive definite, where $J(x)$ is given in Eq. (3.1). □

Remark 7. Corollary 1 provides an effective criterion for testing the (asymptotic) stability of any equilibrium of Hopfield neural networks with delays described by (1.2). This criterion constitutes necessary and sufficient conditions, as long as $\tau\beta\|T_2\|<1$. □

V. AN EXAMPLE

To illustrate the applicability of some of the preceding results, we consider the system

$$\dot{x}(t)=-Cx(t)+TS(x(t-\tau)) , \tag{5.1}$$

where $x\in\mathbb{R}^2$, $C=\text{diag}\{1.1,1.2\}$, and $S(x)=[s_1(x_1),s_2(x_2)]^T$ such that

$$s_1(x_1)=\frac{2}{\pi}\tan^{-1}\left[\frac{1.4\pi}{2}x_1\right]$$

and

$$s_2(x_2)=\frac{2}{\pi}\tan^{-1}\left[\frac{1.5\pi}{2}x_2\right] ,$$

where T is a symmetric matrix given by

$$T=\begin{bmatrix}-0.2 & 1\\ 1 & -0.1\end{bmatrix} .$$

System (5.1) has three equilibria given by $x_{e_0}=(0,0)^T$, $x_{e_1}=(0.2140,0.2091)^T$, and $x_{e_2}=(-0.2140,-0.2091)^T$. The classical method of analyzing the stability of (5.1) is to linearize (5.1) about each of its equilibria. For example, the linearization of (5.1) at the equilibrium x_{e_1} is given by

$$\dot{y}(t)=-Cy(t)+TS_0y(t-\tau) , \tag{5.2}$$

where $y=x-x_{e_1}$, C and T are the same as in (5.1), and

$$S_0=\begin{bmatrix}1.1462 & 0\\ 0 & 1.2071\end{bmatrix} .$$

As mentioned earlier, there are no effective existing results of testing the stability of system (5.2). The results in [14] cannot be applied in the present case, since system (5.2) cannot be decomposed into two one-dimensional subsystems which is essential to the derivation of the results in [14].

Although there exist in the literature some sufficient conditions for the asymptotic stability of linear time-delay systems given by

$$\dot{x}(t)=Ax(t)+Bx(t-\tau) , \tag{5.3}$$

where A,B are constant $n\times n$ matrices, these results are in general very restrictive when applied to system (5.2). For example, the results of [23] yield

$$2\tau(\|A\|+\|B\|)\left[\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}\right]^{1/2}\|PB\|<1 , \tag{5.4}$$

as a condition for the asymptotic stability of the equilibrium $x=0$ of system (5.3), where P is the positive-definite matrix such that $(A+B)^TP+P(A+B)=-I$, and I is the identity matrix. When the bound given in (5.4) is applied to system (5.2), we obtain that the equilibrium $y=0$ (or $x=x_{e_1}$) is asymptotically stable if the delay $\tau<0.014$. Thus, by the classical method of linearization, we know that the equilibrium x_{e_1} of system (5.1) is asymptotically stable if $\tau<0.014$.

If we apply Corollary 1 to system (5.1), it is easily shown that $J(x_{e_1})$ and $J(x_{e_2})$ are positive definite, and $J(x_{e_0})$ is not positive definite, where $J(x)$ is defined by Eq. (3.1). Thus, we know by Corollary 1 that when $\tau<0.579$, x_{e_1} and x_{e_2} are (asymptotically) stable equilibria of system (5.1) while x_{e_0} is not (asymptotically) stable. Therefore, for the present example, Corollary 1 provides stability conditions which are significantly less restrictive than existing results. Additionally, Theorem 1 shows that system (5.1) is globally stable when $\tau<0.579$.

VI. CONCLUDING REMARKS

In this paper, we considered the local stability as well as the global stability of Hopfield neural networks with delays given by system (1.2). We showed that if the condition $\tau\beta\|T_2\|<1$ is satisfied, then Hopfield neural networks with delays and corresponding Hopfield neural networks without delays given by system (1.1) have identical asymptotically stable equilibria, and both networks are globally stable [i.e., any trajectory of system (1.2) approaches some equilibrium of (1.2)]. In addition, we proved that if the same bound is satisfied, then any equilibrium x_e of system (1.2) is asymptotically stable if and only if $J(x_e)$ is positive definite. A specific example was given to demonstrate the applicability of some of our results.

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