Diffusion approximation for a dissipative random medium and the applications

Koichi Furutsu
Nakato 4-15-3, Musashi-Murayama, Tokyo 208, Japan

Yukio Yamada
Mechanical Engineering Laboratory, Agency of Industrial Science and Technology, Ministry of International Trade and Industry, 1-2 Namiki, Tsukuba, Ibaraki 305, Japan

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In almost all of the literature, the diffusion coefficient derived from a conventional transport equation changes with the absorption coefficient when there is a dissipation in the medium. The situation is also the same for the time-independent diffusion equation. In recent numerical simulations made for a biological-mechanical purpose, it happened that the absorption coefficient was increased up to one tenth of the total scattering cross section per unit volume; thereby it was strongly suggested that the diffusion coefficient should preferably be independent of the absorption coefficient. The purpose of this paper is to show theoretically that this is definitely the case. Moreover, several basic equations for applications to optical tomography and photon migration are given.

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I. INTRODUCTION

The space-time transport equation can be reduced to a diffusion equation in the region where the space-time change of the angular distribution function is sufficiently small within the range of the wave coherence distance. Here, when there is a dissipation in the medium, the conventional diffusion coefficient, as written in almost all of the literature, changes with the absorption cross section per unit volume [1]. The situation is the same for the time-independent diffusion equation [2]. On the other hand, assuming the absorption cross section γ_{ab} is small enough compared to the total scattering cross section per unit volume γ, the diffusion coefficient was shown to be independent of γ_{ab} [3]. This difference is not very important as long as $\gamma_{ab} \ll \gamma$, but there recently occurred a case in which $\gamma_{ab} \sim \gamma \times 10^{-1}$, showing that the difference can cause a remarkable change, and that numerical simulations by the Monte Carlo method strongly suggest γ_{ab}-independent diffusion coefficient to be preferable to the conventionally accepted one [4]. Illustrated in Fig. 1 are examples of the numerical comparison made using the two different diffusion coefficients for some typical data in case of a photon migration in turbid media; therein the broken lines show the results when using the conventional coefficient, while the solid lines show those when using the corrected one. The ratio of the two sets of values is shown in Fig. 2 (see the captions for details).

The purpose of this paper is first to show theoretically by a rigorous procedure that this is definitely the case, and then to introduce several basic equations therefrom when making applications, particularly to areas of optical tomography and photon migration which have been attempted recently for medical purposes [5].

II. CASE OF A HOMOGENEOUSLY RANDOM MEDIUM

We employ the following notations: The space coordinate vector is denoted by $\boldsymbol{\rho}=(\rho_1,\rho_2,\rho_3)$, the time by t, and

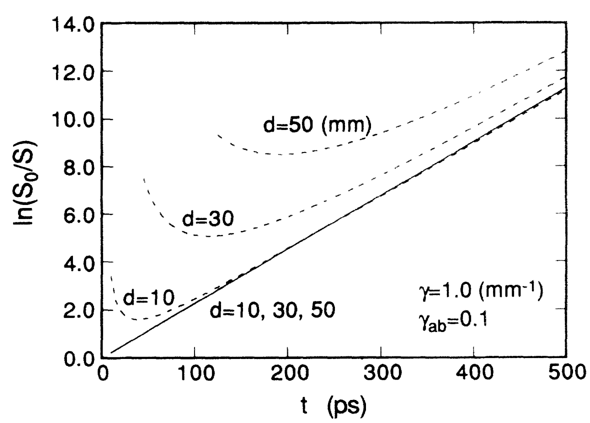

FIG. 1. $\ln(S_0/S)$ in a homogeneously random medium is shown as a function of the time (ps) for a light wave. Here S_0 is the solution of the diffusion equation when $\gamma_{ab}=0$, $\gamma=1\ \mathrm{mm}^{-1}$, $D=1/3\gamma$, $a_1=0$, and when subjected to the initial condition (13), while S is the corresponding solution when $\gamma_{ab}=0.1\ \mathrm{mm}^{-1}$ with the same value of γ. Broken curves are used when S is the solution S_1 using the conventional diffusion coefficient $D=1/3(\gamma+\gamma_{ab})$, while the solid curves (lines) are used when S is the solution S_2 using the corrected diffusion coefficient independent of γ_{ab}. Three sets of curves are shown for the propagation distances $d=10$, 30, and 50 mm, respectively. Note that all solid lines coincide with each other independently of the propagation distances.

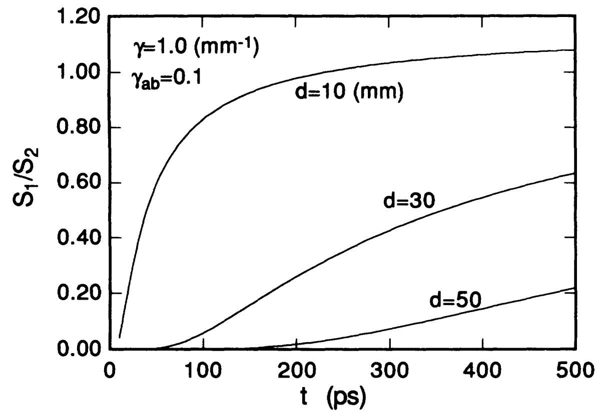

FIG. 2. The ratio S_1/S_2 is shown for the same values of the parameters and propagation distances as in Fig. 1.

$\bar{\rho}=(\rho,ct)$ represents the space-time coordinates together. The spatial differential operator will be denoted by $\nabla=(\partial/\partial\rho_1,\partial/\partial\rho_2,\partial/\partial\rho_3)$ and the time differential operator by $\partial_t=c^{-1}\partial/\partial t$. The space unit vector $\mathbf{\Omega}=(\Omega_1,\Omega_2,\Omega_2)$ with $\mathbf{\Omega}^2=1$ is also used, and $I(\mathbf{\Omega},\bar{\rho})$ designates the angular distribution function of the wave, expressing the intensity of wave propagating in the direction $\mathbf{\Omega}$ at $\bar{\rho}$.

The distribution function $I(\mathbf{\Omega},\bar{\rho})$ is obtained as a solution of the space-time transport equation of the form

$$[\partial_t+\mathbf{\Omega}\cdot\nabla+\gamma_t]I(\mathbf{\Omega},\bar{\rho}) = \int d\mathbf{\Omega}'\sigma(\mathbf{\Omega}|\mathbf{\Omega}')I(\mathbf{\Omega}',\bar{\rho})+J_c(\mathbf{\Omega},\bar{\rho}) \; . \quad (1)$$

Here $\sigma(\mathbf{\Omega}|\mathbf{\Omega}')$ designates the scattering cross section per unit angle and per unit volume, giving rise to scattering of the wave propagating in the direction $\mathbf{\Omega}'$ into $\mathbf{\Omega}$, and will be assumed to have the form $\sigma(\mathbf{\Omega}\cdot\mathbf{\Omega}')$ with $\mathbf{\Omega}\cdot\mathbf{\Omega}'=\sum_{j=1}^{3}\Omega_j\Omega_j'$ in the following. The extinction coefficient γ_t is conveniently divided into two parts according to

$$\gamma_t=\gamma+\gamma_{ab}, \quad \gamma=\int d\mathbf{\Omega}\,\sigma(\mathbf{\Omega}\cdot\mathbf{\Omega}') \; , \quad (2)$$

where γ is the total scattering cross section and γ_{ab} is the absorption cross section. The term $J_c(\mathbf{\Omega},\bar{\rho})$ provides the source of the wave.

Here we introduce a solution of Eq. (1), $S(\mathbf{\Omega},\rho|t-t'|\mathbf{\Omega}',\rho')$, when $J_c(\mathbf{\Omega},\bar{\rho})=0$ and subject to the initial condition

$$S(\mathbf{\Omega},\rho|t-t'=0|\mathbf{\Omega}',\rho')=\delta(\mathbf{\Omega}-\mathbf{\Omega}')\delta(\rho-\rho') \; . \quad (3)$$

Hence

$$[\partial_t+\mathbf{\Omega}\cdot\nabla+\gamma_t]S(\mathbf{\Omega},\rho|t-t'|\mathbf{\Omega}',\rho') = \int d\mathbf{\Omega}''\sigma(\mathbf{\Omega}|\mathbf{\Omega}'')S(\mathbf{\Omega}'',\rho|t-t'|\mathbf{\Omega}',\rho') \; , \quad (4)$$

which enables the solution of Eq. (1) to be expressed by

$$I(\mathbf{\Omega},\bar{\rho})=c\int_{-\infty}^{t}dt'\int d\rho' d\mathbf{\Omega}' S(\mathbf{\Omega},\rho|t-t'|\mathbf{\Omega}',\rho') \times J_c(\mathbf{\Omega}',\bar{\rho}') \; , \quad (5)$$

where $\bar{\rho}'=(\rho',ct')$. The proof becomes straightforward by substituting expression (5) into Eq. (1) and subsequently using condition (3).

In fact, Eq. (5) can be written in the form

$$I(\mathbf{\Omega},\bar{\rho})=\int d\bar{\rho}' d\mathbf{\Omega}'\tilde{S}(\mathbf{\Omega},\bar{\rho}|\mathbf{\Omega}',\bar{\rho}')J_c(\mathbf{\Omega}',\bar{\rho}')$$

(where $d\bar{\rho}'=cd\rho' dt'$) in terms of a Green's function defined by

$$\tilde{S}(\mathbf{\Omega},\bar{\rho}|\mathbf{\Omega}',\bar{\rho}')=\begin{cases} S(\mathbf{\Omega},\rho|t-t'|\mathbf{\Omega}',\rho'), & t>t' \\ 0, & t<t' \; , \end{cases} \quad (6)$$

which is the solution of Eq. (4) with the source term

$$c^{-1}\delta(\mathbf{\Omega}-\mathbf{\Omega}')\delta(\rho-\rho')\delta(t-t') \; , \quad (7)$$

in consequence of condition (3).

The solution S of Eq. (4) can be expressed in terms of the solution of when $\gamma_{ab}=0$, say S_0, by

$$S(\mathbf{\Omega},\rho|t-t'|\mathbf{\Omega}',\rho') = \exp[-c(t-t')\gamma_{ab}]S_0(\mathbf{\Omega},\rho|t-t'|\mathbf{\Omega}',\rho') \; . \quad (8)$$

Here, in view of (2), S_0 is the solution of

$$[\partial_t+\mathbf{\Omega}\cdot\nabla+\gamma]S_0(\mathbf{\Omega},\rho|t-t'|\mathbf{\Omega}',\rho') = \int d\mathbf{\Omega}''\sigma(\mathbf{\Omega}|\mathbf{\Omega}'')S_0(\mathbf{\Omega}'',\rho|t-t'|\mathbf{\Omega}',\rho') \quad (9)$$

subjected to the same initial condition as (3).

In the diffusion region where the space-time change of S_0 (not S) is negligibly small for a change of $\bar{\rho}=(\rho,ct)$ within the range of the order γ^{-1}, the total intensity of S_0, defined by

$$S_0(\rho|t-t'|\rho')=\int d\mathbf{\Omega}\, S_0(\mathbf{\Omega},\rho|t-t'|\mathbf{\Omega}',\rho') \; , \quad (10)$$

is the solution of the conventional diffusion equation (Appendix A)

$$[\partial_t-3^{-1}(1-a_1)^{-1}\gamma^{-1}\nabla^2]S_0(\rho|t-t'|\rho')=0 \quad (11)$$

in a nondissipative medium, with the parameter a_1 defined by

$$a_1=\gamma^{-1}\int d\mathbf{\Omega}\,\mathbf{\Omega}\cdot\mathbf{\Omega}'\sigma(\mathbf{\Omega}\cdot\mathbf{\Omega}'), \quad (12)$$

and subjected to the initial condition

$$S_0(\rho|t-t'=0|\rho')=\delta(\rho-\rho'), \quad (13)$$

which results from condition (3) with (10). The diffusion equation for $S(\rho|t-t'|\rho')$ is consequently given, from (8), by [3]

$$[\partial_t+\gamma_{ab}-3^{-1}(1-a_1)^{-1}\gamma^{-1}\nabla^2]S(\rho|t-t'|\rho')=0 \; , \quad (14)$$

showing that the diffusion coefficient is the same as when no absorption is assumed ($\gamma_{ab}=0$) and, therefore, that it differs from the conventional one which has been given by $3^{-1}[(1-a_1)\gamma+\gamma_{ab}]^{-1}$ in the present notations [1].

On the other hand, time-independent solutions, as obtained at the limit $t\rightarrow\infty$ for time-dependent sources of the form

$$J_c(\mathbf{\Omega},\bar{\rho})=\begin{cases} J_c(\mathbf{\Omega},\rho), & t>0 \\ 0, & t<0, \end{cases} \quad (15)$$

can be obtained as solutions of a time-independent equation, directly, as follows: From expression (5), we observe that substitution of (8) and (15) yields

$$I(\mathbf{\Omega},\boldsymbol{\rho},t)=\int d\boldsymbol{\rho}'d\mathbf{\Omega}'c\int_0^t dt'\exp[-c(t-t')\gamma_{ab}]\times S_0(\mathbf{\Omega},\boldsymbol{\rho}|t-t'|\mathbf{\Omega}',\boldsymbol{\rho}')J_c(\mathbf{\Omega}',\boldsymbol{\rho}')\ , \quad (16)$$

and hence that, as $t\to+\infty$,

$$I(\mathbf{\Omega},\boldsymbol{\rho})\equiv I(\mathbf{\Omega},\boldsymbol{\rho},t=\infty)=\int d\boldsymbol{\rho}'d\mathbf{\Omega}'\tilde{S}(\mathbf{\Omega},\boldsymbol{\rho}|\mathbf{\Omega}',\boldsymbol{\rho}')J_c(\mathbf{\Omega}',\boldsymbol{\rho}'), \quad (17)$$

where

$$\tilde{S}(\mathbf{\Omega},\boldsymbol{\rho}|\mathbf{\Omega}',\boldsymbol{\rho}')=c\int_0^\infty dt\,\exp[-ct\gamma_{ab}]S_0(\mathbf{\Omega},\boldsymbol{\rho}|t|\mathbf{\Omega}',\boldsymbol{\rho}')\ . \quad (18)$$

Here, using the governing equation (9) for S_0 with the same initial condition as (3) for S, the definition (18) leads to an equation of $\tilde{S}(\mathbf{\Omega},\boldsymbol{\rho}|\mathbf{\Omega}',\boldsymbol{\rho}')$ as

$$[\mathbf{\Omega}\cdot\nabla+\gamma_{ab}+\gamma]\tilde{S}(\mathbf{\Omega},\boldsymbol{\rho}|\mathbf{\Omega}',\boldsymbol{\rho}')=\int d\mathbf{\Omega}''\sigma(\mathbf{\Omega}|\mathbf{\Omega}'')\tilde{S}(\mathbf{\Omega}'',\boldsymbol{\rho}|\mathbf{\Omega}',\boldsymbol{\rho}')+\delta(\mathbf{\Omega}-\mathbf{\Omega}')\delta(\boldsymbol{\rho}-\boldsymbol{\rho}'), \quad (19)$$

with the aid of partial integration. Equation (19) is just the equation of the Green function for time-independent solutions of Eq. (1) ($\partial_t I=0$), as it should be from expression (17).

The total intensity of the wave, $I(\boldsymbol{\rho})$, is obtained by the Ω integration of (17) as

$$I(\boldsymbol{\rho})=\int d\mathbf{\Omega}\,I(\mathbf{\Omega},\boldsymbol{\rho})=\int d\boldsymbol{\rho}'\tilde{S}(\boldsymbol{\rho}|\boldsymbol{\rho}')J_c(\boldsymbol{\rho}')\ . \quad (20)$$

Here

$$\tilde{S}(\boldsymbol{\rho}|\boldsymbol{\rho}')=\int d\mathbf{\Omega}\,\tilde{S}(\mathbf{\Omega},\boldsymbol{\rho}|\mathbf{\Omega}',\boldsymbol{\rho}')\ , \quad (21)$$

$$J_c(\boldsymbol{\rho})=\int d\mathbf{\Omega}\,J_c(\mathbf{\Omega},\boldsymbol{\rho})\ , \quad (22)$$

and, in the diffusion region, use of Eqs. (11) and (18) leads to a diffusion equation of $\tilde{S}(\boldsymbol{\rho}|\boldsymbol{\rho}')$ as

$$[\gamma_{ab}-3^{-1}(1-a_1)^{-1}\gamma^{-1}\nabla^2]\tilde{S}(\boldsymbol{\rho}|\boldsymbol{\rho}')=\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')\ , \quad (23)$$

with the aid of partial integration and the initial condition (13). Note that the diffusion coefficient in (23) is again independent of the absorption coefficient γ_{ab}. It should be noted, however, that, as γ_{ab} increases, integral (18) tends to be determined mostly by the t integration over the range $t\sim c^{-1}\gamma_{ab}^{-1}$, while the solution S_0 of the diffusion equation (11) can be utilized with a sufficient accuracy only for the range $t\gg c^{-1}\gamma^{-1}$ in the integrand. Hence this leads to the condition $\gamma_{ab}\ll\gamma$ for the time-independent diffusion equation (23) to be available, in contrast to the time-dependent diffusion equation (14) which is valid even when $\gamma_{ab}\sim\gamma$, although for such a large value of γ_{ab}, use of the original transport equation (1) cannot be justified.

III. CASE OF AN INHOMOGENEOUSLY RANDOM MEDIUM

So far the medium has been assumed to be homogeneous, so that the cross section $\sigma(\mathbf{\Omega}|\mathbf{\Omega}')$ is independent of the space coordinates $\boldsymbol{\rho}$. Hereafter, we consider the case in which the cross section is inhomogeneous in space with the form $\sigma(\mathbf{\Omega}|\boldsymbol{\rho}|\mathbf{\Omega}')=\sigma(\mathbf{\Omega}\cdot\mathbf{\Omega}',\boldsymbol{\rho})$. Thus, in Eq. (2), γ_t, γ, and γ_{ab} are now functions of $\boldsymbol{\rho}$, and

$$\gamma(\boldsymbol{\rho})=\int d\mathbf{\Omega}\,\sigma(\mathbf{\Omega}\cdot\mathbf{\Omega}',\boldsymbol{\rho})\ . \quad (24)$$

To find an expression of $S(\mathbf{\Omega},\boldsymbol{\rho}|t-t'|\mathbf{\Omega}',\boldsymbol{\rho}')$ similar to (8) with the factor S_0 which is the solution of the transport equation (9) when $\gamma_{ab}=0$, we write it in the form (Appendix B is devoted to deriving basic equations in short form which can be easier to follow)

$$S(\mathbf{\Omega},\boldsymbol{\rho}|t-t'|\mathbf{\Omega}',\boldsymbol{\rho}')=\int d\mathbf{\Omega}''d\boldsymbol{\rho}''S_0(\mathbf{\Omega},\boldsymbol{\rho}|t-t'|\mathbf{\Omega}'',\boldsymbol{\rho}'')\times\Gamma(\mathbf{\Omega}'',\boldsymbol{\rho}''|t-t'|\mathbf{\Omega}',\boldsymbol{\rho}')\ . \quad (25)$$

Here $\Gamma(\Omega'',\boldsymbol{\rho}''|t-t'|\mathbf{\Omega}',\boldsymbol{\rho}')$ is an unknown factor to be determined by substituting expression (25) into the original transport equation (4) for S. Hence, on using Eq. (9), we find an equation for Γ as

$$\int d\mathbf{\Omega}''d\boldsymbol{\rho}''[S_0(\mathbf{\Omega},\boldsymbol{\rho}|t-t'|\mathbf{\Omega}'',\boldsymbol{\rho}'')\partial_t+\gamma_{ab}(\boldsymbol{\rho})S_0(\mathbf{\Omega},\boldsymbol{\rho}|t-t'|\mathbf{\Omega}'',\boldsymbol{\rho}'')]\Gamma(\mathbf{\Omega}'',\boldsymbol{\rho}''|t-t'|\mathbf{\Omega}',\boldsymbol{\rho}')=0\ , \quad (26)$$

with the initial condition

$$\Gamma(\mathbf{\Omega},\boldsymbol{\rho}|t-t'=0|\mathbf{\Omega}',\boldsymbol{\rho}')=\delta(\mathbf{\Omega}-\mathbf{\Omega}')\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')\ . \quad (27)$$

In the diffusion approximation, $S_0(\mathbf{\Omega},\boldsymbol{\rho}|t-t'|\mathbf{\Omega}',\boldsymbol{\rho}')$ is reduced to $S_0(\boldsymbol{\rho}|t-t'|\boldsymbol{\rho}')$, defined by Eq. (10), which is the solution of the diffusion equation (11), or in the present case in which γ and σ are spatial functions of $\boldsymbol{\rho}$,

$$[\partial_t-\nabla_j D(\boldsymbol{\rho})\nabla_j]S_0(\boldsymbol{\rho}|t-t'|\boldsymbol{\rho}')=0\ , \quad (28)$$

subjected to the initial condition (13) (Appendix A). Here

$$D(\boldsymbol{\rho})=3^{-1}(1-a_1)^{-1}\gamma^{-1}(\boldsymbol{\rho}) \quad (29)$$

is the diffusion coefficient and is independent of the absorption coefficient γ_{ab}. The diffusion versions of Eqs. (25) and (26) are also obtained by the $\mathbf{\Omega}$ integration; hence, in terms of the notation

$$\Gamma(\boldsymbol{\rho}|t|\boldsymbol{\rho}')=\int d\mathbf{\Omega}\,\Gamma(\mathbf{\Omega},\boldsymbol{\rho}|t|\mathbf{\Omega}',\boldsymbol{\rho}')\ , \quad (30)$$

$$\Gamma(\boldsymbol{\rho}|t=0|\boldsymbol{\rho}')=\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')\ , \quad (31)$$

Eqs. (25) and (26) lead to

$$S(\boldsymbol{\rho}|t-t'|\boldsymbol{\rho}')=\int d\boldsymbol{\rho}''S_0(\boldsymbol{\rho}|t-t'|\boldsymbol{\rho}'')\Gamma(\boldsymbol{\rho}''|t-t'|\boldsymbol{\rho}')\ , \quad (32)$$

$$\int d\rho''[S_0(\rho|t-t'|\rho'')\partial_t+\gamma_{ab}(\rho)S_0(\rho|t-t'|\rho'')] \times\Gamma(\rho''|t-t'|\rho')=0 , \quad (33)$$

and the function $\Gamma(\rho|t|\rho')$ can in principle be found by solving Eq. (33) with the known solutions $S_0(\rho|t|\rho')$ of Eq. (28). Hereafter, we will often understand Eqs. (32) and (33) to be given in terms of the matrix elements of a product of two ρ coordinate matrices $S_0(t-t')$ and $\Gamma(t-t')$, say, defined by the elements $S_0(\rho|t-t'|\rho')$ and $\Gamma(\rho|t-t'|\rho')$, respectively.

The elements $S(\rho|t-t'|\rho')$ can be obtained directly, however, as the solution of another diffusion equation, i.e.,

$$[\partial_t+\gamma_{ab}(\rho)-\nabla_j D(\rho)\nabla_j]S(\rho|t-t'|\rho')=0 , \quad (34)$$

which differs from the conventional diffusion equation in that the diffusion coefficient $D(\rho)$ is perfectly independent of γ_{ab}. The proof is straightforward by using expression (32) with the aid of Eqs. (28) and (33); and the condition (13) is ensured for S by virtue of the initial condition (31) imposed on Γ.

In the special case in which γ_{ab} is a constant over all space, Eq. (33) is reduced to

$$[\partial_t+\gamma_{ab}]\Gamma(\rho|t-t'|\rho')=0 , \quad (35)$$

hence the solution, subject to Eq. (31), is

$$\Gamma(\rho|t-t'|\rho')=\exp[-c(t-t')\gamma_{ab}]\delta(\rho-\rho') . \quad (36)$$

Thus the expression of $S(\rho|t-t'|\rho')$ obtained from (8) is reproduced by Eq. (32).

IV. SCATTERING BY AN ISOLATED ABSORBER

In the optical tomography and photon migration which have recently been attempted for medical purposes, scattering by an isolated absorber as embedded in a random medium is a basic problem to be investigated, and this section is devoted to preparing basic equations for the scattering of this sort to the diffusion approximation.

To facilitate the following equation formulation, we write the diffusion equation (34), on setting $t'=0$, in the ρ coordinate-matrix form

$$\partial_t S(t)=HS(t), \quad H=H_0-H_1 . \quad (37)$$

Here $S(t)$ is a ρ matrix with the matrix elements $S(\rho|t|\rho')$,

$$H_0=\nabla_j D\nabla_j, \quad H_1=\gamma_{ab} , \quad (38)$$

which are also regarded as ρ matrices, with the operators ∇_j having the matrix elements $\nabla_j\delta(\rho-\rho')=-\nabla'_j\delta(\rho-\rho')$; here γ_{ab} is assumed to be nonzero only over a small region in the space constituting an isolated absorber, while, in the case when γ_{ab} has a constant part, that part is to be included in $S_0(t)$ by using Eq. (8). The diffusion equation (28) for S_0 is likewise written by

$$\partial_t S_0(t)=H_0 S_0(t) , \quad (39)$$

The matrices $S(t)$ and $S_0(t)$ are connected by the relation (32), which is now expressed by

$$S(t)=S_0(t)\Gamma(t) . \quad (40)$$

Another relation is

$$S_0(t-t')S_0(t')=S_0(t) , \quad (41)$$

which can be shown directly by using the formal solution of Eq. (39) subjected to the initial condition (13), i.e.,

$$S_0(t)=\exp(ctH_0), \quad S_0(t=0)=1 . \quad (42)$$

Thus, from Eq. (41),

$$S_0(t-t')=S_0(t)S_0^{-1}(t') . \quad (43)$$

The matrix $\Gamma(t)$, as defined by the elements $\Gamma(\rho|t|\rho')$, is governed by Eq. (33), which can be rewritten in the matrix form

$$[\partial_t+\mathcal{H}_1(t)]\Gamma(t)=0, \quad \Gamma(t=0)=1 , \quad (44)$$

$$\mathcal{H}_1(t)=S_0^{-1}(t)\gamma_{ab}S_0(t) , \quad (45)$$

where γ_{ab} is a diagonal ρ matrix with the elements $\gamma_{ab}(\rho)\delta(\rho-\rho')$. Here, when γ_{ab} is a constant independent of ρ, the solution of Eq. (44) is given by

$$\Gamma(t)=\exp(-ct\gamma_{ab}), \quad \mathcal{H}_1=\gamma_{ab} , \quad (46)$$

being a reproduction of Eq. (36).

Integration of (44) with respect to t yields

$$\Gamma(t)=1-c\int_0^t dt'\mathcal{H}_1(t')\Gamma(t') , \quad (47)$$

and the substitution into Eq. (40) leads to an integral equation for $S(t)$, with $\gamma'_{ab}=c\gamma_{ab}$, as

$$S(t)=S_0(t)-\int_0^t dt' S_0(t-t')\gamma'_{ab}S(t') , \quad (48)$$

with the aid of relation (43). The iterative solution of Eq. (48) is written in a power series of γ'_{ab} as

$$S(t)=S_0(t)-\int_0^t dt_1 S_0(t-t_1)\gamma'_{ab}S_0(t_1) + \int_0^t dt_1\int_0^{t_1} dt_2 S_0(t-t^1)\gamma'_{ab}S_0(t_1-t_2) \times\gamma'_{ab}S_0(t_2)- \cdots . \quad (49)$$

The solution can be obtained in a compact form by making the Fourier transformation with respect to the time. With the transform $\tilde{S}(\omega)$ defined by

$$\tilde{S}(\omega)=\int_0^\infty dt\, e^{-i\omega t}S(t) , \quad (50)$$

which is an analytic function of ω in the lower-half plane, and which Fourier inversion

$$S(t)=(2\pi)^{-1}\int_{-\infty-i0}^{\infty-i0} d\omega\, e^{i\omega t}\tilde{S}(\omega) , \quad (51)$$

ensures $S(t<0)=0$, Eq. (48) is transformed to

$$\tilde{S}(\omega)=\tilde{S}_0(\omega)-\tilde{S}_0(\omega)\gamma'_{ab}\tilde{S}(\omega) . \quad (52)$$

The solution can be expressed by

$$\tilde{S}(\omega)=\tilde{S}_0(\omega)-\tilde{S}_0(\omega)\tilde{T}_{ab}(\omega)\tilde{S}_0(\omega) , \quad (53)$$

in terms of a scattering matrix $\tilde{T}_{ab}(\omega)$ of γ'_{ab}, defined by

$$\tilde{T}_{ab}(\omega)\tilde{S}_0(\omega)=\gamma'_{ab}\tilde{S}(\omega) , \tag{54}$$

or, upon substitution of Eq. (53) into the right-hand side, by

$$\tilde{T}_{ab}(\omega)=\gamma'_{ab}[1-\tilde{S}_0(\omega)\tilde{T}_{ab}(\omega)] . \tag{55}$$

Hence, in the ρ matrix form, we obtain

$$\tilde{T}_{ab}(\omega)=[1+\gamma'_{ab}\tilde{S}_0(\omega)]^{-1}\gamma'_{ab} , \tag{56}$$

which includes the multiple scattering effect of all orders between the absorber and the surrounding random medium. The series solution (49) is reproduced by the Fourier inversion of (53) with the expansion

$$\tilde{T}_{ab}(\omega)=\gamma'_{ab}-\gamma'_{ab}\tilde{S}_0(\omega)\gamma'_{ab}+\cdots ; \tag{57}$$

here, when D is a constant, the matrix elements of $\tilde{S}_0(\omega)$, $\tilde{S}_0(\rho|\omega|\rho')$, are given by

$$\tilde{S}_0(\rho|\omega|\rho')=(4\pi cD|\rho-\rho'|)^{-1}\times\exp[-(i\omega/cD)^{1/2}|\rho-\rho'|] , \tag{58}$$

where $\pi/2\geq \arg(i\omega)\geq -\pi/2$.

V. ATTENUATION IN A SLOWLY CHANGING MEDIUM OF γ_{ab}

When γ_{ab} is constant over all the space, the total attenuation is given by $\Gamma(t)$ of (46) to yield the same expression of $S(t)$ as that from Eq. (8). When γ_{ab} is nonzero only over a small region, on the other hand, it works as an absorber and behaves like a scatterer. In this case, $S(t)$ is given, as the solution of integral equation (48), by (49) in a power series of γ_{ab}, or by (53) in terms of the Fourier transform $\tilde{S}(\omega)$ with respect to the time, upon introduction of the scattering matrix $\tilde{T}_{ab}(\omega)$ of γ_{ab}, defined by (56) in the same fashion as in case of a scatterer.

Considered in this section is another extreme case in which γ_{ab} changes sufficiently slowly in the space, while the diffusion coefficient D is assumed to be constant. We start with the basic equation (44) for $\Gamma(t)$. Here, from (45),

$$\mathcal{H}_1(t)=S_0^{-1}(t)\gamma_{ab}(\rho)S_0(t)=\gamma_{ab}[S_0^{-1}(t)\rho S_0(t)] , \tag{59}$$

by virtue of the relation

$$S_0^{-1}(t)\rho^n S_0(t)=[S_0^{-1}(t)\rho S_0(t)]^n,\quad n=1,2,3,\ldots , \tag{60}$$

as applied to a power series expansion of $\gamma_{ab}(\rho)$ with respect to ρ. Hence we can write

$$\mathcal{H}_1(t)=\gamma_{ab}[\hat{\rho}(t)], \tag{61}$$

with a ρ matrix $\hat{\rho}(t)$, defined by

$$\hat{\rho}(t)=S_0^{-1}(t)\rho S_0(t) , \tag{62}$$

which, upon using expression (42) with (38), gives

$$\hat{\rho}(t)=\exp(-ctD\nabla^2)\rho\exp(ctD\nabla^2)=\rho-ctD[\nabla^2,\rho]+\tfrac{1}{2}(ctD)^2[\nabla^2,[\nabla^2,\rho]]-\cdots , \tag{63}$$

in terms of the notation $[A,B]=AB-BA$. Here the third- and higher-order terms of the series become zero, hence,

$$\hat{\rho}(t)=\rho-2ctD\nabla , \tag{64}$$

$$[\hat{\rho}(t),\hat{\rho}(t')]\neq 0,\quad t\neq t' . \tag{65}$$

Thus, with the expression (61), the solution of Eq. (44) can be expanded in a series in the same fashion as Eq. (49) for $S(t)$, or written more generally in the form

$$\Gamma(t)=P\exp\{-c\int_0^t dt'\gamma_{ab}[\hat{\rho}(t')]\} . \tag{66}$$

Here the symbol P designates the time-ordered product defined for any matrices $A(t)$ and $B(t)$ by

$$P[A(t)B(t')]=\begin{cases}A(t)B(t'), & t>t'\\ B(t')A(t), & t<t' .\end{cases} \tag{67}$$

To find the resulting $S(t)$ according to Eq. (40) or, more explicitly,

$$S(\rho|t|\rho')=\int d\rho'' S_0(\rho|t|\rho'')\Gamma(\rho''|t|\rho') , \tag{68}$$

we observe that

$$S_0(\rho|t|\rho'')=(4\pi ctD)^{-3/2}\exp[-(4ctD)^{-1}(\rho-\rho'')^2] , \tag{69}$$

being the solution subject to Eq. (13) in the homogeneously random medium, and also that its spatial change is made mostly by the exponential factor. Therefore, as will be seen below, we can approximate $\Gamma(t)$ of (66) with the operator $\hat{\rho}(t)$ of (64), by the matrix elements

$$\Gamma(\rho''|t|\rho')\simeq\exp\left\{-c\int_0^t dt'\gamma_{ab}(\rho''+ct'\mathbf{u})\right\}\delta(\rho''-\rho') \tag{70}$$

similar in form to (36). Here the vector $\mathbf{u}$ is defined by

$$\mathbf{u}=(\rho-\rho'')/ct , \tag{71}$$

which results from the exponential factor in (69) by operating ∇ [involved in $\hat{\rho}(t')$] from the right-hand side, hence by replacing $\nabla\to-\overleftarrow{\partial}/\partial\rho''\simeq(\partial/\partial\rho'')[(\rho-\rho'')^2/4ctD]$. This approximation is possible when $\gamma_{ab}(\rho)$ changes sufficiently slowly compared to the spatial change of $S_0(t)$.

Thus, substituting (70) into (68), the result is obtained in the form

$$S(\rho|t|\rho')=S_0(\rho|t|\rho')A(\rho|t|\rho') . \tag{72}$$

Here

$$A(\rho|t|\rho')=A(\rho'|t|\rho)=\exp\left\{-c\int_0^t dt'\gamma_{ab}[\rho'+(t'/t)(\rho-\rho')]\right\} \tag{73}$$

means the attenuation coefficient relative to S_0, and subject to the invariance against the interchange of ρ and ρ', as may be shown by changing the variable of integration t' to $t''=t-t'$. In the time integral in (73), γ_{ab} in the integrand changes along the line from the point ρ' to ρ, i.e., from the source to the point of observation along the direction of the averaged power flux.

Equation (72) indicates, since the factor A is perfectly free from the diffusion coefficient D, that the scattering and absorption are made independently of each other, as long as the attenuation coefficient γ_{ab} changes sufficiently slowly in the space. Even when D slowly changes in the space as γ_{ab} does, result (70) remains unchanged by redefining $\mathbf{u}$ more generally by

$$\mathbf{u}=-2D\nabla S_0/S_0(\rho|t|\rho'') \ . \tag{74}$$

Hence $\mathbf{u}$ is manifestly in the direction of the power flux [Eq. (A12)], and the resulting change in Eq. (73) is straightforward. Here, more exactly, expansion (64) for $\hat{\rho}(t)$ is replaced by

$$\hat{\rho}(t)=\rho-ct(D\nabla+\nabla D)+O((ctD)^2) \ , \tag{75}$$

where the higher-order terms are nonzero only when $\nabla D\neq 0$. Thus, to an approximation similar to the geometrical optics, we obtain

$$\mathbf{u}=2D\nabla\phi(\rho|t|\rho'') \ , \tag{76}$$

where ϕ is a solution of the Hamilton-Jacobi equation from the diffusion equation, i.e.,

$$-\partial_t\phi=D(\nabla\phi)^2; \tag{77}$$

$\mathbf{u}$ of (71) is reproduced from the solution in the special case.

VI. SUMMARY AND DISCUSSION

When the attenuation coefficient γ_{ab} is constant over all the space, solution of the transport equation subjected to the initial condition (3) is given by (8), in which γ_{ab} is involved only in the exponential factor, the other factor S_0 being the solution when the medium is purely nondissipative. To the diffusion approximation, this S_0 is replaced by the corresponding solution of the diffusion equation (11) subjected to the initial condition (13). The diffusion coefficient is given by (29) independently of whether the medium is dissipative or not, in contrast to what is commonly accepted in most of the literature. In case of the time-dependent diffusion equation (14), this is mathematically true no matter whether the dissipation is large, while, in the case of the time-independent diffusion equation (23), it is subject to the restriction $\gamma_{ab}\ll\gamma$. When the medium is inhomogeneously random, on the other hand, the corresponding solution is given by (32), in which S_0 is a solution of the diffusion equation (28) and the factor Γ, when defined as the matrix elements of a ρ matrix $\Gamma(t)$, is the solution of Eq. (44) with $\mathcal{H}_1(t)$ by (45). The resulting S is a solution of another diffusion equation (34), which differs from the conventional equation in that the diffusion coefficient is perfectly independent of γ_{ab}.

When $\gamma_{ab}\neq 0$ only over a small region in the space, it works as an isolated absorber and behaves like a scatterer, causing a basic problem to be investigated in the optical tomography and photon migration which have recently been attempted for medical purposes. The integral equation (48) provides a basic equation to be solved in this case, and the solution is given by (49) in a power series of γ_{ab}, or by (53) for the Fourier transform $\tilde{S}(\omega)$ with respect to the time, in terms of the scattering matrix $\tilde{T}_{ab}(\omega)$ of γ_{ab}, defined by (56) in ρ matrix form; it includes the multiple scattering effect of all orders between the absorber and the surrounding medium. Also considered in Sec. V is another extreme case in which γ_{ab} changes sufficiently slowly in the space, on first assuming a constant diffusion coefficient. The basic equation to be solved in this case is Eq. (44) with (45) for the attenuation factor $\Gamma(t)$, and the solution is given by (70) under the condition that $\gamma_{ab}(\rho)$ changes sufficiently slowly compared to the spatial change of the original S_0 in the nondissipative medium. The resulting $S(t)$ is given in form (72) with the attenuation factor $A(t)$ of (73). Here, since the factor $A(t)$ is entirely free from the diffusion coefficient D, this result indicates that the scattering and absorption are made independently of each other, as long as γ_{ab} changes sufficiently slowly in the space. Even when D slowly changes in the space as γ_{ab} does, the result (70) remains unchanged by redefining $\mathbf{u}$ more generally by Eq. (74), which expresses $\mathbf{u}$ in terms of the power flux; the resulting change in the final result (73) is straightforward. Thus $\mathbf{u}$ is given by (76) with ϕ, which is a solution of the Hamilton-Jacobi equation (77) from the diffusion equation, in the same fashion as in the geometrical optics.

APPENDIX A: DIFFUSION EQUATION IN A NONDISSIPATIVE MEDIUM

To derive a diffusion equation of $S=S_0$ from the transport equation (9), we first prepare low-order eigenfunctions of the original $\sigma(\boldsymbol{\Omega}|\boldsymbol{\Omega}')$, which are [3]:

$$\int d\boldsymbol{\Omega}\sigma(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')=\gamma \ , \tag{A1}$$

$$\int d\boldsymbol{\Omega}\Omega_j\sigma(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')=a_1\gamma\Omega_j', \quad j=1,2,3 \ , \tag{A2}$$

$$\int d\boldsymbol{\Omega}\Omega_j\Omega_k\sigma(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')=\gamma[b\Omega_j'\Omega_k'+\tfrac{1}{2}(1-b)(\delta_{jk}-\Omega_j'\Omega_k')]. \tag{A3}$$

Here Eqs. (A1) and (A2) indicate that the uniform distribution 1 and Ω_j are the eigenfunctions with the eigenvalues γ and $a_1\gamma$, respectively, where a_1 is the constant defined by (12). The same interpretation is also possible for Eq. (A3), i.e., a linear combination of $\Omega_j\Omega_k$ and δ_{jk} is also an eigenfunction of it. Relations (A1)–(A3) also hold true for the present $\sigma(\boldsymbol{\Omega}|\rho|\boldsymbol{\Omega}')$. We also introduce a set of notations

$$S(\rho|t|\rho')=\int d\boldsymbol{\Omega}\, S(\boldsymbol{\Omega},\rho|t|\boldsymbol{\Omega}',\rho') \ , \tag{A4}$$

$$S_j(\rho|t|\rho')=\int d\boldsymbol{\Omega}\,\Omega_j S(\boldsymbol{\Omega},\rho|t|\boldsymbol{\Omega}',\rho') \ , \tag{A5}$$

$$S_{ijk\ldots}(\rho|t|\rho')=\int d\boldsymbol{\Omega}\Omega_i\Omega_j\Omega_k\cdots S(\boldsymbol{\Omega},\rho|t|\boldsymbol{\Omega}',\rho') \ . \tag{A6}$$

Hence integration of Eq. (9) with respect to $\boldsymbol{\Omega}$ leads to

$$\partial_t S(\rho|t-t'|\rho')+\nabla_j S_j(\rho|t-t'|\rho')=0 \ , \tag{A7}$$

in consequence of relation (A1) with the summation convention for the same subscript. In the same way, multiplication of Eq. (9) by $\mathbf{\Omega}_k$ and followed by $\mathbf{\Omega}$ integration yields

$$[\partial_t+(1-a_1)\gamma(\boldsymbol{\rho})]S_k(\boldsymbol{\rho}|t-t'|\boldsymbol{\rho}')+\nabla_j S_{jk}(\boldsymbol{\rho}|t-t'|\boldsymbol{\rho}')=0\ , \tag{A8}$$

where use has been made of relation (A2). The next order equation is

$$[\partial_t+\tfrac{3}{2}(1-b)\gamma]S_{kl}+\nabla_j S_{jkl}=2^{-1}(1-b)\gamma\delta_{kl}S\ . \tag{A9}$$

These equations are not equations of closed form, and are formally the same as those given in Ref. [3] for a homogeneously random medium, except that all the γ_{ab} terms are presently zero.

In the diffusion region where the space-time change of S is negligibly small for a change of $\bar{\rho}=(\boldsymbol{\rho},ct)$ within the range of the order γ^{-1}, we observe in Eq. (A9) that $|\gamma^{-1}\partial_t|\ll 1$ and $|\gamma^{-1}\nabla_j|\ll 1$, and therefore that terms with ∂_t and ∇_j are negligible in leading to the approximate relation

$$S_{kl}\simeq 3^{-1}\delta_{kl}S\ . \tag{A10}$$

Hence the substitution into (A8) and succeeding rearrangement leads to

$$S_k=-(1-a_1)^{-1}\gamma^{-1}[3^{-1}\nabla_k S+\partial_t S_k] \tag{A11}$$

$$\simeq -3^{-1}(1-a_1)^{-1}\gamma^{-1}\nabla_k S\ , \tag{A12}$$

upon neglect of the term of $\gamma^{-1}\partial_t S_k$ which is negligible compared to S_k on the left-hand side; and, therefore, also that $|S_k|\ll|S|$ in view of $|\gamma^{-1}\nabla_k|\ll 1$. Thus Eq. (A7) is reduced, upon substitution of (A12) and in terms of the diffusion coefficient $D(\boldsymbol{\rho})$ defined by

$$D(\boldsymbol{\rho})=3^{-1}(1-a_1)^{-1}\gamma^{-1}(\boldsymbol{\rho})\ , \tag{A13}$$

to the diffusion equation

$$[\partial_t-\nabla_j D(\boldsymbol{\rho})\nabla_j]S(\boldsymbol{\rho}|t-t'|\boldsymbol{\rho}')=0\ , \tag{A14}$$

subjected to the initial condition (13).

The flux vector of the wave, S_k, $k=1,2,3$, is defined by Eq. (A5), and is given to the diffusion approximation by Eq. (A12). When the medium is deterministic and homogeneous in the range $\rho_3>0$, solution S is subject to the boundary condition of the form [3]

$$-D\frac{\partial}{\partial\rho_3}S=ZS,\quad \rho_3=0\ . \tag{A15}$$

Here, when the boundary is free from the reflection, $Z=\frac{1}{2}$ (more exactly, $Z=0.7104$, Milne's value) and, generally, Z takes a value within the range from 0 to $\frac{1}{2}$, depending on the boundary reflection-transmission matrix, including cases of a rough boundary.

APPENDIX B: DERIVATION OF BASIC EQUATIONS IN MATRIX FORM

We first introduce a coordinate matrix $S(t-t')$, defined by the matrix elements $S(\mathbf{\Omega},\boldsymbol{\rho}|t-t'|\mathbf{\Omega}',\boldsymbol{\rho}')$ labeled by the coordinates $\mathbf{\Omega}$ and $\boldsymbol{\rho}$; the matrix $S_0(t-t')$ is also defined in the same fashion. Hence the transport equation (4) can be written in the matrix form

$$\partial_t S(t-t')=HS(t-t'),\quad H=H_0-H_1\ . \tag{B1}$$

Here

$$H_0=-\mathbf{\Omega}\cdot\nabla-\gamma+\sigma,\quad H_1=\gamma_{ab}\ , \tag{B2}$$

where σ is defined by the matrix elements

$$\sigma(\mathbf{\Omega},\boldsymbol{\rho}|\mathbf{\Omega}',\boldsymbol{\rho}')=\sigma(\mathbf{\Omega}|\boldsymbol{\rho}|\mathbf{\Omega}')\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')\ , \tag{B3}$$

γ and γ_{ab} are diagonal matrices with the elements $\gamma(\boldsymbol{\rho})\delta(\mathbf{\Omega}-\mathbf{\Omega}')\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')$, etc. The corresponding equation for $S_0(t-t')$ is

$$\partial_t S_0(t-t')=H_0 S_0(t-t')\ , \tag{B4}$$

and subject to the initial condition (3), i.e.,

$$S(t-t'=0)=S_0(t-t'=0)=1\ . \tag{B5}$$

Hence the solution of Eq. (B4) can be written by

$$S_0(t-t')=\exp[c(t-t')H_0]\ . \tag{B6}$$

Here, with a new matrix $\Gamma(t-t')$, we write $S(t-t')$ in the form

$$S(t-t')=S_0(t-t')\Gamma(t-t') \tag{B7}$$

$$=\exp[c(t-t')H_0]\Gamma(t-t'); \tag{B8}$$

expression (B7) represents Eq. (25). Substitution of (B8) into (B5) leads to the initial condition for Γ as

$$\Gamma(t-t'=0)=1\ , \tag{B9}$$

which represents Eq. (27).

To find the governing equation of Γ, we observe that, from Eqs. (B7) and (B8),

$$\partial_t S(t-t')=H_0 S(t-t')+S_0(t-t')\partial_t\Gamma(t-t')\ , \tag{B10}$$

which, upon the substitution into Eq. (B1), leads to

$$[S_0(t-t')\partial_t+\gamma_{ab}S_0(t-t')]\Gamma(t-t')=0\ , \tag{B11}$$

which represents Eq. (26).

The diffusion version of these equations is given by Eqs. (37)–(42).

[1] See, for example, J. J. Duderstadt and L. J. Hamilton, *Nuclear Reactor Analysis* (Wiley, New York, 1976), Chap. 4.

[2] See, for example, K. M. Case and P. F. Zweifel, *Linear Transport Theory* (Addison-Wesley, New York, 1967), Chap. 8.3.

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