Periodicity and chaos in a one-dimensional dynamical model of earthquakes

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We study the homogeneous one-dimensional dynamic Burridge-Knopoff stick-slip model for an earthquake fault with periodic boundary conditions. The dissipation is tuned such that the system is asymptotic to elasticity at all wavelengths. For an inhomogeneous distribution of prestress, after an initial transient that displays a power-law distribution of fracture sizes for cases of weak dissipation, the earthquake sequences settle into a periodic state of through-going ruptures; the power-law distributions are attributable to persistence effects. The transition time from the precursory chaotic regime to the periodic state generally increases with an increasing ratio of transverse to longitudinal spring constants, but nonmonotonically so in individual simulations. These results argue in favor of the importance of a mechanism for generating localization to suppress through-going events, if these models are to be used in earthquake simulations.

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In view of the Gutenberg-Richter power-law distribution [1] for earthquakes, Bak and Tang [2] proposed that the earth's crust is in a state of self-organized criticality, a state that is only reached through the dynamics of the fracture process. Models of the dynamics of a smooth one-dimensional (1D) fault [3], and smooth 2D systems as cellular automata [2,4] have successfully reproduced the Gutenberg-Richter power law, a not unexpected result since scale-independent models might be expected to produce scale-independent distributions. Hence it has been argued that the spatiotemporal complexity of slip events is a generic feature of mechanical fault models.

This view has been challenged by Rice [5], who studied a laterally homogeneous 2D fault model based on continuum quasistatic elasticity with a rate- and state-dependent friction law that introduces a characteristic slip distance Δs. The fault is imbedded in a 3D elastic medium and has periodic boundary conditions. Numerical simulations show that if the computational grid size is small compared to Δs, only periodic slip events on the fault are observed. However, for a computational grid size larger than Δs, chaotic spatiotemporal slip behavior is observed. These results suggest that the display of chaotic behavior on the Burridge-Knopoff model is due to the use of a discrete system with $\Delta s = 0$.

We test this proposal by studying in detail the 1D Burridge-Knopoff (BK) [6] model with spatially homogeneous fracture strengths and periodic boundary conditions. The details of the model are described by Knopoff *et al.* [7]. The model consists of L identical masses m, interconnected by identical longitudinal springs with spring constant k. To mimic tectonic plate motions, the system is driven uniformly at a slow, constant rate by one of two rigid plates which is connected by identical leaf springs of constant l attached to each block (Fig. 1).

Once the force on a block exceeds the static friction, sliding is initiated and the frictional resistance on the block drops instantly to the dynamic friction (Fig. 2); hence, $\Delta s = 0$. Crack dynamics is simulated by the coherent motion of several blocks. Blocks that are in motion cause the stress on their stationary neighbors to increase and may cause the neighbors to move, thereby causing a crack to grow. A particle stops when its velocity vanishes; at this instant the static friction is reestablished. The dynamic friction is high enough that no particle can move in a direction opposite to that of the driving plate. The slip of a particle satisfies the difference equation,

$$m\ddot{u}_n + k(2u_n - u_{n+1} - u_{n-1}) + lu_n = \tau_n = F_n - \alpha\dot{u}_n, \tag{1}$$

where τ_n is the stress drop, F_n is the difference between the prestress on the nth particle and the dynamic friction, and $\alpha\dot{u}_n$ is a radiation damping term which can be regarded here as a velocity dependent dynamic friction [6]. Knopoff *et al.* [7] show that if $\alpha = (4lm)^{\frac{1}{2}}$, the system is asymptotic to dispersion-free elasticity in the continuum limit.

Once a crack nucleates, it can only be stopped by fluctuations in the prestress, since the breaking strength is uniform. Whether crack growth is arrested or not depends on the stress at the crack tip. If a crack transfers a large stress to the crack tip, the crack is likely to grow; thus large cracks are more likely to develop if the trans-

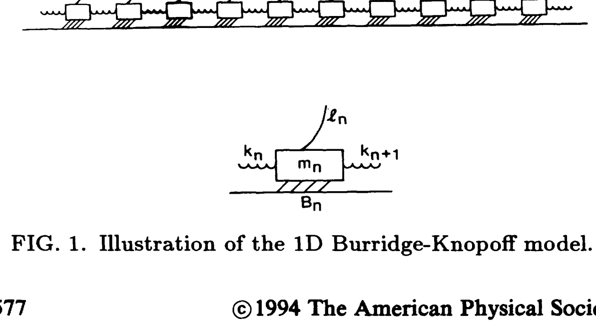

FIG. 1. Illustration of the 1D Burridge-Knopoff model.

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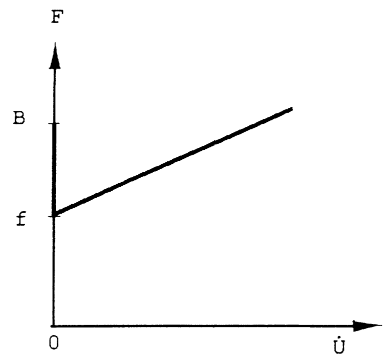

FIG. 2. Friction-velocity relation for a moving block.

fer of stress is large. The amount of stress transferred to the crack tip is a function of l/k and the crack length. If $l = 0$, all the stress drop on the crack is redistributed to the crack tips. Therefore, if $l = 0$ the stress concentration at the tips is proportional to the crack length, a conclusion consistent with 1D continuum elasticity. In this limit, there exists a critical crack length above which the crack cannot be stopped and will grow to the limit of the boundary to the system for any distribution of fluctuations in the prestress. For nonzero values of l, some of the stress drop will be dissipated through the leaf springs and through radiative energy loss; for sufficiently large cracks, the stress concentration at the tips no longer scales with crack length but instead it approaches a value that only depends on the ratio l/k. In this case, in terms of a scaling of fluctuations in the prestress by the length of the crack, a large crack can be more easily stopped than a small one; thus, one might expect to see a roundoff in the distribution of sizes for large cracks.

The final stress F_n on the ruptured segment is relatively smooth even though the prestress distribution may be rough. After a throughgoing rupture that tears the chain from end to end, which we call a runaway event, the stress on the entire crack will be more or less uniform. Since the tectonic stress increases uniformly with time, the model will develop solely periodic runaways thereafter. Runaways are more likely to occur if l/k is small, since in this case large cracks are more likely to form. We test numerically to assess Rice's hypothesis to see if there is an observable transition from periodic to chaotic behavior as the value of l/k increases from zero.

We have carried out simulations for systems with 64, 128, 256, 512, and 1024 masses and various values of l/k. The infinite impedance contrasts of unbreakable and free end conditions impose dramatic inhomogeneities on the system. In accordance with our goal of maximizing homogeneity, we use periodic boundary conditions; in this case the system mimics an infinite homogeneous lattice with a finite correlation length.

The dynamical equations (1) are solved as a piecewise linear system by means of an eigenvector expansion method developed by Knopoff *et al.* [7]. This scheme minimizes numerical errors in the integration of the equations of motion.

For systems with the same lattice lengths, the same initial random prestress is used. While the initial conditions cannot be made identical from a lattice of one length to a lattice of another, we make the initial conditions as uniform as possible by randomizing them for the largest lattice and decimating these initial values by factors of two for each reduction in lattice size by a factor of two. Two examples of earthquake sequences are shown in Figs. 3 and 4 for $l/k = 0.10$ and 0.38, in the case $L = 256$. The linear extent of an individual fracture is shown as a vertical line at the time of the event. For the smaller value of l/k (Fig. 3), after a chaotic episode, the system settles into a periodic runaway state after 1651 events. For the larger value of l/k (Fig. 4), the chaotic state of the system appears to persist, at least from the display of the first 4000 events. However, even this simulation eventually settles into a periodic state for this value of l/k as well, but now after more than four million events.

The relaxation time to the periodic state is difficult to specify in detail but, as is to be expected, it increases

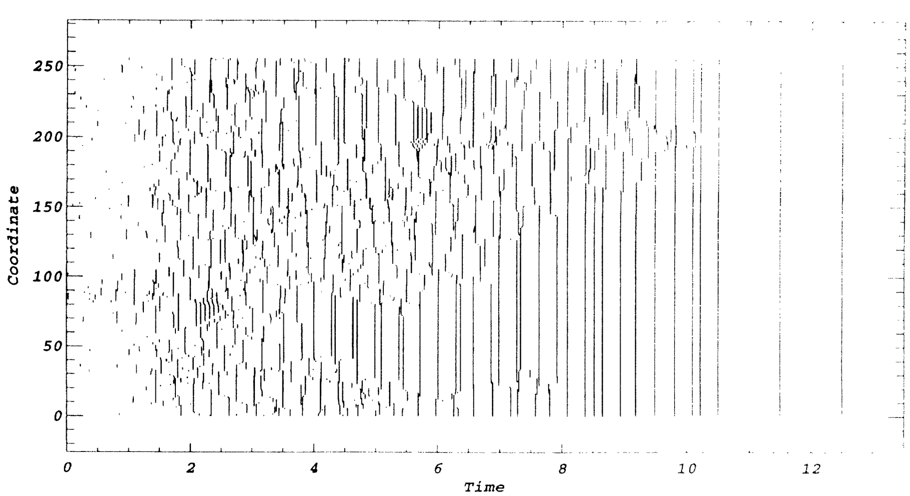

FIG. 3. Space-time patterns of simulated earthquakes for $l/k = 0.10, L = 256$. The dynamical history settles into a state of periodic runaways after 1651 events. Several runaways are shown. The initial prestress distribution is random.

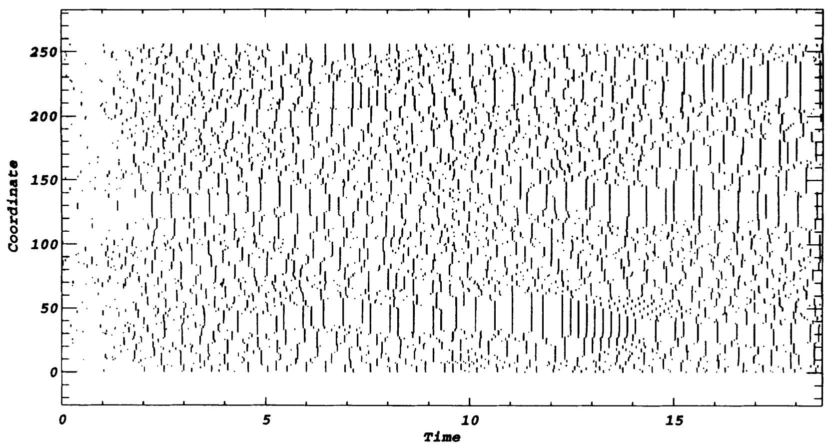

FIG. 4. Space-time patterns of simulated earthquakes for $l/k = 0.38, L = 256$. The first 4000 events are shown. The initial distribution of prestress is the same as in Fig. 3.

with increasing lattice length. Figure 5 is a display of the number of event steps before the periodic state appears *for the same initial condition*, as a function of the ratio l/k. We find that the relaxation time to the periodic state increases roughly exponentially with l/k, but with large fluctuations. The fluctuations are due to the irregular evolution through the L-dimensional lattice space toward the periodic attractor as the ratio l/k is changed. The curves for different values of L are not superposable because of correlations with length scale L, as well as because of variations among the initial conditions. Although the amplitudes of the fluctuations appear to increase for larger values of l/k, there is no hint that the system is self-organizing into a perpetual chaotic state, or what is equivalent, that the system never reaches the periodic state; put another way, if the exponential property in Fig. 5 extends indefinitely to large values of l/k, we cannot conclude that there is a phase transition at some characteristic value of l/k that will forbid the appearance of a periodic state.

As remarked, for $l/k = 0.38$, periodic runaways set in after more than 4×10^6 events for our random distribution of prestress. The lengths of the first 2.56×10^6 events in this sequence display a good power-law distribution over much of the range of sizes (Fig. 6). The distributions of event sizes display a rolloff at a fracture length that also depends on l/k, being about $X_c = 150$ for the case $l/k = 0.38$. The rolloff is independent of lattice size L and hence the system is not scale-independent, i.e., the Gutenberg-Richter power-law distribution is not obtained to all scale sizes up to the lattice size; the distribution of Fig. 6 is better fit by a gamma distribution $X^{-p}\exp(-X/X_c)$ than by a power law. If the distribution has a rolloff for all values of $(l/k, L)$, then all lattices whatever their length may be, must ultimately develop a runaway event at a time determined by the tail of the distribution.

The distribution of crack lengths as a function of l/k for fixed L are shown in Fig. 7. For small values of the dissipation parameter l/k, the system exhibits well-developed power-law behavior with rolloff over a wide range of crack sizes. The exponent p is a function of the ratio l/k. With increasing values of l/k, the distribution becomes steeper and the portion of the distribution that describes the power law disappears. As l/k increases, the time to the periodic runaway increases exponentially, as

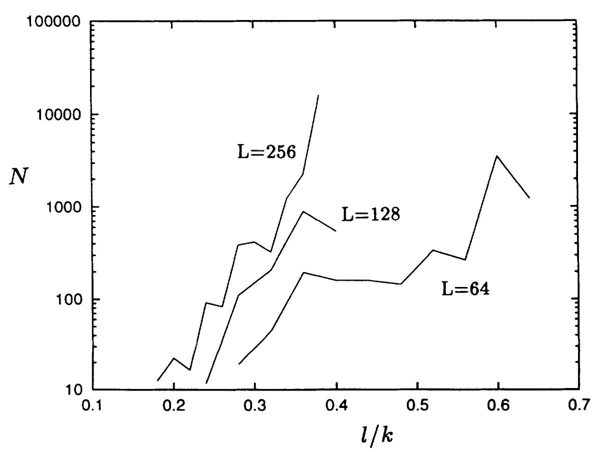

FIG. 5. Number of events before the periodic state is reached. The initial distributions of prestress are the same as in Fig. 3 or are decimations thereof.

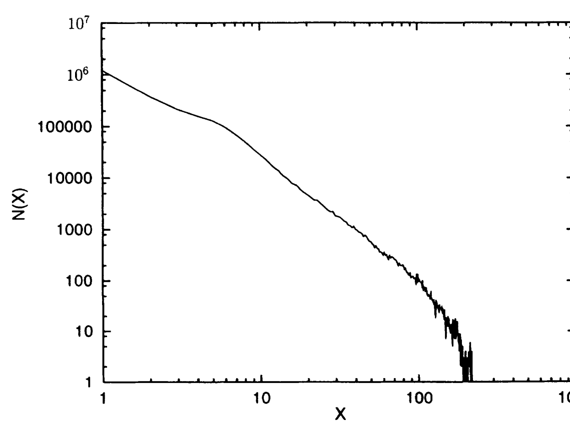

FIG. 6. Crack length distribution for the first 2 560 000 events. $l/k = 0.38, L = 256$.

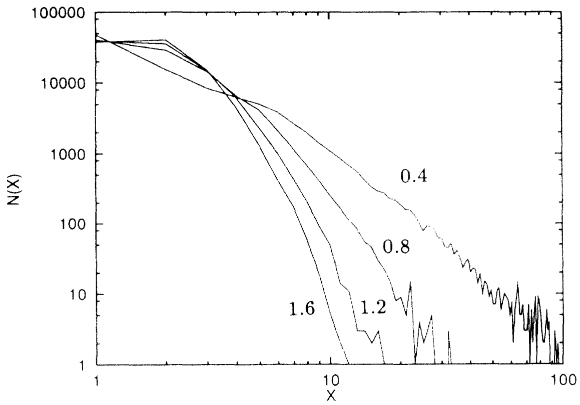

FIG. 7. Crack length distribution for the first 100 000 events as a function of l/k. $L = 256$.

remarked. Inspection of the space-time pattern of fractures gives the reason for the disappearance of the power law. For small values of l/k, the patterns have a persistent character as a consequence of the nature of the healing condition, $\dot{u}_n = 0$. A fractured segment has a relatively uniform stress after cessation of motion. A long fracture will therefore fracture again with approximately the same length as before at about the same sites, thereby having a tendency to form a repetitive cluster of events, all with about the same length. These cracks are stopped by fluctuations in stress at the ends and are thus likely to grow or shorten by roughly one lattice site at the ends of the fractures in successive events. For larger values of l/k, there is less stress transferred to the edge of a crack, and hence a smaller probability of developing a large fracture and, thus, a smaller likelihood of forming persistent clusters. Thus, we suppose that the absence of a power-law distribution for the larger values of l/k describes self-organization with relatively few persistent events, whereas the power-law distribution characterizes the persistent states for smaller values of l/k. Thus, the events in the sequences are not independent and the power laws describe the properties of the clusters of repetitive events, namely, the number of clusters of a given set of lengths and the number of events in each cluster. It is evident from Fig. 7 that the time to periodicity of a lattice whose length is much greater than the characteristic rolloff length for a particular value of l/k will be very long and increases dramatically with increasing lattice size.

Whereas in Fig. 4, we have shown the space-time evolution of the first 4000 events in the case $l/k = 0.38$, $L = 256$, in Fig. 8 we show the history for the last 2000 events before the ultimate runaway at event 4.5×10^6 (approximately). The succession of fractures has now developed into a series of repetitive long events that dominate most of the lattice. With increasing time, the repetitive long cracks occupy an increasing part of the remainder of the space-time regime by progressive erosion of the space-time region occupied by the remaining small events, until a crack of length $L = 256$ develops. The system is periodic after that. Thus, there is a dramatic shift in the last stages of pattern evolution to a distribution that has a significant peak near the lattice size; of course culmination in a runaway event is the last ingredient of the peak in the distribution before the onset of the periodic phase.

Does the system evolve gradually from the state shown in Fig. 4 to that in Fig. 8 or is there some intermediate pattern? A partition of the central 2.5×10^6 events into ten equal time intervals show that the distribution is virtually identical in each. Thus, we infer that the system is self-organizing into a metastable but transient state that begins shortly after initialization, and persists for a long time without significant change; the terminal phase begins when, with low probability of occurrence, a very long crack develops whose successors in a persistent cluster ultimately consume the entire lattice. The self-organization in the metastable transient appears to be independent of the initial conditions for random initial conditions. The rolloff, as implied by the discussion above, is set by the ratio l/k and not by the size of the lattice for sufficiently large L.

Our results show that this dynamical BK model with

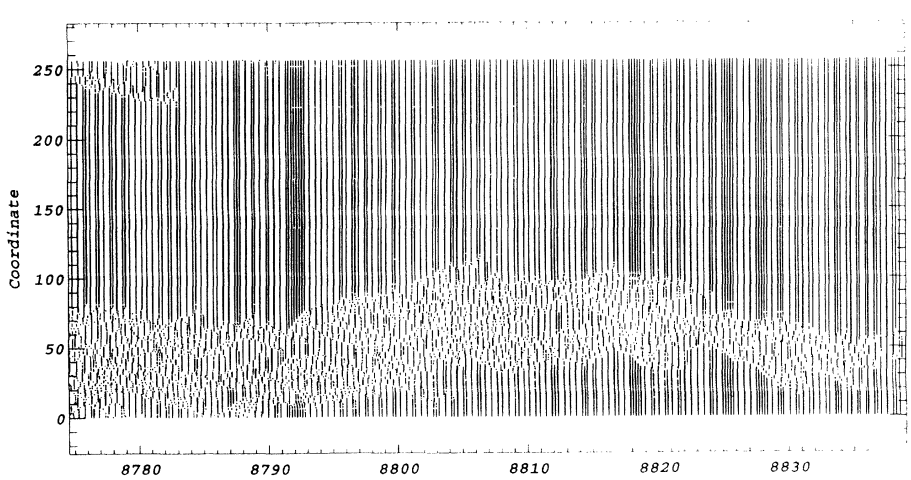

FIG. 8. Space-time patterns of simulated earthquakes for $l/k = 0.38, L = 256$. Initial condition is the same as in Fig 3. The last 2000 events before the system develops periodic runaways are shown.

simulated radiation damping will eventually settle into a periodic state for any lattice of finite size with periodic boundary conditions. The time may be very long before the onset of the periodic state, and the events in the transient state prior to the transition to periodicity appear to have power-law distributions, for small dissipative parameter but not for large; the power law arises because of persistent clustering.

The behavior after runaway depends on the way in which the system is reset by the runaway. In these models, the periodic runaway property without intervening small scale seismicity is a direct consequence of the smooth stress on the rupture site after the fracture event. With other models of the internal dynamics of fractures that leave irregular stresses in their aftermath, the fracture sequence after a runaway can be more complex. Thus the runaway event is the last in a series of self-organizing events whose character depends on the conditions either after the immediately preceding runaway or at the start of the calculation. In some cases, these initial conditions lead to a rich transient sequence before the next runaway and in others, the extended transient can be absent; in the model described in this paper, both forms of behavior are displayed. Other transient sequences are possible.

It is questionable whether models that develop periodic runaway events simulate earthquakes. In a runaway event, one end of the fault slips in a manner that is identical to that at the other end for periodic boundary conditions. In real earthquakes, large events are stopped by excessive barrier stresses established by geometrical influences that give an upper limit to the sizes of earthquakes.

Runaway events are a property of these lattice systems as well as of the continuum systems described by Rice and the two types of fault models give results consistent with each other whether $\Delta s = 0$ or not. We suppose that systems with periodic end conditions will ultimately display runaways if the computations are extended to sufficiently long times. Thus spatiotemporal complex slip distributions should not be regarded as a generic feature of the nonlinear dynamics of a smooth fault, whether we model this fault from the framework of continuum mechanics or from lattice dynamical systems, but rather should be regarded as part of an evolutionary transient process.

To account for the complex histories observed in real earthquake sequences, our simulations strongly suggest that one has to take into account the effects of geometrical disorder along a fault and hence that inhomogeneity is a critical property of earthquake models, a conclusion also reached by Knopoff *et al.* [7] and Rice [5].

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