Controlling hyperchaos in a multimode laser model

Pere Colet,* Rajarshi Roy, and Kurt Wiesenfeld
School of Physics, Georgia Institute of Technology, Atlanta, Georgia 30332-0430
(Received 25 April 1994)

We show that it is possible to stabilize remnants of periodic orbits in a hyperchaotic system (with more than one positive Lyapunov exponent) using a single feedback parameter. Numerical simulations of a multimode laser with an intracavity frequency doubling crystal using occasional proportional feedback indicate that the controlled periodic orbits are locally unstable in only one coordinate, even though the system has three positive Lyapunov exponents in this parameter regime. This situation should be common in dynamical systems of the coupled oscillator type with many degrees of freedom.

PACS number(s): 05.45.+b, 42.50.Ne

Globally coupled oscillator arrays have been the focus of much recent attention. Aside from their use in mean field approximations [1], globally coupled arrays arise naturally in a variety of physical contexts, including mechanical examples such as the loaded string [2], electrical circuits such as Josephson junction arrays in series [3] and parallel [4] configurations, and optical systems such as laser arrays [5] and multimode lasers [6]. Work on these systems has tended to focus on the possibility of achieving simple (periodic or steady state) dynamics, either through mutual (spontaneous) synchronization or through the application of an externally imposed master drive. An alternative approach is to borrow a page from the recent work on control of chaos and introduce a small feedback to achieve *nearly* periodic dynamics [7].

In this article, we investigate the control of chaos in a particular globally coupled oscillator system, namely, the multimode Nd:YAG laser (where Nd:YAG denotes neodymium-doped yttrium aluminum garnet) with an intracavity frequency doubling crystal [8]. Recent experiments showed that both the steady state and periodic orbits could be stabilized in this laser using the method of occasional proportional feedback (OPF) [9] in the chaotic regime [10]. Interestingly, the chaos in this laser is not low dimensional—the "hyperchaotic" attractor [11] has three positive Lyapunov exponents in the parameter regime explored here—yet control is achieved using just a single parameter for feedback. While single parameter control might be expected to work if the chaos has a sufficiently high symmetry, in our problem the chaotic attractor has no symmetry. In view of the practical desirability of single parameter control, it is important to investigate why OPF is successful in this case. Our numerics suggest that it is because the target orbit has only one unstable direction. That is, we can control this hyperchaotic system provided there are embedded orbits of "low dimensional instability": we expect this result to hold in general. This is of practical importance because, while high dimensional chaos is common, in many cases the most desirable orbits have a single unstable direction. We return to this point at the end of the article.

In the schemes developed to date, control has been applied usually to stabilize unstable steady states or periodic orbits either isolated or embedded in a chaotic attractor. The idea is to compensate, typically by making small corrections on a control parameter, for the deviation of the trajectory followed by the system from the selected fixed point or periodic orbit. Here we present a case where what is stabilized is not an unstable periodic orbit but the *remnant* of a periodic orbit in which a slow dynamics is responsible for a departure from the orbit. An appropriate control scheme can compensate for this slow dynamics and stabilize this orbit. We show that this type of control can be achieved in a model for a multimode Nd:YAG laser with an intracavity nonlinear crystal by using the OPF scheme.

The dynamics of a multimode Nd:YAG laser with an intracavity potassium titanyl phosphate (KTP) frequency doubling crystal can be described in terms of the rate equations for the intensity I_k and gain G_k associated with each mode [12],

$$\tau_c \frac{dI_k}{dt} = \left[G_k - \alpha_k - g\varepsilon I_k - 2\varepsilon \sum_{j\neq k} \mu_{jk} I_j \right] I_k ,$$
$$\tau_f \frac{dG_k}{dt} = \gamma - \left[1 + I_k + \beta \sum_j I_j \right] G_k , \qquad (1)$$

where $k=1,\ldots,N$. Here, τ_c is the cavity round trip time (0.2 ns), τ_f is the fluorescence lifetime of the Nd^{3+} ion (240 μs), α_k is the cavity loss parameter for the kth mode, γ is the small signal gain, which is related to the pump rate, and g is a geometrical factor dependent on the phase delays due to the YAG and the KTP crystals as well as on the angle between the YAG and KTP fast axes. Here we take $g=0.1$. Each cavity mode can be polarized only in one of two orthogonal directions (X,Y). For modes j having the same polarization as the kth mode, $\mu_{jk}=g$, while $\mu_{jk}=(1-g)$ for modes having or-

*Permanent address: Departament de Física, Universitat de les Illes Balears, E-07071 Palma de Mallorca, Spain.

thogonal polarization. The nonlinear coefficient ε is associated with the conversion efficiency of the fundamental intensity into doubled intensity by the KTP crystal ($\varepsilon = 5\times10^{-6}$). β is the cross-saturation parameter related to the competition among the different longitudinal modes. In general its value will be different for each pair of cavity modes, but for simplicity we assume the same value of β for all the mode pairs ($\beta=0.7$).

We have numerically integrated Eq. (1) in the case of five coexisting modes in the cavity, four polarized in the X direction and one in the orthogonal Y direction. This situation has been experimentally observed [12,13]. The values of the losses were assumed to be equal for all the modes with $\alpha_k=0.01$. There is a natural relaxation oscillation frequency in the system related to the rate of interchange of energy between the field and the atoms. With an increase of pump power above threshold, the system shows a stable steady state, periodic behavior, and, finally, chaos. Chaotic behavior arises as a consequence of the global coupling between the longitudinal modes through the nonlinear process of sum frequency generation.

Figure 1 shows the total and individual mode intensities for a chaotic time trace for $\gamma/\alpha=5$. For these parameter values, three of the ten Lyapunov exponents for the system were computed to be positive for a time trajectory of length 20 ms. Their values were $\lambda_1=14.1\ \mathrm{ms}^{-1}$, $\lambda_2=5.4\ \mathrm{ms}^{-1}$, and $\lambda_3=1.9\ \mathrm{ms}^{-1}$. An estimate of the Lyapunov dimension using the Kaplan-Yorke conjecture [14] gave a value of $D_L\sim5.4$. On the face of it, one may expect that controlling chaos in this system would require active feedback on three different parameters, or a more complex scheme of feedback that retains information about parameter perturbations in prior steps [15]. However, our simulations demonstrate stabilization with feedback on a single parameter; the mechanism revealed here may be responsible for stabilization of some periodic orbits observed in the laboratory [10]. As mentioned earlier, an alternative explanation for this unexpected good fortune may be that owing to an underlying symmetry, control of a single parameter effectively influences several degrees of freedom simultaneously. This explanation is ruled out, however: Fig. 1(b) shows that despite the symmetry of the underlying differential equations, no symmetry is present in the attracting solution (i.e., the individual modes evolve quite differently). Occasionally, the system lands very close to an unstable periodic orbit, at which time it is possible to control the chaos. In this example, the orbit we stabilize has a period five times that of the relaxation oscillations of the total intensity.

We apply the OPF technique in the same way as done in the experiment [10]. The total laser output intensity I is sampled with a given periodicity T and its value is compared with a fixed reference value I_{ref}. Then, during a short time $\Delta t \ll T$, the pump rate is modified to $\gamma=\gamma_0+a(I-I_{\mathrm{ref}})$ where γ_0 is the ambient pump level and a is a small proportionality constant. After the interval Δt and until the next sampling time, the pump takes

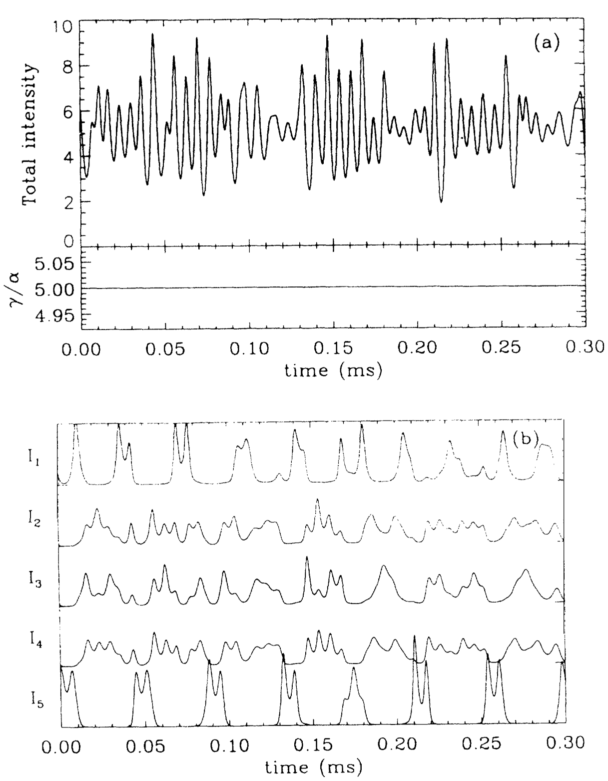

FIG. 1. Chaotic time series for pump parameter $\gamma/\alpha=5.0$. (a) Total intensity and applied pump level; (b) each of the five mode intensities.

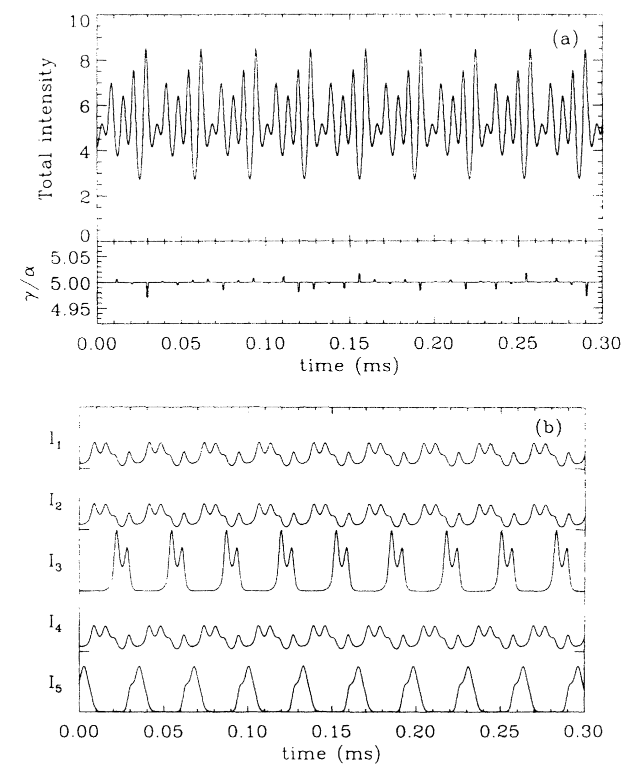

FIG. 2. Time series in the presence of occasional proportional feedback applied to the pump parameter γ/α, showing the stabilization of the remnant of a period-five orbit. (a) Total intensity and applied pump level; (b) intensity output of each mode.

its ambient level $\gamma = \gamma_0$. Thus there are four parameters $(I_{\text{ref}}, \Delta t, a, T)$, which allows considerable flexibility in achieving control. (These four parameters are constants; dynamical control is implemented only through γ.) As a practical matter, one finds that T has to be close to some multiple or submultiple of the underlying relaxation oscillation period.

Figure 2(a) shows the time series for the total intensity when the control is applied, with $T = 9.0$ μs, $\Delta t = T/10$, and $a = 0.008$. Also shown is the time series of the applied feedback on γ. The amplitude of applied pulses is at most 1% of γ_0. We have set I_{ref} close to the mean value of the total intensity (that is, $I_{\text{ref}} = 5$), so that there are approximately an equal number of positive and negative kicks.

Figure 2(b) shows the time series of the intensity for each mode. Of the four modes polarized in the X direction, three of them (1, 2, and 4) oscillate in phase. The symmetry is not complete; the other X-polarized mode behaves quite differently. Physically, due to the competition between different modes for the available gain, one sees that I_3 has maxima when I_1, I_2, I_4 are near minima,

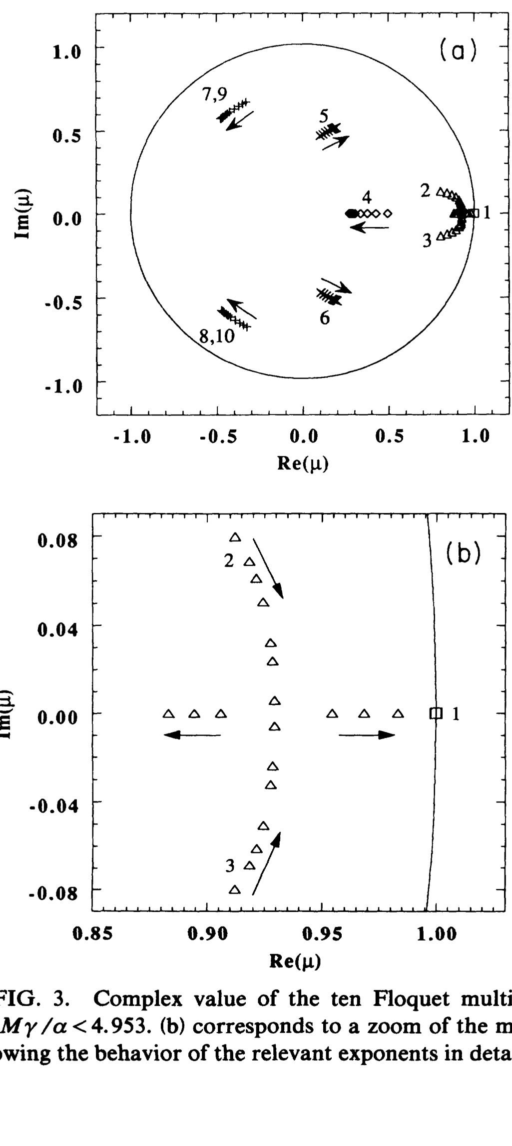

FIG. 3. Complex value of the ten Floquet multipliers for $4.9M\gamma/\alpha < 4.953$. (b) corresponds to a zoom of the main figure showing the behavior of the relevant exponents in detail.

and that the Y-polarized mode has largest intensity when all of the modes 1–4 have low intensities.

In our simulations, this periodic orbit can be stabilized indefinitely, unless the control is turned off, in which case the system spirals away from the periodic orbit and returns to the original chaotic attractor. Just as in the experiments [10], it has proved impossible to stabilize the dynamics by applying a simple periodic perturbation [16] rather than one dependent in some way on the output.

It turns out that for these parameter values, the periodic orbit is stable when $\gamma/\alpha < 4.953$. We have calculated all ten of the Floquet multipliers for this periodic orbit for different values of the gain in the stable region. The results are shown in Fig. 3, for the range $4.9 < \gamma/\alpha < 4.953$, which shows that the orbit undergoes a bifurcation. Because of the symmetry of the orbit there are two equal pairs of complex conjugate Floquet multipliers (in the left-hand plane). The interesting behavior comes from the complex conjugate pair closest to the unit circle, shown in detail in Fig. 3(b). The pair collide at $\gamma/\alpha = 4.951$, and then split up along the positive real axis, one towards the origin and the other away. The latter multiplier crosses the unit circle, signaling a bifurcation. Generically this will be a saddle-node bifurcation, where a stable and an unstable orbit coalesce and disappear; we can rule out a symmetry breaking bifurcation because the degenerate multipliers (7,9 and 8,10, in Fig. 3) remain inside the unit circle. Consequently, for $\gamma/\alpha > 4.953$ there is no unstable periodic orbit, but rather the remnant of one. However, for values of the gain that are not too far above the critical value the orbit drifts only a small amount each iteration of the local Poincaré map.

It is precisely in this situation that one can think of stabilizing the orbit by applying small perturbations on a control parameter that close this nonperiodic orbit, making it periodic. This is what the OPF control perturbations do in our system. Despite the fact that the chaotic attractor has three positive Lyapunov exponents, *locally* the orbit is unstable only in one direction. This is why a simple control method works for this hyperchaotic system.

We have also looked at the choice of values for the control parameters to stabilize the orbit. To characterize the periodicity of the stabilized orbit we measure the value of the five maxima in each period and compute the variances σ_i of these maxima over many periods. The lower values of σ_i correspond to better stabilization, i.e., the periodic orbit has less fluctuations. Figure 4 shows the mean of these five variances $\langle \sigma \rangle$ for different values of the control period T. The lower values of $\langle \sigma \rangle$ are achieved at values of T which are rational fractions of the relaxation oscillation period. The places where the value of $\langle \sigma \rangle$ is found to be very large correspond to regions where stabilization cannot be achieved. It can also be seen from Fig. 4 that, in general, the stabilization achieved when the reference value I_{ref} is placed close to the mean of the total intensity is better than when it is placed far away from it. Figure 5 shows the dependence of $\langle \sigma \rangle$ on Δt for two different values of a and fixed values of $T = 9$ μs and $I_{\text{ref}} = 5$. It can be seen that the

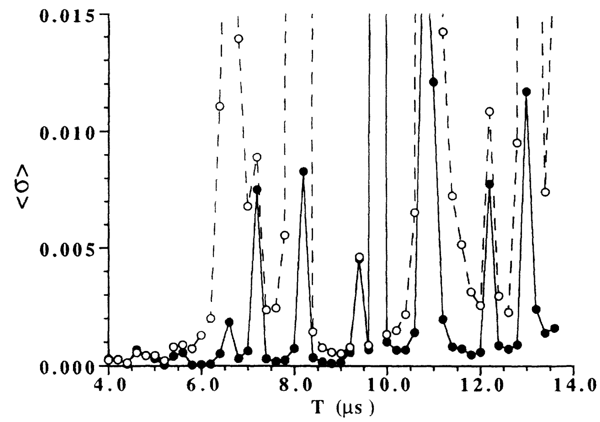

FIG. 4. Dependence of $\langle \sigma \rangle$ on the control period T for $I_{\rm ref}=5$ (solid line connecting filled circles) and for $I_{\rm ref}=10$ (dashed line connecting empty circles). For each value of T, Δt was taken to be $\Delta t = 0.1T$.

stabilization improves for smaller values of Δt until a limiting value is reached, after which stabilization is not possible. The larger the size of the proportionality constant a, the smaller the threshold value of Δt to achieve stabilization. Figure 5 indicates that stabilization can be achieved for a large interval of values of Δt and a, but the best stabilization is achieved for a certain minimum correction. As the size of the correction increases the quality of the stabilization degrades.

As an indication of the quality of control achieved, Figs. 6(a) and 6(b) show the power spectrum of the total intensity for the chaotic orbit and the stabilized orbit. Although the chaotic spectra show some peaks, these are embedded in a large noisy background. After application of the stabilization method the energy emitted has been dramatically concentrated in a few frequencies, as one can expect from stable periodic operation.

Summarizing, we are led to the following picture. It is possible to control a hyperchaotic system with OPF feed-

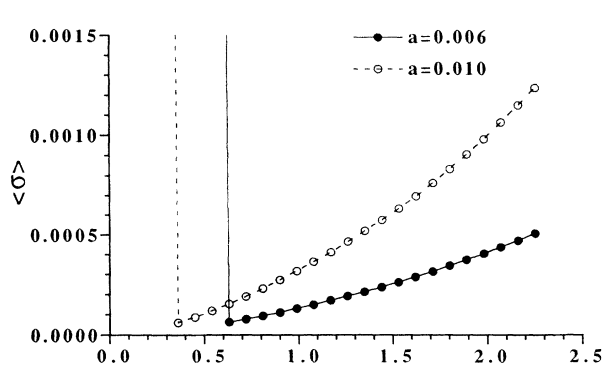

FIG. 5. Dependence of $\langle \sigma \rangle$ on Δt for $T=9$ μs, $I_{\rm ref}=5$, and two values of a: $a=0.006$ (solid line connecting filled circles) and $a=0.010$ (dashed line connecting empty circles).

back on a single parameter. This is because embedded within the chaos there are remnants of periodic orbits with just one unstable direction. This situation could be a rather general one for higher dimensional chaotic systems. Unstable orbits with unstable manifolds of dimension greater than one will presumably require a correspondingly greater number of parameters, or a more complex algorithm to be dynamically controlled.

In general, the control mechanism described here does not require the existence of symmetry. However, in problems of the coupled-oscillator type the presence of symmetry should make this scenario rather common. In these problems, the interacting elements are nominally identical, so that synchronized (or partially synchronized) solutions are highly symmetric. Such orbits are often the most desirable since they reflect a high degree of coherence among the oscillators. A symmetric attractor may lose stability in one of two ways, either tangent to the symmetric subspace (with multiple positive Lyapunov exponents) or tangent to the unsymmetric subspace. In the latter case, the orbit is unstable but nevertheless "in-phase attracting," that is, desynchronizing perturbations are locally damped out. In our multimode laser example,

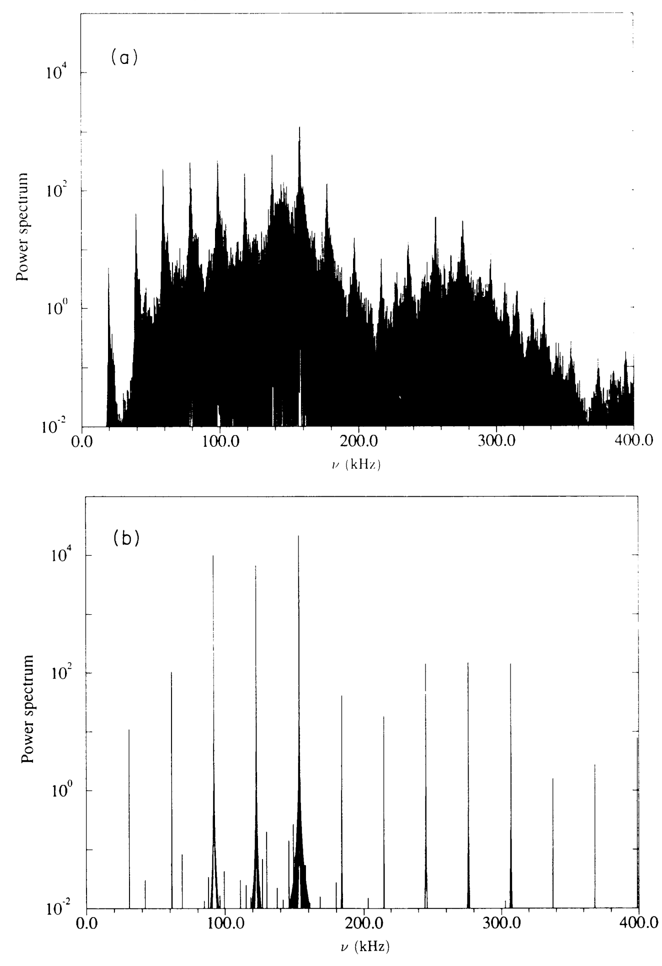

FIG. 6. (a) Power spectrum of the total intensity for a 59 ms chaotic time trace corresponding to $\gamma/\alpha=5$; (b) power spectrum of the total intensity when the remnant of the period-five orbit is stabilized, calculated over the same amount of time.

the chaotic attractor is entirely desynchronized, but the local periodic orbit (remnant) has three modes synchronized. A similar case of an unstable periodic state with in-phase attracting dynamics was reported recently in a study of Josephson junction arrays [17].

P.C. acknowledges the Dirección General de Investigación Científica y Técnica (DGICYT) (Spain) for support from the MEC-Fulbright program, R.R. was supported by NSF Grant No. ECS-9114232, and K.W. by ONR Grant No. N00014-91-J-1257.

[1] Y. Kuramoto and I. Nishikawa, J. Stat. Phys. **49**, 569 (1987); S. H. Strogatz, C. M. Marcus, R. M. Westervelt, and R. E. Mirollo, Phys. Rev. Lett. **61**, 2380 (1988); D. Golumb, D. Hansel, B. Shraiman, and H. Sompolinsky, Phys. Rev. A **45**, 3516 (1992); K. Kaneko, Physica D **41**, 137 (1990).

[2] M. J. McKenna, R. L. Stanley, and J. D. Maynard, Phys. Rev. Lett. **69**, 1807 (1992).

[3] P. Hadley, M. R. Beasley, and K. Wiesenfeld, Phys. Rev. B **38**, 8712 (1988); K.Y. Tsang and I. B. Schwartz, Phys. Rev. Lett. **68**, 2265 (1992); S. Watanabe and S. H. Strogatz *ibid*. **70**, 2391 (1993).

[4] A. Pance and M. J. Wengler, IEEE Trans. Appl. Supercond. **3**, 2481 (1993); J. A. Stern, H. G. LeDuc, and J. Zmuidzinas, *ibid*. **3**, 2485 (1993).

[5] Although most studies of laser arrays consider local coupling, global coupling has also been investigated; see, for example, M. Silber, L. Fabiny, and K. Wiesenfeld, J. Opt. Soc. Am. B **10**, 1121 (1993). For a review, see V. V. Likhanskii and A. P. Napartovich, Usp. Fiz. Nauk **160** (3), 101 (1990) [Sov. Phys. Usp. **33**, 228 (1990)].

[6] See, for example, M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974), Chap. 9.

[7] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990); T. Shinbrot, C. Grebogi, E. Ott, and J. A. Yorke, Nature **363**, 411 (1993).

[8] T. Baer, J. Opt. Soc. Am. B **3**, 1175 (1986).

[9] E. R. Hunt, Phys. Rev. Lett. **67**, 1953 (1991).

[10] Rajarshi Roy, T. Murphy, Jr., T. D. Maier, Z. Gills, and E. R. Hunt, Phys. Rev. Lett. **68**, 1259 (1992); Z. Gills, C. Iwata, R. Roy, I. B. Schwartz, and I. Triandaf, *ibid*. **69**, 3169 (1992).

[11] R. C. Hilborn, *Chaos and Nonlinear Dynamics* (Oxford, New York, 1994), p. 174.

[12] C. Bracikowski and Rajarshi Roy, Chaos **1**, 49 (1991).

[13] R. Roy, C. Bracikowski, and G. E. James, in *Recent Developments in Qauntum Optics*, edited by R. Inguva (Plenum, New York, 1993).

[14] J. Kaplan and J. A. Yorke, in *Functional Differential Equations and Approximation of Fixed Points*, edited by Heinz O. Peitgen and H.-C. Walther, Lecture Notes in Mathematics Vol. 730 (Springer-Verlag, Berlin, 1979), p. 204.

[15] D. Auerbach, C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. **69**, 3479 (1992).

[16] Y. Braiman and I. Goldhirsch, Phys. Rev. Lett. **66**, 2545 (1991).

[17] S. Nichols and K. Wiesenfeld, Phys. Rev. E **48**, 2569 (1993).

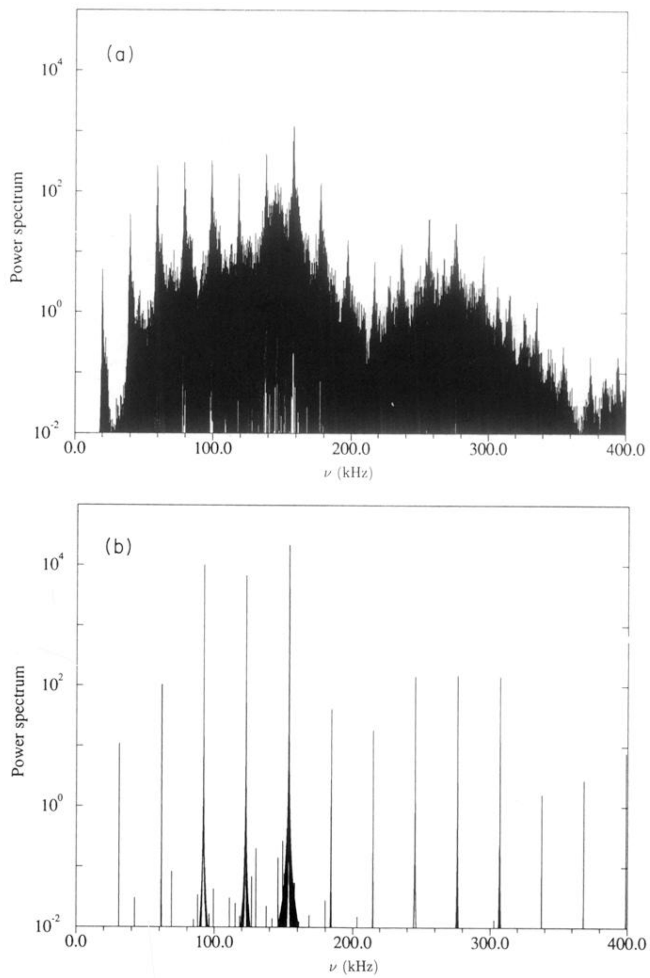

FIG. 6. (a) Power spectrum of the total intensity for a 59 ms chaotic time trace corresponding to $\gamma/\alpha=5$; (b) power spectrum of the total intensity when the remnant of the period-five orbit is stabilized, calculated over the same amount of time.