Stochastic resonance in an autonomous system with a nonuniform limit cycle

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In a recent numerical study, Gang *et al.* [Phys. Rev. Lett. **71**, 807 (1993)] presented the first example of stochastic resonance in an autonomous system. They considered a two-variable model in which a limit cycle is born as some parameter is varied. Their numerical experiments revealed various noise-induced effects, including a noise-induced shift in the frequency of the limit-cycle oscillations, and noise-induced oscillations in the absence of a deterministic limit cycle. We show that both of these effects are simple consequences of the nonuniformity of the motion along the limit cycle.

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The response of dynamical systems to noise is an active field of study [1]. In particular, the phenomenon of stochastic resonance [2] continues to attract considerable attention [3]. In its simplest form, stochastic resonance occurs in a bistable system driven by a periodic external force. The periodic force raises the potential wells alternately; when the noise is sufficiently strong, the particle can jump over the potential barrier. The resulting particle motion is coherent with the driving force. Stochastic resonance has recently been observed experimentally in a system where the noise was purely thermal [4], as well as in some biological systems [5–7]. In addition, it has been found that noise can generate coherent motion in globally coupled maps [8] and globally coupled oscillators [9].

In a recent Letter, Gang *et al.* [10] presented numerical simulations of a two-dimensional autonomous system for which the inclusion of noise generates stochastic resonance. The novel aspect of their results is that stochastic resonance occurred in the absence of an external periodic force. The model considered in [10] exhibits a stable limit cycle for certain values of a control parameter. In this regime, the power spectrum has a δ function at the frequency of the limit-cycle oscillations. Noise was found to broaden the peak and to shift it to higher frequencies. In another regime of the control parameter, the deterministic system does not have a limit cycle, yet noise was found to induce a peak in the power spectrum at a definite frequency. This frequency was also observed to shift to higher values for increasing noise strength. Furthermore, a plot of the height of the peak versus the noise strength showed a clear resonancelike behavior.

In this Brief Report, we reexamine the model of Gang *et al.* [10] and point out that their numerical results have a simple explanation. Written in polar coordinates, the system is

$$\dot{r} = r(1 - r^2) + q_1(t), \quad (1)$$

$$\dot{\theta} = b - r^2\cos(2\theta) + q_2(t), \quad (2)$$

where b is the control parameter and $q_1(t), q_2(t)$ are white noise terms. In the absence of noise, the phase portrait depends on b as follows. For $b > 1$ the system has a stable limit cycle with $r = 1$ and with period $\int_0^{2\pi} d\theta/[b - \cos(2\theta)]$. For $b < 1$ this limit cycle no longer exists; instead we have four fixed points on the circle $r = 1$ with $\theta = \pm\frac{1}{2}\arccos(b)$. Two of those fixed points are stable (see Fig. 1) and the other two are unstable. The stable and unstable fixed points approach each other as b approaches 1. They collide at $b = 1$ and disappear for $b > 1$.

First consider the case $b < 1$. Without noise, the system will settle in one of the two stable fixed points. The effect of noise is to kick the system away from a stable fixed point. When b is close to 1, the stable and unstable fixed points are close together. Therefore, if the noise is strong enough, the system will occasionally be kicked to the far side of the unstable fixed point, after which it will flow rapidly around the circle towards the other stable fixed point. Near this second fixed point, the same scenario will apply again. This leads to coherent motion around the circle with a certain frequency. It is now clear why increasing noise will increase the resulting frequency: the system is kicked "over the hump" more often, and thereby skips the slowest part of the circle.

To test this idea we have simulated the equations above, where for simplicity we have added a noise term only to the equation for θ ($q_1 = 0$). In this case we are simulating the motion of the system on the circle $r = 1$. The noise term is generated by choosing a random num-

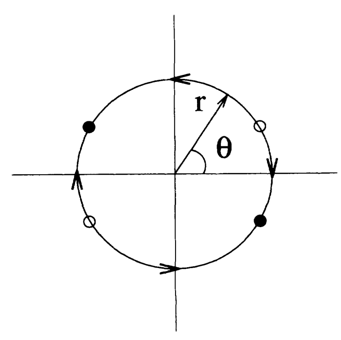

FIG. 1. The phase portrait of the system with two stable fixed points (solid circles), two unstable fixed points (open circles), and the flow on the circle $r = 1$.

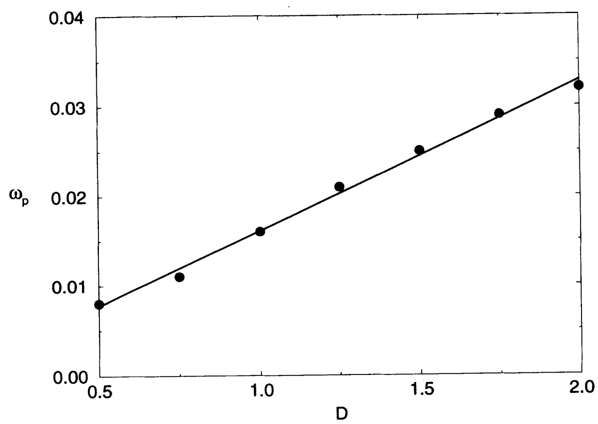

FIG. 2. The frequency ω_p of the peak in the power spectrum as a function of the noise strength D for $b = 0.99$.

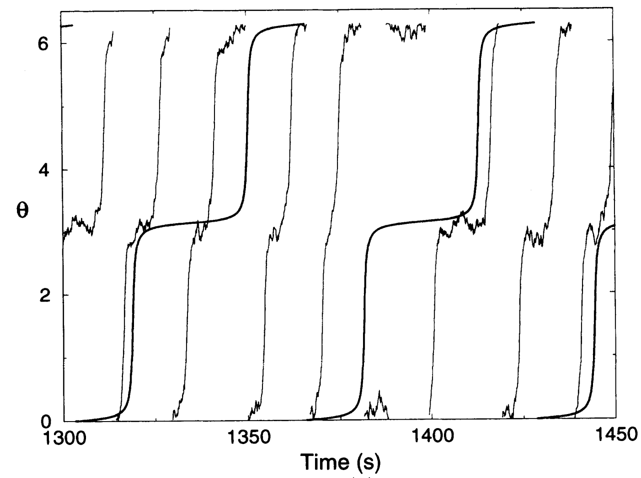

FIG. 3. The time series of the system in the case of a limit cycle ($b = 1.005$). The thick line is the noiseless case while the thin line is for $D = 0.01$.

ber from a uniform distribution between -1 and 1, and then multiplying it by a noise strength D. We then numerically integrated the system, using the same method as in [10], and calculated the position ω_p of the peak in the power spectrum [11]. The result is plotted in Fig. 2. For small noise strength, ω_p increases roughly linearly with D. This is as expected from the arguments above. If one doubles the noise strength, one expects that the system will be kicked out of the stable fixed point at twice the rate, and hence the frequency will double.

Now suppose the system has a limit cycle ($b > 1$). The frequency shift of the peak is also explained by the above picture. Without noise, and for b close to 1, the motion around the limit cycle is highly nonuniform; the system spends a long time passing through the slow regions at $\theta = 0$ and $\theta = \pi$, which are the ghosts of the former fixed points. The effect of the noise is to help the system to skip these slow regions. Consequently, the frequency will increase in the presence of noise.

This can be seen most easily in Fig. 3, where we have plotted the time series of θ with and without noise. The thick line shows θ without noise: the system spends relatively long times near $\theta = 0$ and $\theta = \pi$. In contrast, when we add noise (in this case, $D = 0.01$), we find the time series plotted as the thin line in Fig. 3. The system spends less time in the slow regions of the limit cycle and hence the resulting frequency is increased.

Although the system studied here has two slow regions (because of a symmetry in the model), the same effects would occur even if there were only one [12]. The crucial property of the system is that a saddle-node bifurcation occurs on an invariant cycle, thereby creating a limit cycle along which the motion is strongly *nonuniform*. This mechanism for the creation of a limit cycle is known as an infinite-period bifurcation; it is common in physical, biological, and chemical systems [13]. Therefore the effects discussed here should be experimentally observable in a variety of systems. On the other hand, one would not expect to find this type of stochastic resonance in systems whose oscillations are created by a Hopf bifurcation, since those systems have relatively uniform motion on their limit cycles.

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[1] P. Jung, Phys. Rep. **234**, 175 (1993).
[2] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, 453 (1981).
[3] For a review, see F. Moss, in *An Introduction to Some Contemporary Problems in Statistical Physics*, edited by G. H. Weiss (SIAM, Philadelphia, in press).
[4] A. Simon and A. Libchaber, Phys. Rev. Lett. **68**, 3375 (1992).
[5] J. Douglass, L. Wilkens, E. Pantazelou, and F. Moss, Nature **365**, 337 (1993).
[6] H. Braun, H. Wissing, K. Schafer, and M. C. Hirsh, Nature **367**, 270 (1994).
[7] K. Wiesenfeld, D. Pierson, E. Pantazelou, C. Dames, and F. Moss, Phys. Rev. Lett **72**, 2125 (1994).
[8] G. Perez and H. A. Cerdeira, Phys. Rev. A **46**, 7492 (1992).
[9] V. Hakim and W.-J Rappel (unpublished).
[10] H. Gang, T. Ditzinger, C.Z. Ning, and H. Haken, Phys. Rev. Lett. **71**, 807 (1993).
[11] We have used Euler's forward integration method with a time step of $dt = 0.01$. To obtain a reasonably smooth power spectrum, we have averaged over the spectra of 500 different time series.
[12] For an experimental realization of this dynamics in a sensory neural system, along with a theory of stochastic resonance for excitable systems, see [7].
[13] S. H. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering* (Addison-Wesley, Reading, MA, 1994).