Stabilizing unstable periodic orbits in fast dynamical systems

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We present a technique for stabilizing unstable periodic orbits in low-dimensional dynamical systems that allows for control over a large domain of parameters. The technique uses a continuous feedback loop incorporating information from many previous states of the system in a form closely related to the amplitude of light reflected from a Fabry-Pérot interferometer. We demonstrate that the approach is well suited for practical implementation in fast systems by stabilizing a chaotic diode resonator driven at 10.1 MHz.

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In many cases of practical importance, it is desirable to render a chaotic system periodic by applying only small perturbations to some accessible system parameter. An efficient scheme for achieving such control was proposed by Ott, Grebogi, and Yorke (OGY) [1]. The key idea is to take advantage of the unstable periodic orbits (UPO's) embedded in the chaotic attractor. As the system approaches an UPO, the strength of the perturbations required to keep it there vanishes, so that the smallness of the feedback signal is limited only by the noise level in the system.

In the simplest implementation of the OGY idea, a variable is measured as the system passes through a surface of section, the difference from a known fixed point value of the surface of section map is determined, and a control parameter is adjusted accordingly. Several variations of this method [2] have been successfully applied to many experimental systems with natural frequencies ranging from 10^{-2} Hz to 10^5 Hz [3]. None of these techniques, however, can be scaled up to the significantly higher frequencies encountered, for example, in high-speed electronic or optical systems. They require accurate sampling of a variable at discrete times in order to compare it with a reference value and involve discontinuous adjustments of the control parameter [4].

An alternative implementation, recently introduced by Pyragas [5], employs continuous feedback designed to synchronize the current state of a system and a time-delayed version of itself, with the time delay equal to one period of the desired orbit. We refer to this method of control as "time-delay autosynchronization" (TDAS). The feedback in TDAS does not require rapid switching or sampling, nor does it require a reference signal corresponding to the desired orbit. Unfortunately, the domain of system parameters over which control can be achieved via TDAS is limited [5]. The method fails for highly unstable orbits.

In this paper, we introduce a generalization of TDAS that is capable of extending the domain of effective control significantly. We refer to the scheme as "extended TDAS" (ETDAS). We show, for the discrete analogue of ETDAS, how utilization of the information from *many* previous states of the system allows control to be maintained for arbitrarily highly unstable fixed points. While the use of many previous states of the system or many past iterates of its surface of section map has been considered by others [6–8], ETDAS has the advantage of being easy to implement in high-speed systems. As evidence, we present experimental results on the control of a chaotic diode resonator operating at 10.1 MHz.

For a dynamical system with a measurable variable ξ and an UPO of period T, ETDAS prescribes the continuous adjustment of an available system parameter by a feedback signal

$$\epsilon(t) = \gamma \left[\xi(t) - (1-R) \sum_{k=1}^{\infty} R^{k-1} \xi(t-kT) \right], \qquad (1)$$

where $0 \leq R < 1$ and T. The case $R = 0$ corresponds to TDAS, the scheme investigated by Pyragas [5]. We will see that in the limit $R \to 1$ UPO's with arbitrarily large negative Floquet multipliers can be stabilized. We emphasize that, for any R, $\epsilon(t)$ vanishes when the system is on the UPO, since $\xi(t-kT) = \xi(t)$ for all k. Thus, whenever ETDAS is successful *there is no power dissipated in the feedback loop.*

The factor that limits the domain of control for a given R is *not* simply the continuity of the feedback: stabilization is known to be possible using continuous feedback of the difference between the current state of the system and the desired position on the UPO [9]. Rather, the problem stems from the use of a comparison to a past state of the system instead of an ideal reference state. To understand the latter effect, it is helpful to study a simple discrete version of ETDAS. We consider a fixed point x^* of a single-variable map, $x_{n+1} = f(x_n; \mu)$, where μ is an accessible control parameter that is adjusted on each iteration by an amount ϵ_n. The feedback is given by

$$\epsilon_n = \gamma \left[x_n - (1-R) \sum_{k=1}^{\infty} R^{k-1} x_{n-k} \right]; \qquad (2)$$

$$= \gamma(x_n - x_{n-1}) + R\,\epsilon_{n-1}. \qquad (3)$$

The first form is obviously analogous to Eq. (1), while the second, equivalent form is the easiest to study analytically.

For concreteness, we consider the controlled logistic map,

$$f(x;\epsilon) = (\mu + \epsilon)x(1-x), \qquad (4)$$

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though the analysis clearly applies to any fixed point of a smooth, single-variable map. Results concerning trajectories that stay close to a fixed point x^* are universal properties depending only on the quantity $dx^*/d\epsilon$ and the Floquet multiplier $\nu \equiv f'(x^*)$.

The logistic map has a fixed point at $x^* = (\mu - 1)/\mu$ which is unstable for $\mu > 3$. To determine the values of γ that render this fixed point stable for a given R, the controlled map is linearized about x^*. Letting $y_n = x_n - x^*$ and $e_n = y_n - y_{n-1}$, Eqs. (3) and (4) yield

$$\begin{pmatrix} y_{n+1} \\ e_{n+1} \end{pmatrix} = \begin{pmatrix} \nu & \beta \\ \nu - 1 & \beta + R \end{pmatrix} \begin{pmatrix} y_n \\ e_n \end{pmatrix}. \qquad (5)$$

where $\nu = 2 - \mu$ and $\beta \equiv \gamma(1-\nu)(dx^*/d\epsilon) = \gamma(\mu-1)/\mu^2$. Stability of the fixed point requires that $y_n \to 0$ as $n \to \infty$. A standard analysis of Eq. (5) reveals that the domain of stable control in the (ν, β) plane is bounded by the curves $\beta' < 1$, $2\beta' + \nu' > -1$, and $\nu' < 1$, where $\nu' \equiv \nu + R - R\nu$ and $\beta' \equiv \beta + R\nu$ [10]. The maximum value of $|\nu|$ for which control can be maintained corresponds to $\nu' = -3$, which in turn corresponds to $\nu = (-3-R)/(1-R)$. For $R = 0$ (the original TDAS) one finds that control cannot be maintained for $\nu < -3$ [6]. The maximum value of $|\nu|$ can be made *arbitrarily large*, however, by choosing R close enough to unity. The value of β required for stabilization at this ν is $(1+R)^2/(1-R)$, which also becomes large. Figure 1 compares the domains of effective control for the logistic map with $R = 0.5$ and $R = 0$, clearly displaying the advantage of ETDAS.

The domain of effective control obtained with ETDAS may be compared to the corresponding domain for the most straightforward implementation of the OGY idea that uses comparison to a known fixed point [1]. If we take $\epsilon_n = \gamma(x_n - x^*)$, control is achieved in the infinite strip defined by $|\beta + \nu| < 1$. Using ETDAS, the limit $R \to 1$ yields precisely the same domain, though only for $\nu < 1$.

The rate of convergence to the fixed point for the linearized map [Eq. (5)], defined by $\epsilon_n \sim \exp(\alpha n)$ for large n, is given by $\alpha = \ln\sqrt{\beta'}$ if $D < 0$ and $\alpha = \ln[(-\beta' - \nu' + \sqrt{D})/2]$ otherwise, where $D \equiv (\beta' + \nu')^2 - 4\beta'$. Figure 2 shows the rate of convergence toward the fixed point as a function of γ for fixed μ at $R = 0$ and 0.5. (The fastest convergence for a given μ is obtained at $D = 0$.) The similarity between the shape of the $R = 0$ curve and plots

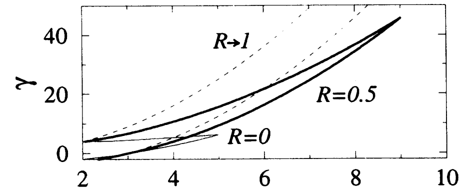

FIG. 1. Enhancement of control using ETDAS, illustrated with the parameters appearing in Eqs. (3) and (4). The region outlined in heavy lines produces a stable fixed point of the controlled map with $R = 0.5$. For comparison, the dashed lines (light lines) show the stable region for $R \to 1$ ($R = 0$).

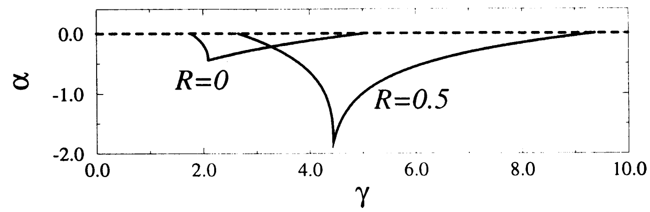

FIG. 2. Convergence properties of the controlled logistic map. Rate of convergence for $R = 0$ and 0.5 as a function of feedback gain γ, shown for $\mu = 3.7$.

of the maximum Lyapunov exponent in continuous systems controlled by TDAS [5] suggests that the discrete map problem does indeed contain many of the crucial ingredients of the continuous problem.

The basin of attraction for convergence to the fixed point for the controlled nonlinear map [Eqs. (3) and (4)] is finite. It is found that the system does not always converge to the fixed point (and ϵ_n remains large) for certain initial conditions. We expect, however, that the fixed point generally can be reached by a tracking procedure beginning in a regime where the desired behavior is stable [11]. Some care may be required, as it appears that the basin of attraction becomes smaller as R is increased.

The generalization of these results to higher order cycles becomes somewhat complicated. It is clear that any unstable q cycle of a discrete map $f(x)$ can be stabilized by applying ETDAS to $g(x) = f^q(x)$. The appropriate analogy to continuous control, however, is to adjust the control parameter on every iteration of f, using $\epsilon_n = \gamma(x_n - x_{n-q}) + R\epsilon_{n-q}$. With no control, the four-cycle becomes unstable at $\mu = 3.54\ldots$. Numerical studies show that with $R = 0$ control of the four-cycle of the logistic map control cannot be maintained above $\mu \simeq 3.62$, but with $R = 0.5$ control can be maintained up to $\mu \simeq 3.75$.

While our analysis clearly demonstrates the potential usefulness of ETDAS, it is not entirely transparent why it is so successful. A frequency-domain analysis of the controlled system provides a partial answer. Obviously, there should be no feedback when the system is on the UPO and hence the transfer function $F(\omega)$ of the ETDAS feedback generator must satisfy $F(0) = 0$. On the other hand, sufficient feedback must be generated at all other frequencies. (For the discrete map we define the time for one iteration to be unity and consider only $0 \le \omega \le \pi$.) Let $\omega_u(R_1)$ be the frequency at which the entire controlled system first becomes unstable as β is increased at fixed ν with $R = R_1$, and let $\nu_c(R_1)$ be the largest magnitude of ν for which control is possible with $R = R_1$. A straightforward calculation shows that for $R > R_1$, one obtains more sensitivity in the feedback at $\omega_u(R_1)$, without significantly altering the sensitivity at higher frequencies; $F(\omega)$ becomes flat over a broader range of frequencies below and including $\omega = \pi$. This results in an enhanced domain of control for each ν and an increase in $|\nu_c(R)|$.

Some remarks concerning the relation of ETDAS to other techniques may be helpful. (1) Bielawski *et al.* [6] have shown that a fixed point of a map can be stabilized for arbitrarily large $|\nu|$ by applying TDAS-type feedback only on every other period. This requires, however, that the feedback be switched on and off every period, whereas ETDAS allows stabilization for arbitrarily large ν using a completely continuous feedback. (2) Rollins et al. [7] have studied a control protocol similar to Eq. 3 but with the crucial difference that $\xi_n - \xi_{n-1}$ is replaced by $\xi_n - \xi^*$, where ξ^* is the fixed point value. The analogy to continuous systems in this case would require that a signal executing the desired continuous motion be available for reference (a method also considered by Pyragas [5]) rather than a time-delayed signal taken from the system itself. (3) Flake [8] has considered a scheme similar to ETDAS in which the feedback is proportional to $\xi_n - \frac{1}{N}\sum_{k=1}^{N} \xi_{n-k}$. In the limit $N \to \infty$, this becomes equivalent to the $R \to 1$ limit of ETDAS. Unlike ETDAS, however, direct analog implementation of this scheme requires N delay lines and correspondingly many amplifiers. (4) For ETDAS, no property of the desired UPO need be determined in advance except its period [5,6]. In periodically driven systems, where the period of the orbit is determined by the driving, no features of the UPO need ever be explicitly determined.

To confirm the effectiveness of ETDAS in a continuous system and demonstrate its practicality [12], we have used it to control the dynamics of a chaotic electrical ciruit: a diode resonator [13] driven at 10.1 MHz (corresponding to a drive period under 100 nsec). The ETDAS algorithm was implemented using analog circuitry as shown in Fig. 3. The voltage drop $V(t)$ across the resistor (proportional to the current flowing through the circuit) was used as the accessible dynamical variable. It was sensed with a high-impedance buffer so that the resonator was not disturbed.

The infinite series in Eq. (1) was generated with a *single* delay line whose output, weighted by R, was summed with its input $V(t)$. The propagation time through this loop (including the amplifier propagation delays) was precisely adjusted to the period T of desired UPO. The voltage in the loop was sensed by a buffer, weighted by $(1-R)$, subtracted from $V(t)$, and amplified by an amount γ to produce the ETDAS feedback voltage $V_\epsilon(t)$. Finally, $V_\epsilon(t)$ passed through a switch, was summed with the the drive voltage (denoted by $V_d \sin \omega t$), and injected into the resonator.

We chose to study the period-4 orbit because it could not be controlled beyond $V_d \approx 2.3$ V using the original TDAS algorithm ($R = 0$). For increasing R, the resonator could be controlled for higher and higher drive voltages. Successful control was indicated by the observation of a small feedback voltage $[V_\epsilon(t) < 0.005\, V_d]$ and the observation of a stable, periodic form of $V(t)$. For parameters corresponding to successful control, it was found that the system always approached the same orbit independent of its state when control was initiated at an arbitrary time, in contrast to the convergence behavior observed under some conditions for the logistic map.

Figure 4(a) shows $V(t)$ corresponding to the period-4 orbit when $R = 0.65$, $\gamma = -2.1$, and $V_d = 2.4$ V together with the associated ETDAS feedback voltage. Analysis of the return maps of the controlled and uncontrolled systems reveal that the stabilized orbit is indeed an UPO of the uncontrolled system and not an orbit created by the feedback circuitry. We found that the smallness of V_ϵ was affected by slight distortion in the delay line which we minimized by predistorting the wave form with a single-pole filter. It is seen in Fig. 4(b) that the maximum value of V_ϵ is no greater than 0.4% of V_d which is comparable to the noise level in the system (0.1%).

Figure 4(b) illustrates that ETDAS significantly increases the domain of control of the period-4 orbit in

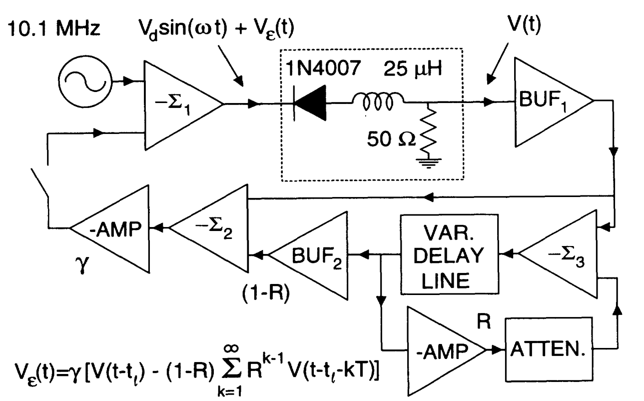

FIG. 3. Analog implementation of ETDAS used to control a diode resonator. Components: high-impedance buffers (BUF), inverting op amp ($-$AMP), inverting, summing op amps ($-\Sigma$), attenuator (ATTEN.), and low-loss variable delay line. The time lag t_ℓ between BUF_1 and $-\Sigma_1$ was ≈ 10 nsec.

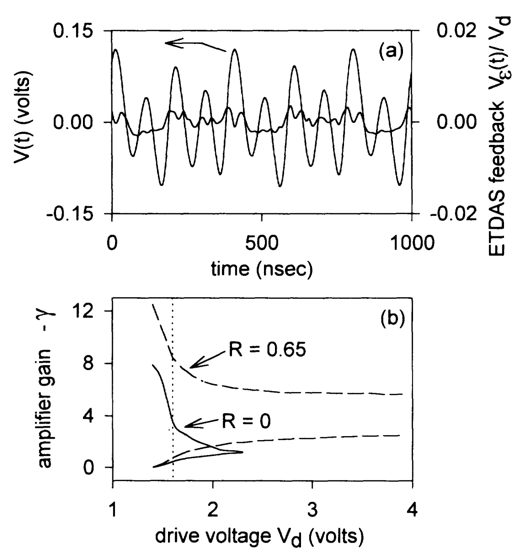

FIG. 4. Control of an unstable four-cycle of a diode resonator. (a) The controlled orbit and the ETDAS feedback voltage at $V_d = 2.4$ V and $R = 0.65$. (b) The parameter region where a stable four-cycle is obtained for $R = 0$ (the original TDAS) and $R = 0.65$. The dashed vertical line marks the values of V_d at which the four-cycle becomes unstable in the absence of feedback. The criterion for successful control is when the maximum feedback drops below $0.005\, V_d$.

comparison to the original TDAS scheme. It is seen that a larger range of feedback gains γ provide control at a given V_d and that control is possible for all values of V_d for $R = 0.65$. These results are consistent with expectations based on the discrete-map analysis.

Our analysis and experimental evidence suggest that ETDAS is a promising approach to the stabilization of UPO's in systems with high frequency chaotic oscillations. It is especially attractive since it lends itself naturally to an all-optical implementation. Note that Eq. (1) represents precisely the signal reflected from an interferometer consisting of mirrors with reflectivity R, spaced such that the round-trip transit time in the cavity is equal to the period of the UPO [14]. We speculate that a Fabry-Perot interferometer could be used to implement ETDAS and suppress deterministic chaotic fluctuations that occur on the nanosecond time scale in laser diodes.

Note added in proof. After submission of this manuscript we became aware of another mention of a version of ETDAS by Abed, Weng, and Chen [15].

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