Symbolic analysis of attractor geometry for the Lozi map

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Symbolic dynamics of the Lozi map is described for both positive and negative Jacobians. Based on the ordering rules of symbolic sequences, the geometrical structure of foliations associated with attractors is constructed for some typical cases. The critical parameters between one- and two-piece attractors are also given.

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I. INTRODUCTION

Symbolic dynamics provides almost the only rigorous way to understand all possible motions of dynamical systems. Symbolic dynamics of one-dimensional (1D) maps on the interval is well understood [1,2]. Recent progress is mainly in the extension from 1D to two-dimensional (2D) maps. Two-often-studied 2D models are the Hénon map [3] and its piecewise linear version, the Lozi map [4]. By considering forward and backward foliations, which are a natural generalization of stable and unstable manifolds [5], a 2D map may be decomposed into two coupled 1D maps. There is a procedure to construct a "good" partition for the Hénon map from tangencies between forward and backward foliations. For the Lozi map the situation is much simpler since a partition is given by the definition of the map at the beginning. The symbolic description of the partition line determines the admissibility conditions for allowed orbits [6,7]. Furthermore, foliations are well ordered according to their symbolic sequences. Symbolic dynamics helps us to understand the geometry of attractors. It is known that for relatively weak dissipation a two-piece attractor may exist even if the fixed point possesses homoclinic points. In Ref. [8] an approximate criterion for the critical parameters between one- and two-piece attractors was proposed. In this paper we shall perform a symbolic analysis of the problem to obtain a better estimation.

The paper is organized as follows: In Sec. II we give a brief summary of symbolic dynamics of the Lozi map. In Sec. III we then apply the symbolic dynamics to discuss a simple example, the boundary for the existence of a finite attractor. Section IV is the main part of the paper, where we determine critical parameters between one- and two-piece attractors by means of symbolic sequences. Based on ordering rules of foliations, we sketch the geometry of attractors. Finally, in Sec. V, we make some concluding remarks.

II. SYMBOLIC DYNAMICS OF THE LOZI MAP

The Lozi map is defined by

$$x'=y, \quad y'=1+bx-a|y| \ . \tag{1}$$

It is expected that the Lozi map exhibits certain topological similarities to the Hénon map.

We assign the letter L to the half plane $x<0$, R to $x>0$, and C to $x=0$. For a given initial point (x_0,y_0), we may encode its orbit according to the sign of y coordinates as

$$\ldots s_{\bar{m}}\ldots s_{\bar{2}}s_{\bar{1}}\bullet s_0 s_1 s_2\ldots s_n\ldots ,$$

where s_0 (R or L) indicates the sign of y_0, s_n corresponds to the nth image, and $s_{\bar{m}}$ corresponds to the mth preimage. The "present" position is indicated by a bullet, which divides the doubly infinite sequence into two semi-infinite sequences, the backward sequence $\ldots s_{\bar{m}}\ldots s_{\bar{2}}s_{\bar{1}}\bullet$ and the forward sequence $\bullet s_0 s_1 s_2\ldots s_n\ldots$. A straight line segment on which all the points have the same forward sequence $\bullet s_n s_{n+1}\ldots$ forms a forward foliation. Assume that the line equation for this foliation is

$$y-k_n x=\xi_n \ . \tag{2}$$

The slope k_n and intercept ξ_n may be calculated from the forward sequence by means of the recursion relations [7]

$$\xi_n=k_n(\xi_{n+1}-1)/b \ , \tag{3a}$$

$$k_n=b/(a\epsilon_n+k_{n+1}) \ , \tag{3b}$$

where ϵ_n denotes the sign of the y coordinate of any point on the foliation, and k_{n+1} and ξ_{n+1} correspond to the shifted sequence $\bullet s_{n+1}s_{n+2}\ldots$. Similarly, for the backward sequence $\ldots s_{n-1}s_n\bullet$, we have the line equation

$$y-h_n x=\eta_n \tag{4}$$

and the relations

$$\tilde{h}_n\equiv -b/h_n=b/(a\epsilon_n+\tilde{h}_{n-1}) \ , \tag{5a}$$

$$\eta_n=1+\tilde{h}_{n-1}\eta_{n-1} \ . \tag{5b}$$

Foliations are well ordered according to their symbolic sequences. More specifically, a bundle of forward foliations intersecting with some backward foliation is ordered as

$$\bullet E_r R\ldots > \bullet E_r L\ldots, \quad \bullet O_r R\ldots < \bullet O_r L\ldots , \tag{6}$$

where finite strings E_r and O_r consisting of letters R and L contain an even and an odd number of R, respectively.

The ordering rules for backward sequences are different for different signs of b. For $b>0$ the rules are

$$\ldots RE_l\bullet > \ldots LE_l\bullet, \quad \ldots RO_l\bullet < \ldots LO_l\bullet , \tag{7}$$

where strings E_l and O_l contain an even and an odd number of L, respectively. For $b<0$ the ordering rules are similar to (6), i.e.,

$$\ldots RE_r\bullet > \ldots LE_r\bullet, \quad \ldots RO_r\bullet < \ldots LO_r\bullet . \tag{8}$$

The above ordering rules can be understood from the "local" ordering that both eigenvalues of the fixed point R^∞ (and L^∞) for the Lozi map with $b<0$ are negative (and positive), while for the map with $b>0$ both are of opposite signs, and only the stable eigenvalue of the fixed point R^∞ and the unstable eigenvalue of L^∞ are positive.

Since both forward and backward sequences are well ordered, tangencies between them put restrictions on allowed sequences. (For a piecewise linear map the term "tangency" still makes sense. For example, we may call the case when two backward foliations $QR\bullet$ and $QL\bullet$ meet with some forward foliation $\bullet P$ at a single point a tangency). A point of tangency on the partition line y axis ($C\bullet$) may symbolically be represented as $QC\bullet P$. The sequence UV where $U\bullet$ is between $QR\bullet$ and $QL\bullet$, and $\bullet V > \bullet P$, must be forbidden by the tangency $QC\bullet P$. (This is the meaning of a pruning or forbidden rectangle in the symbolic plane [6,7,9].) Consider a finite set of tangencies $\{Q_iC\bullet P_i\}$. If the shift of a sequence $\ldots s_{k-1}\bullet s_k s_{k+1}\ldots$ satisfies the condition that the backward sequence $\ldots s_{k-2}s_{k-1}\bullet$ is not between $Q_iR\bullet$ and $Q_iL\bullet$, and at the same time $\bullet P_i > \bullet s_k s_{k+1}\ldots$ for some i, then this shift is not forbidden by any tangencies, due to the well-ordering property of foliations. Thus, we may say that the shift is allowed according to that tangency. If all shifts of the sequence are allowed according to the set of tangencies, then the sequence is admissible. Based on the above described symbolic dynamics of the Lozi map, we shall study some typical cases in the Secs. III and IV.

III. BOUNDARY FOR THE EXISTENCE OF A FINITE ATTRACTOR

When varying the parameters of the map, the structure of the attractor is changed. For $b>0$ the smallest and greatest backward sequences are $R^\infty\bullet$ and $R^\infty L\bullet$, while the border forward sequences are $\bullet RL^\infty$ and $\bullet L^\infty$. For a fixed b at a critical value a_c of a, the foliation $\bullet RL^\infty$ is tangent to $R^\infty\bullet$ and $R^\infty L\bullet$ at the y axis. The tangency or the tangent point may be denoted by $R^\infty C\bullet RL^\infty$. The tangency condition is

$$\xi(\bullet RL^\infty)=\eta(R^\infty\bullet) , \tag{9}$$

from which the value $a_c=2-b/2$ can be determined [8]. If a is further increased, there will no longer be any finite attractor. Images ($R^\infty CRL^k\bullet L^\infty$) of the tangent point $R^\infty C\bullet RL^\infty$ are on the stable manifold $\bullet L^\infty$, while its preimages ($R^\infty\bullet R^kL^\infty$) are on the unstable manifold $R^\infty\bullet$. Foliations associated with these points are arranged according to the ordering rules as sketched in Fig. 1.

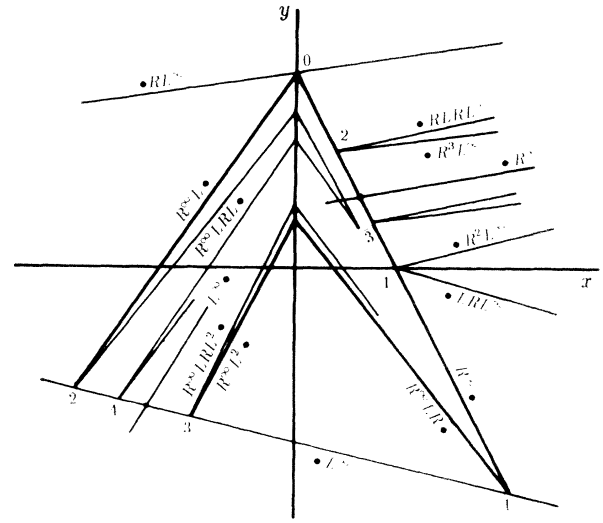

FIG. 1. Manifolds and their symbolic sequences at the critical situation of heteroclinic tangency $R^\infty C\bullet RL^\infty$ for $b>0$. The scale is not relevant. Only the relative position has a meaning. Heteroclinic points are numbered according to their sequential order on the orbit.

For $b<0$ the border forward sequences are still the same as those for $b>0$, but the border backward sequences are now $L^\infty\bullet$ and $L^\infty R\bullet$. The boundary for the existence of a finite attractor then becomes the tangency $L^\infty C\bullet RL^\infty$. This tangent point, some of its images and preimages, and foliations associated with them are shown in Fig. 2. From

$$\begin{gathered} k\equiv k(\bullet L^\infty)=\tilde{h}(L^\infty\bullet)=(a-\sqrt{a^2+4b})/2 , \\ \eta(\tilde{L}^\infty\bullet)=\frac{1}{1-k}=\xi(\bullet RL^\infty)=\frac{b}{(k+a)(k+b)} , \end{gathered} \tag{10}$$

the condition for the critical a_c is

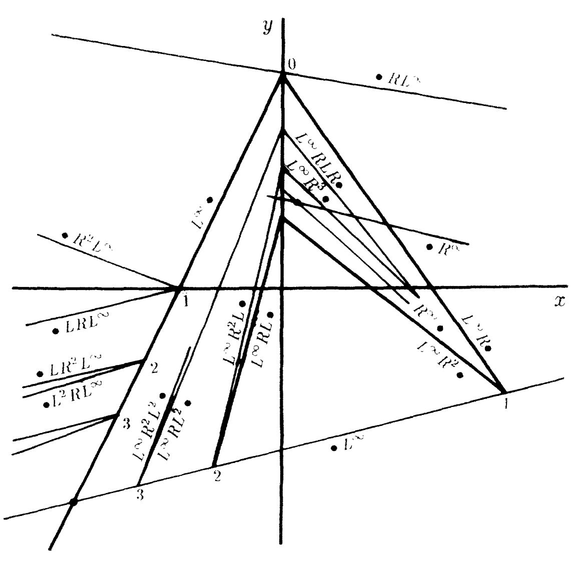

FIG. 2. Manifolds and their symbolic sequences at the critical situation of homoclinic tangency $L^\infty C\bullet RL^\infty$ for $b<0$.

TABLE I. Admissibility of periodic sequences up to period 7. Here the letter X stands for R or L, and only nonrepeating strings of the sequences are given. A shorthand notation is used. If the kth shift of the periodic sequence $P^\infty \bullet P^\infty$ is allowed or forbidden by a tangency T, we write the criterion as kT or $k\bar{T}$, respectively.

Sequence	Period	Admissibility	Criterion
X	1	Allowed	$0K$
RL	2	Allowed	$0J1J$
$RLRR$	4	Allowed	$0J1J2J3J$
RLR^3X	6	Allowed	$0J1J2J3J4J5J$
RLR^4X	7	Allowed	$0J1J2J3J4J5J6J$
RLR^2X	5	Allowed	$0J1J2J3J4J$
RLR^2LRX	7	Forbidden	$0\bar{J}$
RLX	3	Allowed	$0I1K2K$
RL^2RLR	6	Allowed	$0H1K2K3H4K5K$
RL^2RLRX	7	Forbidden	$0\bar{I}$
RL^2RX	5	Forbidden	$0\bar{I}$
RL^2R^3X	7	Forbidden	$0\bar{I}$
RL^2R^2X	5	Forbidden	$0\bar{I}$
RL^2R^2X	7	Allowed	$0I1K2K3K4I5K6K$
RL^2X	4	Allowed	$0H1K2K3K$
RL^3RLX	7	Allowed	$0H1K2K3K4I5K6K$
RL^3RX	6	Forbidden	$0\bar{I}$
RL^3R^2X	7	Forbidden	$0\bar{I}$
RL^3X	5	Allowed	$0G1K2K3K4K$
RL^4RX	7	Forbidden	$0\bar{I}$
RL^4X	6	Allowed	$0G1K2K3K4K5K$
RL^5X	7	Allowed	$0G1K2K3K4K5K6K$

$$2a^3+3a^2b-4a^2-8ab-4b^2=0 \ . \tag{11}$$

For the combination of $a=1.77798$ and $b=-0.60526$ satisfying relation (11), there are five tangencies on the y axis:

$$G\colon\ L^\infty R^2L^2C\bullet RL^6R^2L^2RL\ldots$$

$$H\colon\ L^\infty RLC\bullet(RL^3R)^2LR^2\ldots$$

$$I\colon\ L^\infty R^2C\bullet(RL^2)^3LRL^2\ldots$$

$$J\colon\ L^\infty R^4LRC\bullet RLR^2(LR)^3RLR\ldots$$

$$K\colon\ L^\infty RC\bullet R^5LR^4LR^2\ldots ,$$

from which the admissibility of all periodic orbits up to order 7 can be examined. The result is given in Table I. Furthermore, from the tangencies G and K it can be verified that any sequences consisting of only the segments RL^4 and RL^5 are always allowed, so chaotic orbits can be constructed with them.

IV. CRITICAL PARAMETERS BETWEEN ONE- AND TWO-PIECE ATTRACTORS

We shall consider only the case of $b>0$ since discussions for $b<0$ are similar. For the unstable periodic orbit P^∞ symbolic sequences of its unstable and stable manifolds are respectively of the types $P^\infty W\bullet$ and $\bullet WP^\infty$, where W stands for some finite string. Since an attractor is associated with unstable manifolds of periodic orbit on the attractor, a necessary condition for P^∞ to belong to the attractor is the existence of a homoclinic point or a sequence $P^\infty UP^\infty$, where U is a nonblank string. Assume now that the periodic orbit P^∞ is on an attractor. If no heteroclinic points from P^∞ to another periodic orbit Q^∞ exist, or sequences $P^\infty VQ^\infty$ are forbidden for any finite string V, then Q^∞ cannot be on the attractor. However, a heteroclinic point from Q^∞ to Q^∞, or a sequence $Q^\infty UP^\infty$ may still exist, which means that unstable manifolds of Q^∞ will be attracted to stable manifolds of P^∞, and hence finally to the attractor.

It is emphasized that in two-dimensional maps the symbolic dynamics is specified by an infinite number of parameters, one for each tangency on the partition line [6]. The condition that the fixed point R^∞ possess homoclinic points is necessary for the existence of a one-piece attractor, but it is not sufficient. If no sequences of the type $(RL)^\infty UR^\infty$ exist, the attractor cannot be a one-piece attractor. In Ref. [8] the criterion for an attractor to change from one- to two-piece form is that the x coordinate of the point $(LR)^\infty CR^2\bullet$ coincide with that of the point $(R^3LRL)^\infty\bullet(R^3LRL)^\infty$. For $b=0.5$ the criterion gives $a^*=1.556\ldots$, which is close to the value $a^*=1.555\ldots$ from the computer simulation. However, the criterion is not so directly related to manifolds. As discussed in the above, we may estimate a^* by looking for the appearance or disappearance of sequence $(RL)^\infty WR^\infty$ for a certain length of string W. The procedure is as follows: The upper tip of the unstable manifold $(RL)^\infty\bullet$ is $(x,y)=[0,\eta((RL)^\infty\bullet)]$. From Eq. (5) it can be derived that

$$\begin{aligned} &\tilde{h}((RL)\infty\bullet)\equiv\tilde{h}=(-a+\sqrt{a^2-4b})/2,\\ &\eta^{-1}((RL)^\infty\bullet)=1-b+(1-a)\tilde{h} \ . \end{aligned} \tag{12}$$

We then generate the forward sequence of the point for decreasing a, starting from a rather large a below the value a_c. For example, for $b=0.5$ one can see the forward sequence

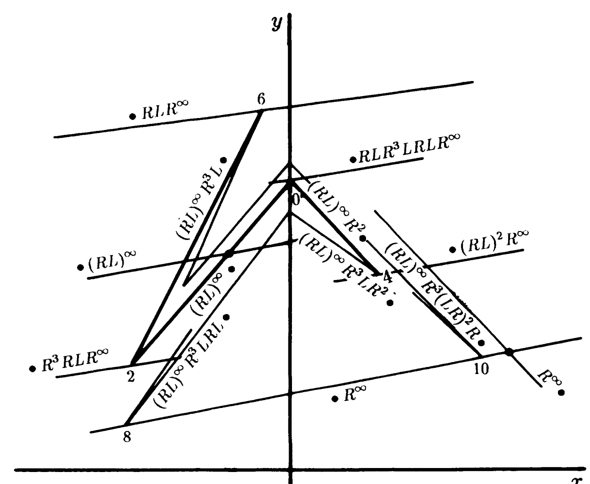

FIG. 3. Manifolds and their symbolic sequences at the heteroclinic tangency $(LR)^\infty C\bullet RLR^3LRLR^\infty$ at a rather large b. This is a case close to the critical transition from a one- to a two-piece attractor.

$\bullet RLR^3LRLR^{11}L\ldots$ at $a=1.5637$

$\bullet RLR^3LRLR^{16}L\ldots$ at $a=1.5638$

$\bullet RLR^3LRLR^{10}L\ldots$ at $a=1.5639$.

From the ordering rules we know that the forward sequence $\bullet RLR^3LRLR^\infty$ is between $a=1.5637$ and 1.5638. The real value is $a=1.563\,7947$, for which some stable and unstable manifolds are sketched in Fig. 3. Here the string $W=RLR^3LRL$ is of the length 8. Increasing the length of W, we may obtain better estimates. For example, the forward sequence $\bullet(RLR^3L)^2RLR^\infty$ is between $a=1.5578$ and 1.5579, and $\bullet(RLR^3L)^3RLR^\infty$ between $a=1.556\,98$ and 1.556 99. These forward sequences are related to the orbit $(RLR^3L)^\infty$ of period 6. An even better estimate is given by $\bullet(RLR^3L)^2(RL)^3(R^3LRL)^2R^kL\ldots$, with $k=21$ and 22 at $a=1.555\,682\,108$ and 1.555 682 109. Since the parameters are close to those of the homoclinic tangency $R^\infty RLR^\infty$, most sequences consist of only the segments RR and RL. An even number of successive R in a sequence will indicate the appearance of R^∞ at a nearby value of a.

For $b=0.38$ we find the forward sequence

$\bullet RLR^5LR^{14}L\ldots$ at $a=1.4822$,

and

$\bullet RLR^3LRLR^{14}L\ldots$ at $a=1.4807$.

The latter sequence is the smaller of the two, so the tangency between its manifold and the unstable manifold $(RL)^\infty\bullet$ happens at the smaller a, which then dominates the estimation of the critical a. At $b=0.3675$ we find two close values of $a=1.4728$ and 1.4727 for sequences $\bullet RLR^5LR^{12}L\ldots$ and $\bullet RLR^3LRLR^{14}L\ldots$, respectively. Further reducing b will stop the appearance of sequences of the latter type. In other words, the crossing between $(LR)^\infty\bullet$ and $\bullet RLR^3LRLR^\infty$ is now forbidden by some tangency. For $b=0.25$ we find the forward sequence $\bullet RLR^5LR^{13}L\ldots$ at $a=1.4424$, and $\bullet RLR^5LR^{14}L\ldots$ at $a=1.4425$. An estimate from $\bullet RLR^5LR^{28}L\ldots$ is $a=1.442\,468\,9$. Some manifolds at the tangency $(LR)^\infty CRLR^5LRLR^\infty$ are shown in Fig. 4.

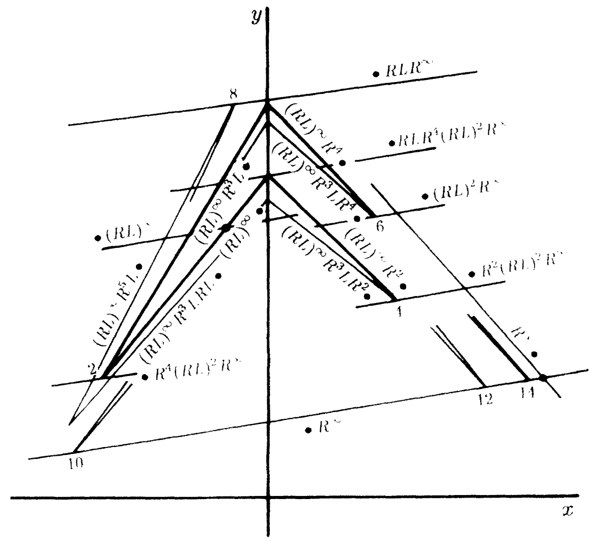

FIG. 4. Manifolds and their symbolic sequences at the heteroclinic tangency $(LR)^\infty C\bullet RLR^5LRLR^\infty$ for a rather small b.

V. CONCLUSIONS

In the above we have described symbolic dynamics of the Lozi map for both $b>0$ and $b<0$. The map with a positive Jacobian ($b<0$) plays an important role for analysis of dynamics in differential systems [10]. Generally, symbolic dynamics reflects topological properties of systems, and is not very directly relevant to metric properties. Although it does not solve problems concerning the stability of an attractor, it is still of much help in understanding the structure of attractors. Based on symbolic dynamics, we have determined the critical parameters between one- and two-piece attractors. When some tangencies on an attractor are known, we can obtain the geometrical structure of the attractor from the ordering rules of foliations.

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