Multifractality and multiscaling in collision cascades

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It is shown that self-ion collisional cascades exhibit multifractality. The structure of the cascades was studied by analyzing the length distribution $n(l, L)$ of the free flight path l traveled by particles during the cascade evolution. It was found that the collisional cascade can be partitioned into an infinity of subsets, each one characterized by a fixed value of $x = \frac{\ln l}{\ln L}$ (where L is the system size) and a distinct value of the fractal dimension $\phi(x)$. The mechanism for multifractality based on an underlying multiplicative process is illustrated on a simple geometrical model. The multiscaling structure of the function $n(l, L)$ is discussed.

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I. MOTIVATIONS

In the present paper, we investigate the multifractal structure of self-ion collision cascades in the framework of the binary collision approximation (BCA). Our aim is to clarify the origin of multifractality in collision cascades and obtain a scaling form for the length distribution function on the basis of the multifractal spectra.

An energetic ion penetrating into a solid loses part of its kinetic energy through elastic collisions with the target atoms. The energy transferred into atomic motion gives rise to a collision cascade. From the viewpoint of the cascade geometry the scattered and the recoiled particles are indistinguishable. Thus, the whole system can be considered as a treelike geometrical object which is composed of trajectories of the moving particles and points where the collisions occurred. The limitations of the present technology preclude the possibility of following the cascade process by direct measurements, thus we study multifractal properties of collision cascades generated by Monte - Carlo models in the BCA approximation.

The basic idea of multifractality is that by defining a distribution on the object (fractal measure) a richer structure can be revealed. Namely, the whole set can be partitioned into an infinite hierarchy of subsets with own fractal dimensions, and the spectrum of these dimensions (instead of a single dimension) gives the full characterization of the object. Moreover, within the framework of the multifractal formalism the scaling behavior of the distribution function can also be formulated and interpreted [1, 2]. This characterization stresses again the necessity of considering a spectrum of exponents in order to describe the scaling properties of fractal systems. In several cases, the basic macroscopic quantities of the structures can be obtained as moments of the associated distribution [2] so the study of their scaling behavior can be unified within the framework of multifractal formalism. Because of these properties, multifractality has proven to be a valuable tool in the study of numerous systems of statistical mechanics. It enables us to understand the complexity and richness of structures created by nonequilibrium processes. Examples can be mentioned from a wide range of phenomena like growth processes [such as diffusion-limited aggregation (DLA), ballistic deposition, and river network], percolation, and fracture [1–10].

Collisional cascades have been investigated recently from the viewpoint of fractal geometry [11–15], but these investigations have been restricted to the study of the self-similarity properties of the cascade, to the determination of its fractal dimension for different interaction potentials, and to the study of cascade-subcascade transitions and spikes. A simple deterministic model was established [11] (called fractal-tree) as an average cascade in the case of inverse power potential of the type $V(r) = G(m)r^{\frac{-1}{m}}, 0 < m \leq 1$. It was found that the fractal dimension of the tree $D = \frac{1}{2m}$ depends only on the parameter of the potential. This result is supported by Monte-Carlo simulations. In the more realistic case of the universal potential [16] the situation is more involved, as for the characterization it is necessary to introduce an average fractal dimension [13] as a function of the bombarding energy and of the mass of the projectile.

We try to get a deeper insight into the cascade structure by analyzing its multifractal properties. It will be shown that the cascade process naturally leads to multifractality within the framework of the BCA approximation, so simple fractal models as mentioned above cannot give a good description.

The basic quantity of our treatment is the length distribution $n(l, L)$ of the free flight path traveled by particles during the cascade evolution. The macroscopic quantities of the cascade, the average total number of collisions N, and the average total free-flight path R can be obtained as the 0th and the 1st moments of this distribution, respectively. Therefore, the scaling behavior of the length distribution determines that of the macroscopic quantities.

The outline of the remainder of this paper is as follows. In Sec. II. the multifractal concept is briefly reviewed. It should be emphasized that our formalism is based on the multifractal formalism established by Amitrano *et al.* [1] and it is slightly different from the concept of Halsey et. al. [17], which uses the box-counting method. In Sec. III a simple geometrical model is presented for collision cas-

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cades as the modification of the model known in the literature. The length distribution of this hierarchical model was calculated at the final collisional generation and it is shown that this distribution is log-binomial leading to multifractality. The full distribution including all the generations is composed of such kind of subsets. We used this model to illustrate how multifractality arises in the cascade and we found a scaling form for $n(l,L)$. It turned out that the fundamental variables of the system are not l and L but $x = \frac{\ln l}{\ln L}$. In Sec. IV numerical results are presented. To test our predictions, we performed Monte Carlo simulations in two different model systems.

II. MULTIFRACTALITY

The basic concepts of multifractality is worth reviewing [1, 3, 17] to connect it with the observables of the cascade and to fix the notations for the remaining part of the paper.

For studying multifractality at first, we have to argue on the system size L. The basic quantity of our treatment is the length l of the free flight path traveled by particles during the cascade evolution. The minimum value of the length of the branches in a cascade $l_{\min}$ is defined by the cutoff energy of the model where the particle is considered to be at rest, the maximum value is the length L of the path traveled by the projectile up to the first collision. L depends only on the bombarding energy $L = L(E_0)$. That is why L can be introduced as an independent variable instead of the bombarding energy; it plays the role of the system size in our considerations.

The complete information about the length distribution is contained in its moments. Let us consider these moments as functions of L:

$$Z(q,L) = \sum_i l_i^q \,; l_{\min} \le l_i \le L, \tag{1}$$

the sum is over all the branches, l_i is the length of the ith branch and $-\infty < q < +\infty$. Let us rewrite (1) as

$$Z(q,L) = \int n(l,L) l^q d \ln l = \int e^{F(l,L)} d \ln l \ , \tag{2}$$

$$Z(q,L) \sim L^{\xi(q)} \ , \tag{3}$$

where $n(l,L) d \ln l$ is the number of branches with length in $[\ln l, \ln l + d \ln l]$ and $F(l,L) = \ln n(l,L) + q \ln l$. It can be seen that the 0th moment is the average total number of collisions $N(L)$ and the 1st moment is the average total free fligth path $R(L)$ in the cascade. Other moments provide useful additional information: higher (lower) moments for example are related to the tail of the distribution corresponding to long (short) branches. The asymptotic relation serves to define the exponents $\xi(q)$ for sufficiently large L, assuming that this behavior exists.

In performing multifractal analysis the lengths of the branches are usually normalized. The normalized length of the ith branch, the normalized moments, and the proper asymptotic relation are as follows:

$$l_i' = \frac{l_i}{Z(1)}; \quad Z'(q) = \frac{Z(q)}{[Z(1)]^q}; \quad Z'(q) \sim L^{-\tau(q)}. \tag{4}$$

We do not use this normalization procedure. Following the approach of Refs. [1] and [17] the integral in (2) can be evaluated by the steepest descent method. If l^* is the value for which $F(l,L)$ has a maximum we have

$$\frac{\partial \ln n(l,L)}{\partial \ln l}\bigg|_{l=l^*} = -q. \tag{5}$$

In general, there is a corresponding value of $l^* = l^*(q)$ for each value of q, l^* being the length of branches giving the maximal contribution to the qth moment of the length distribution. We made the following scaling ansatz:

$$l^* = A(q) L^{-\alpha(q)}, \tag{6}$$

$$n(l^*) = B(q) L^{f(q)}, \tag{7}$$

therefore,

$$Z(q) \sim e^{F(l^*,q)} \sim L^{-[q\alpha - f(q)]}. \tag{8}$$

Using $\frac{\partial}{\partial l} = \frac{\partial q}{\partial l}\frac{\partial}{\partial q}$ and neglecting terms of the order $1/\ln L$ one obtains from (5):

$$\alpha(q) = -\frac{d\xi(q)}{dq} \quad \text{and} \quad \xi(q) = f(q) - q\alpha(q), \tag{9}$$

with α as the singularity exponent. Since l^* takes all the possible values of l as q varies from $-\infty$ to $+\infty$, it can be considered as an independent variable and we denote it by l.

In the way outlined above the exponent $\xi(q)$ characterizing the qth moment of the length distribution decomposes into two factors. $f(q)$ represents the dependence of $n(l,L)$ on L so this is the fractal dimension of the subset of branches with length l. Similarly, α characterizes the manner how $l(q)$ scales with the system size L. If $\xi(q)$ were a linear function, the moments of the distribution could be described by a single exponent and $f(q)$ and $\alpha(q)$ would be constant. This simple type of scaling is usually found in critical phenomena and it is called gap scaling. In our case, $\xi(q)$ is a more complicated function and $f(q), \alpha(q)$ have a nontrivial q dependence. This will be verified both by analytical calculations on a hierarchical model and by numerical simulations. The anomalous behavior of the length distribution is responsible for the infinite set of moment exponents $\xi(q)$.

Using the knowledge of $\alpha(q)$ and $f(q)$, we can write $n(l,L)$ in a scaling form. Let us define the new quantity $x = \frac{\ln l(q)}{\ln L}$, which turns out to be the fundamental variable of our system. It will be shown that $n(l,L)$ is a scaling function of this fundamental variable. For large L we can write $x = \frac{\alpha(q)}{\alpha(\infty)}$ from (6) neglecting the terms of order $1/\ln L$.

It is known from the general theory that α is a monotonous function of q. Thus we can invert $x(q)$, so as $q = q(x)$. This way we are able to express q in terms of physical quantities. We find from (7)

$$n(l,L) = C(x) L^{\Phi(x)}, \tag{10}$$

where

$$\Phi(x) = f(q(x)), \qquad C(x) = B(q(x)). \tag{11}$$

It can be seen that assuming the nonlinearity of $\xi(q)$, the length distribution can be written as a power law in L, but the exponent is length dependent. This scaling form expresses the fact that the collision cascade can be partitioned into subsets, each one being characterized by the value of $x = \frac{\ln l}{\ln L}$. Each subset has an independent fractal dimension $\Phi(x)$ and a singularity exponent α. Similar anomalous scaling behavior was found in growth phenomena, where the growth probability distribution has these features [4–6, 8] in percolation, where the voltage distribution at the percolation threshold exhibits this behavior [2]. The same was observed in fractions [3] and river networks [7] as well.

In the following section we analyze a hierarchical model to illustrate how this type of anomalous scaling behavior arises in collisional cascades.

III. LENGTH DISTRIBUTION OF THE HIERARCHICAL MODEL

In a self-ion collision cascade where the projectile and the target atoms are the same, the scattering angle falls between 0° and 90°. The geometrical model of self-ion collisional cascades established by Cheng. *et al.* as an average cascade [11] is a deterministic one using the single value 45° of the scattering angle (Fig. 1). We have modified this model by allowing any value between 0° and 90° and discuss the model of Cheng as a limiting case. Our model calculation illustrates how the multiplicative mechanism of the cascade process leads to multifractality. Let us denote the ratios of lengths traveled by the scattered and the recoiled particles to the L by l_1 and l_2, where $l_1 < l_2 < 1$. With these similarity ratios a two-scale cascade tree can be generated, which is rigorously self-similar up to the scale of the entire system (L). Our aim was to analyze the total length distribution in the cascade. It turned out to be convenient to analyze at first only one collisional generation. Therefore, we start with considering only one particular generation, the last one, containing the cutoff length of the model $l_{\min}$. In the Nth collisional generation the possible different lengths are $l(i) = Ll_1^i l_2^{N-i}$, $i = 0, 1, ..., N$, and the number of branches with the length $l(i)$ is given by

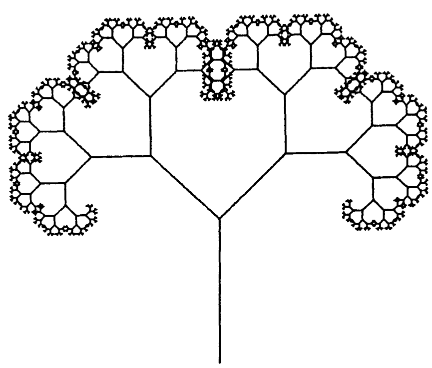

FIG. 1. Fractal tree in the deterministic model of Cheng *et al.* for the similarity ratio $l_0 = 0.7$.

$$n(l(i)) = \binom{N}{i}. \tag{12}$$

This is a simple binomial in i, but i varies logarithmically with l. Thus, we conclude that the length distribution in the Nth generation is log binomial.

The moments of this distribution can be easily calculated:

$$Z(q, L) = L^q \left[l_1^q + l_2^q\right]^N. \tag{13}$$

To calculate the moment exponents $\xi(q)$ we have to connect the generation number with the physical length scale. The cutoff value of the model $l_{\min}$ is fixed, it should be the length of the shortest branch in the tree. In our two-scale cascade tree the shortest branch has the length $l_1^N \times L$. From $l_{\min} = l_1^N \times L$ we get the connection between N and L: $N = \frac{\ln l_{\min} - \ln L}{\ln l_1}$. Then we find

$$\xi(q) = -\frac{1}{\ln l_1} \ln\left[1 + \left(\frac{l_2}{l_1}\right)^q\right]. \tag{14}$$

It should be noted that $\xi(q)$ is a nonlinear function of q, thus an infinite set of exponents is required to describe the scaling behavior of the moments. The exponents describing the scaling of $n(l^*)$ and l^* are given by

$$\alpha(q) = -\frac{\partial}{\partial q}\xi(q) = \frac{\ln(l_2/l_1)}{\ln l_1} \frac{1}{1 + \left(\frac{l_1}{l_2}\right)^q}, \tag{15}$$

$$f(q) = \xi(q) + q\alpha(q)$$
$$= \frac{1}{\ln l_1}\left\{\frac{\ln\left(\frac{l_2}{l_1}\right)^q}{1 + \left(\frac{l_1}{l_2}\right)^q} - \ln\left[1 + \left(\frac{l_2}{l_1}\right)^q\right]\right\}. \tag{16}$$

Now we demonstrate how these exponents can be obtained from the opposite direction by taking first the continuum approximation for $n(l, L)$ and using the approach outlined in Sec. II. For the continuum limit, we used the lowest order form of the Stirling approximation $\ln k! \sim k \ln k - k$ and replaced the sum by its saddle point approximation:

$$Z(q, L) = \sum_{l_i} n(l_i) l_i^q \sim n(l_i^*) l_i^{*q}, \tag{17}$$

where l_i^* satisfies the equation

$$\left.\frac{\partial \ln(n(l))}{\partial \ln l}\right|_{l=l^*} = -q. \tag{18}$$

From (5) and (12) we get

$$i(q) = N \frac{1}{1 + \left(\frac{l_2}{l_1}\right)^q}$$

and

$$l^*(i(q)) = Ll_1^{i(q)} l_2^{N-i(q)} \sim L^{-\alpha(q)}. \tag{19}$$

From these equations we find that

$$\alpha(q) = \frac{\ln(l_2/l_1)}{\ln l_1} \frac{1}{1+\left(\frac{l_1}{l_2}\right)^q}, \tag{20}$$

and from (5) and (12)

$$n(l^*(i(q))) = \binom{N}{i(q)} \sim L^{f(q)}, \tag{21}$$

and the fractal dimension using the Stirling formula is

$$f(q) = \frac{1}{\ln l_1}\left\{\frac{\ln\left(\frac{l_2}{l_1}\right)^q}{1+\left(\frac{l_1}{l_2}\right)^q} - \ln\left[1+\left(\frac{l_2}{l_1}\right)^q\right]\right\}. \tag{22}$$

Finally, substituting (20) and (22) into (9), the moment exponent $\xi(q)$ takes the form

$$\xi(q) = -\frac{1}{\ln l_1}\ln\left[1+\left(\frac{l_2}{l_1}\right)^q\right]. \tag{23}$$

Relations (20), (22), and (23) are identical to (14), (15), and (16). Thus, we have shown that the moment exponents $\xi(q)$ can be obtained correctly from the continuum form of the length distribution by applying the general formalism of Sec. II.

In the continuum limit the length distribution can be cast into a scaling form:

$$n(l,L) = C(x)L^{\Phi(x)}, \tag{24}$$

where

$$\Phi(x) = \frac{1}{\ln l_1}\left[x\ln x + (1-x)\ln(1-x)\right], \tag{25}$$

$$C(x) = \frac{\ln l_{\min}}{\ln l_1}\left[x\ln x - (1-x)\ln(1-x)\right]. \tag{26}$$

Here, $n(l,L)$ shows a power law behavior as a function of L but both the exponent $\Phi(x)$ and the coefficient $C(x)$ depend on the ratio of the logarithms of lengths x. Figure 2 shows $f(q(x))$ for various l_1 values. It is easy to show that $\Phi(x) = f(q(x))$. Furthermore, $\Phi(x)$ does not contain any specific information on the collisional generation considered. It is valid for all sufficiently small $l_{\min}$ values, only $C(x)$ depends on $l_{\min}$.

So far we have considered only one particular generation. The total length distribution in the cascade contains all the collisional generations up to the generation of the cutoff value. To calculate the moments of this total distribution we have to sum over the generations k:

$$Z(q,L) = L^q\sum_{k=0}^{N}[l_1^q + l_2^q]^k = L^q\frac{[l_1^q + l_2^q]^{N+1} - 1}{l_1^q + l_2^q - 1}, \tag{27}$$

with the Nth generation containing the cutoff length $l_{\min}$. The moment exponents can be calculated analytically only in the large and low q limits:

(i) If $q >> 1$, then $Z(q,L) \sim L^q$, thus, $\xi(q) = q$, $\alpha(q) = -1$, and $f(q) = 0$. In this limit the moments $Z(q,L)$ are dominated by the longest branch of length L. (ii) If $q << 1$, then $Z(q,L) \sim L^q\,[l_1^q + l_2^q]^N$, thus, $\xi(q) = -\frac{1}{\ln l_1}\ln\left[1+\left(\frac{l_2}{l_1}\right)^q\right]$. This is identical with (14) so in this region $\alpha(q)$ and $f(q)$ are also identical with (15) and (16), respectively. Thus in this low q limit the last generation dominates the moments. Our arguments above can be generalized to the case when the similarity ratio l_1 varies stochastically according to a probability distribution, similarly to the β model of the turbulent flow [18].

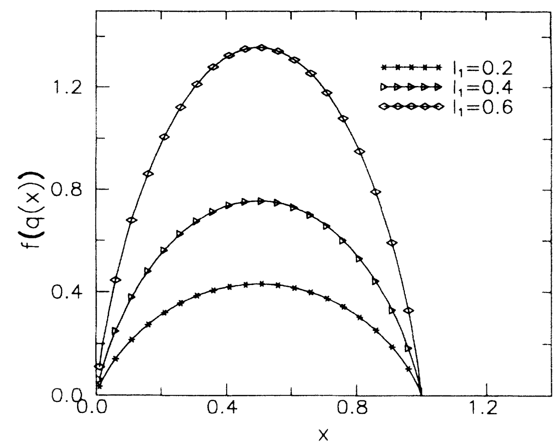

FIG. 2. Function $f(q(x))$ of the idealized model for various l_1 values. The change of l_1 corresponds the change of the scattering angle.

The particular case of $l_1 = l_2 = l_0$ is worth considering, it corresponds to the deterministic model of cascades known in the literature as a case allowing the scattering angle 45° only [11]. In this model the potential $V(r) = G(m)r^{\frac{-1}{m}}$ results in $l_0 = \left(\frac{1}{2}\right)^{2m}$ and the object is a simple fractal with dimension $D = \frac{1}{2m}$ [11]. In this case the total length distribution $n(l,L)$ can be easily calculated since all the branches have the same length $l = l_0^k L$ in the kth generation. So the total length distribution

$$n(l,L) = L^{-\frac{\ln 2}{\ln l_0}}\, l^{\frac{\ln 2}{\ln l_0}} \tag{28}$$

exhibits a simple power law in both L and l. This property leads to constant values for both α and f. If $q > -\frac{\ln 2}{\ln l_0}$ then $\xi(q) = q$ so $\alpha(q) = -1$ and $f(q) = 0$ and the moments are dominated by the longest branch. If $q < -\frac{\ln 2}{\ln l_0}$ then $\alpha(q) = 0, f(q) = -\frac{\ln 2}{\ln l_0}$, this describes how the number of the shortest branches scales and for $l_0 = \left(\frac{1}{2}\right)^{2m}$ we get $f(q) = D = \frac{1}{2m}$, which coincides with the dimension of Cheng for the cascade [11]. Such a simple scaling behavior was found by Meakin *et al.* [6] for polygons, needles, and stars studying the harmonic probability measure on these objects.

The crucial point to be stressed here is that we do not find multifractality in this symmetric case ($l_1 = l_2$), but it naturally arises in the nonsymmetric ($l_1 \neq l_2$) and stochastic cases. In the next section, we present numerical results which verify that collisional cascades exhibit multifractal properties and the multiscaling structure of the distribution function can be understood on the basis of the multifractal spectra. The deterministic model

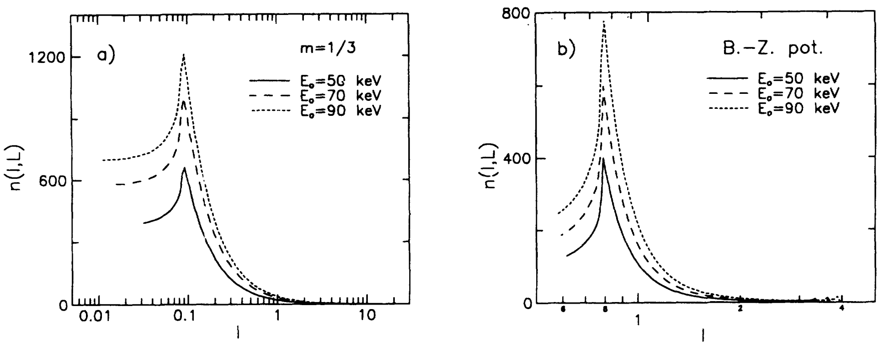

FIG. 3. Length distribution $n(l, L)$ of the model systems for various system sizes L (i.e., bombarding energies E_0). They are normalized by the number of events. For the inverse power law potential the case of $m = 1/3$ is shown as an illustrative example.

of cascades in the literature is based on the facts that in the hard sphere approximation the expectation value of the particle energy (scattered and recoiled) is given by $\bar{E}_p = \frac{1}{2}E$ (which belongs to the 45° scattering angle) and the main free-flight path has the simple energy dependence $L_{kv}(E) = KE^{2m}$ (K is independent of the energy) [11]. Thus the average cascade might be substituted by a deterministic one with the similarity ratio $l_0 = \left(\frac{1}{2}\right)^{2m}$ being the ratio of two successive branches at the scattering angle 45°. This results in the simple scaling behavior of the length distribution. These arguments can be generalized. The geometrical properties of the average cascade can be approximated by a symmetric model ($l_1 = l_2$) if the expectation value of the particle energy has a form $\bar{E}_p = CE$ (C is independent of E) and if the main free-flight path is a homogeneous function of the energy, which means that $L_{kv}(\lambda E) = \lambda^a L_{kv}(E)$, for any value of λ, where a is the order of homogeneity. In this case, the similarity ratio of the substituting deterministic object is $l_0 = C^a$ and its fractal dimension is $D = -\frac{1}{a}\frac{\ln 2}{\ln C}$. Since we have shown that multifractal properties and the scaling structure of the distribution function cannot be understood on the basis of a symmetric model, these models should be considered as a first approximation only.

IV. NUMERICAL ANALYSIS OF THE LENGTH DISTRIBUTION

To test our predictions for the scaling behavior of the length distribution and its moments we have performed numerical simulations in two different model systems distinguished by the interaction potential applied. In the case of the inverse power potential we used the scattering cross section $d\sigma = CE^{-m}T^{-1-m}dT$. In order to get rid of the divergency around $T = 0$ the minimum possible value of the scattering angle was fixed at Θ_0. In the case of the realistic Biersack-Ziegler potential (B.-Z. pot.), the cross-section cannot be obtained in a closed form, thus simulation was based on the potential function itself and the impact parameter b was chosen at random in an interval $b \in [0, b_{\max}]$.

Simulations were performed in the 10 keV–100 keV bombarding energy range with an increment of 10 keV for both systems. The parameter of the inverse power potential (m) was chosen $m = \frac{1}{2}, \frac{1}{3}, \frac{1}{6}$ and the cutoff angle was $\Theta_0 = 5°$. Averages were done over 3000 events at each particular bombarding energy.

Figure 3 shows the length distribution at different system sizes for both model systems. For the evaluation of the length distribution we used logarithmic bining of the length to obtain a nonsingular shape because of the log-binomial-like distribution. We adjusted the horizontal scales of the figures so that the increment in $x = \frac{\ln l}{\ln L}$ is the same for every system size, the left edge of the distribution being defined by $l_{\min} = L^{x_0}$, where x_0 is fixed for a particular model system. It can be seen that all the distributions have long tails at the edge of the longest branches and the sharp maximum in the range of the shorter ones.

To demonstrate that the moments of the length distribution scale independently we calculated the quantity

$$z(q) = \left(\frac{Z(q)}{Z(0)}\right)^{\frac{1}{q}} \tag{29}$$

as a function of the system size L. If the function $\xi(q)$

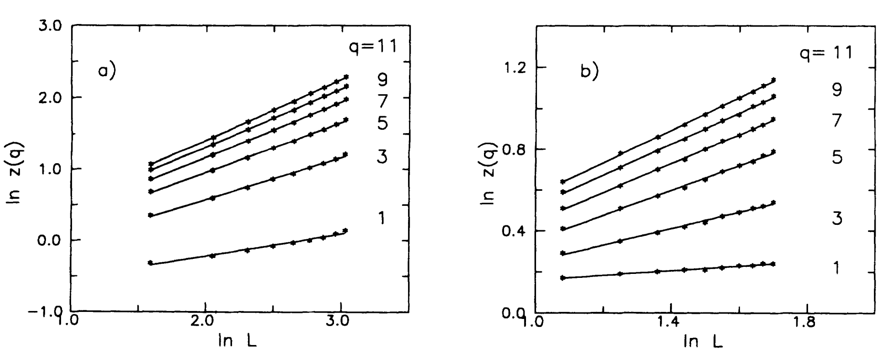

FIG. 4. $z(q)$ vs L for various q values. The fitted straight lines are not parallel, showing that the moments of the length distribution scale independently.

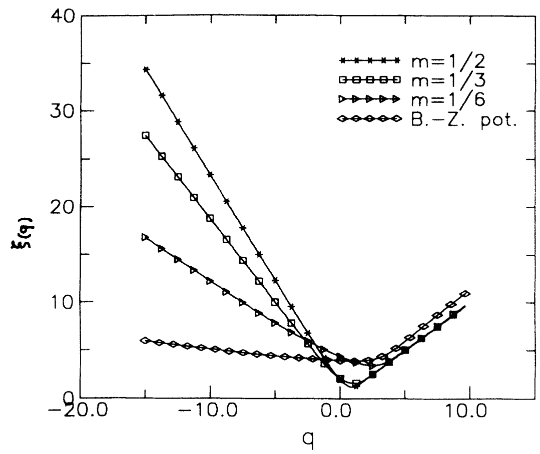

FIG. 5. Scaling exponents $\xi(q)$ of the moments $Z(q)$. In the case of the inverse power law potential $\xi(q)$ depends strongly on m for $q < 0$ and ≈ 0. This nontrivial q-dependence expresses the anomalous scaling of $Z(q)$.

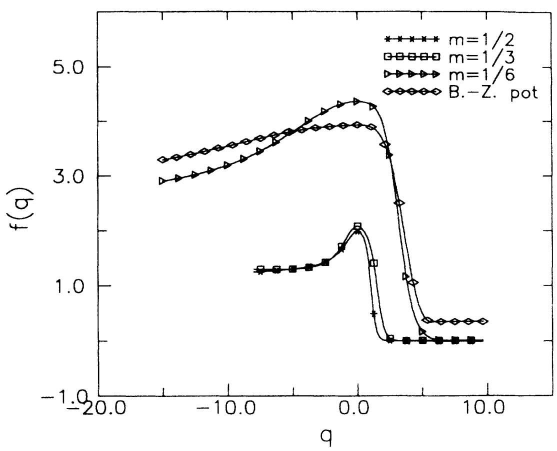

FIG. 6. Fractal dimensions $f(q)$ of the subsets giving the maximum contribution to $Z(q)$. In the case of the inverse power law potential the peak around the maximum of $f(q)$ becomes wider for $m \to \frac{1}{6}$. The curve of the B.-Z. pot. is in an intermediate place.

were linear (constant-gap scaling) the curves of $\ln z(q)$ vs $\ln L$ should be straight lines parallel to each other for different q values. As an illustrative example Fig. 4 shows $z(q)$'s for both systems. It is evident that the moments scale independently.

Then the functions $\xi(q)$, $\alpha(q)$, and $f(q)$ were calculated by using the multifractal formalism of Sec. II. It is worth noting again that $\xi(0)$ and $\xi(1)$ describe how the average total number of collisions $N(L)$ and the average total free-flight path $R(L)$ of the cascade scale as a function of the system size. Figure 5 shows the $\xi(q)$'s and Fig. 6 the associated functions $f(q)$, these are the fractal dimensions of the subsets giving the maximum contribution to the moments $Z(q)$. $f(q=0) = \xi(q=0)$ and the limiting cases $q \to \pm\infty$ describe how the number of the longest and shortest branches scales, respectively.

Besides multifractal analysis our aim was to cast the length distribution into a scaling form on the basis of the multifractal spectra. From the general theory of Sec. II it turned out that for a system showing multifractality the natural variable is $x = \frac{\ln l}{\ln L}$ instead of l and L. So a scaling form can be found for $n(l, L)$ as a function of x, rather than as a function of l and L. This is caused by the log-binomial-like distributions. The scaling form is

$$n(l, L) = C(x) L^{\Phi(x)}, \tag{30}$$

where $\Phi(x)$ should be equal to $f(q(x))$, $C(x)$ depends also only on x.

To prove the validity of this scaling structure two different tests were performed. At first we calculated $f(q)$ from the general theory of Sec. II and then inverting the function $x(q) = \frac{\ln l(q)}{\ln L}$, we obtained $f(q(x))$. Furthermore, we derived $\Phi(x)$ by independent means. Fixing values for $x = \frac{\ln l}{\ln L}$, $\ln n(l, L)$ was calculated as a function of $\ln L$, then straight lines were fitted to these data. $\Phi(x)$ was obtained as the slope of the straight lines at a given x and from the additive term we evaluated the function $C(x)$. Figure 7 shows the comparison of $f(q(x))$ and $\Phi(x)$ obtained independently in both model systems. A reasonable agreement was found between the two curves, they differ only slightly in the vicinity of the maximum.

The second test is the usual data collapse analysis. If the scaling form (30) is valid then $C(x) = n(l, L)/L^{\Phi(x)}$ should depend solely on x, thus calculating it for different

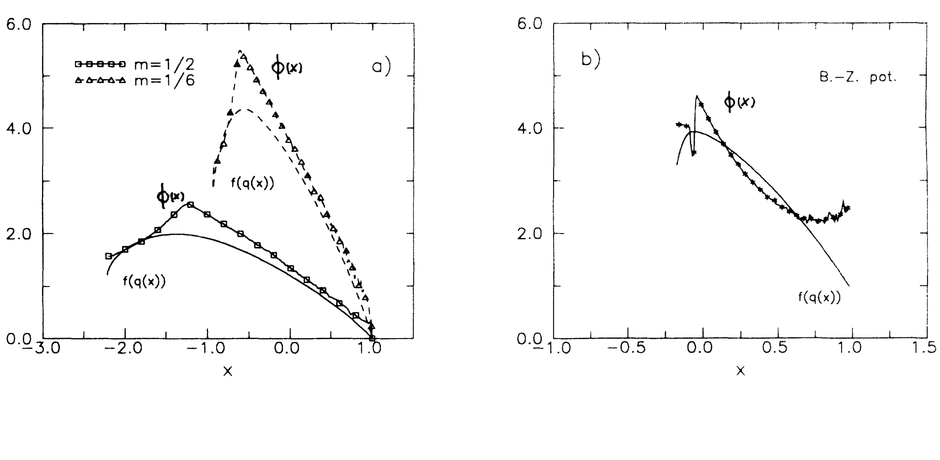

FIG. 7. Comparison of the functions $f(q(x))$ and $\Phi(x)$ obtained independently. Their good agreement verifies the validity of the scaling structure (30).

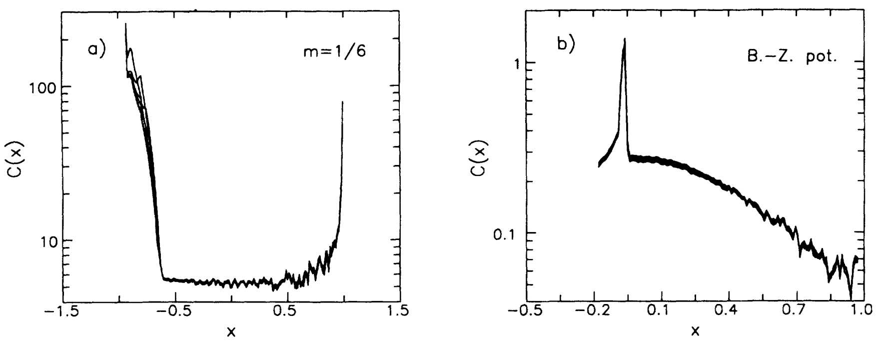

FIG. 8. Results of the data collapse analysis. The excellent collapse of the amplitude functions $C(x)$ is shown for ten different system sizes.

system sizes these curves should collapse onto each other. Figure 8 shows that we found a sufficiently good collapse.

V. DISCUSSION

In the present paper, we have studied the structure of self-ion collision cascades by the means of multifractality. We focused our attention on the length distribution of the free-flight path traveled by particles during the cascade evolution and made calculations in two different model systems. It was presented that the moments of the length distribution scale independently, which is a direct consequence of the fact that the length distribution can be written naturally in terms of the logarithm of the length. It turned out that the cascade tree can be partitioned into fractal subsets corresponding to particular values of $x = \frac{\ln l}{\ln L}$. These fractal subsets are described by the fractal dimensions $\Phi(x)$ and the singularity exponents $\alpha(x)$, respectively. This multifractal structure leads to the anomalous scaling behavior of the distribution function. We illustrated on a simple hierarchical model that multifractality does arise in a collision cascade because of the underlying multiplicative process. It was found that a symmetric model cannot account for multifractality observed in Monte Carlo simulations.

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