Dynamics of a nonlinear lattice with randomly distributed masses

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A nonlinear lattice with randomly distributed masses is considered within the framework of the mean field method. It is stated that the mean field is described by an equation that reduces to the Burgers one in the case of small scale inhomogeneities. The law of the mean field amplitude decay is obtained to be $t^{-1/2}$.

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I. INTRODUCTION

A great number of investigations devoted to the nonlinear lattice dynamics are based on the so-called long wave approximation, which allows reduction of the system of differential-difference equations to the partial differential ones. Detailed discussion of the application of this method to inhomogeneous lattices has been reported recently [1]. As has been shown, it is possible to classify the respective problems on the basis of the relations among the wavelength, the scale, and the amplitude of inhomogeneities. Though the partial differential equation obtained with the help of that technique depends on the scaling of the problem, it has been stated that the equation of the continuum limit is of the Korteveg–de Vries (KdV) type. The advantages of the long wave approximation are evident. First, it is much easier to deal with partial differential equations rather than with discrete ones. Second, in some cases the final nonlinear equation appears to be exactly integrable, which allows one to write down the solution in the explicit form.

The long wave approximation, being essentially based on the well-defined relations among scales, requires the wavelength (or the characteristic size of the lattice excitation) to be much greater than the distance between adjacent particles. In the meantime, it has been demonstrated in [2] that the discreteness can lead to new effects not observed in the stochastic dynamics of the continuum limit. Also, sometimes while studying random structures the *mean field* is the main interest (so, for instance, the divergence of the mean field is associated to the denaturation of DNA [3]). Then, if the resulting partial differential equation is nonintegrable, the long wave approximation is only the first step of the study while the next one must be another approach.

A number of publications have been concerned with the mean field related to the nonlinear stochastic equations. So it has been discovered that after averaging the randomly "perturbed" nonlinear Schrödinger soliton [4] and the KdV soliton [5] described by the exactly integrable stochastic equations display decay of the amplitude according to the law $t^{-3/2}$. As is argued in [6] such a law is stipulated by the type of the random perturbation and is replaced by $t^{-1/2}$ for special kinds of the stochastic term. As a result of numerical experiments it has been found in [7] that the decay of a soliton of the Toda lattice with randomly distributed masses is governed by the power law n^{-p}, n being a number of a site and p a constant from the interval 1–1.2. Then, in [8] the law $t^{-1/2}$ has been obtained for the mean field of a lattice excitation in the continuum approximation when randomness can be treated as external noise.

The aim of the present paper is to provide direct expansion of the averaged solution of the randomly perturbed lattice in the weakly nonlinear limit. It will not be required for the length of the carrier wave to be large. The amplitude of the inhomogeneities will be a small parameter of the approach. As long as the proposed expansion has to take into account small scale fluctuations (and this will be the subject of main interest) it is natural that it will be based on the mean field method (see, e.g., [6]).

The paper is organized as follows. In Sec. II the equation for the mean field and the expression for the fluctuations through the mean field are derived. In Sec. III the lattice with small scale inhomogeneities is discussed. It is shown that in the definite spatial-temporal region the equation for the mean field is reduced to the Burgers equation, which allows us to estimate the law of the amplitude decay. In the last section the outcomes are summarized and discussed.

II. EQUATION FOR THE MEAN FIELD

Let us consider a lattice, the dynamics of which is described by the equation

$$(M+m_n)\frac{d^2q_n}{dt^2}=U'(q_{n+1}-q_n)-U'(q_n-q_{n-1}). \qquad (1)$$

Here q_n is the displacement of the n th particle from the equilibrium position, M is the average mass of a particle, m_n is the deviation of the real mass M_n of the n th particle from the average value: $m_n=M_n-M$, $U(q)$ is the potential of the interaction between adjacent particles, the prime stands for the derivative with respect to the argument: $U'(q)=dU(q)/dq$, and hereafter we do not write explicitly the dependence of the displacement on time. m_n are considered as random numbers.

For our purposes it is convenient to rewrite (1) in terms of the new dependent variable

$$\alpha_n=q_{n+1}-q_n. \qquad (2)$$

Evidently

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$$\frac{d^2\alpha_n}{dt^2} = \frac{1}{M}[U'(\alpha_{n+1}) - 2U'(\alpha_n) + U'(\alpha_{n-1})] - \frac{1}{M}\frac{d^2}{dt^2}\left(m_{n+1}\sum_{l=-\infty}^{n}\alpha_l - m_n\sum_{l=-\infty}^{n-1}\alpha_l\right). \quad (3)$$

Here it is taken into account that we are interested in the solutions of (1) tending to constants at infinity: $q_n \to q_\pm$ at $n \to \pm\infty$, which means that $|\alpha_n| \to 0$ at $|n| \to \infty$.

In the weakly nonlinear case the solution of (3) can be found in the form of the expansion in the series of a small parameter. To be concrete, we will assume that such a parameter μ is defined by the amplitude of fluctuations of the inhomogeneities: $\mu = O(m_n^2/M^2)$. On the other hand, as far as m_n is a random value, it is natural to represent α_n in a form of the sum of the regular (or coherent) and the random (or incoherent) components. Namely, we write down

$$\alpha_n = \alpha_c(n) + \alpha_i(n), \quad (4)$$

where $\langle\alpha_n\rangle = \alpha_c(n)$ and $\langle\alpha_i(n)\rangle = 0$ (hereafter the angular brackets designate averaging over all realizations of the random numbers m_n). Then the expansion in μ has the form

$$\alpha_c(n) = \mu\alpha_c^{(1)}(n) + \mu^2\alpha_c^{(2)}(n) + \cdots, \quad (5a)$$

$$\alpha_i(n) = \mu^{3/2}\alpha_i^{(1)}(n) + \mu^2\alpha_i^{(2)}(n) + \cdots. \quad (5b)$$

The representation (4) means that the solution of the original equation (1) will be obtained in the similar form

$$q_n = \langle q_n\rangle + q_{i,n}. \quad (6)$$

Here $q_{i,n}$ describes the noise generated by the interaction of the wave packet with inhomogeneities.

As is customary, in order to provide the expansion without secular terms one has to use the multiscale method. In our case this means that α_n depends not only on the "fast" time t, but also on the slow time $\tau = \mu t$, which is considered as an independent variable, so that

$$\frac{d^2}{dt^2} \longmapsto \frac{\partial^2}{\partial t^2} + 2\mu\frac{\partial}{\partial\tau}\frac{\partial}{\partial t} \quad (7)$$

(in what follows the consideration is restricted to the lowest orders and that is why higher scales are not discussed). Substitution of (4)–(5b) and (7) into (3) leads to the equations

$$\frac{\partial^2\alpha_c^{(1)}(n)}{\partial t^2} = U_1\Delta_n\alpha_c^{(1)}(n), \quad (8)$$

$$\frac{\partial^2\alpha_i^{(1)}(n)}{\partial t^2} = U_1\Delta_n\alpha_i^{(1)}(n) - \epsilon_{n+1}\frac{\partial^2}{\partial t^2}\sum_{l=-\infty}^{n}\alpha_c^{(1)}(l) + \epsilon_n\frac{\partial^2}{\partial t^2}\sum_{l=-\infty}^{n-1}\alpha_c^{(1)}(l), \quad (9)$$

$$\frac{\partial^2\alpha_c^{(2)}(n)}{\partial t^2} = U_1\Delta_n\alpha_c^{(2)}(n) - 2\frac{\partial}{\partial\tau}\frac{\partial}{\partial t}\alpha_c^{(1)}(n) + U_2\Delta_n[\alpha_c^{(1)}(n)]^2 - \left\{\left\langle\epsilon_{n+1}\frac{\partial^2}{\partial t^2}\sum_{l=-\infty}^{n}\alpha_i^{(1)}(l)\right\rangle - \left\langle\epsilon_n\frac{\partial^2}{\partial t^2}\sum_{l=-\infty}^{n-1}\alpha_i^{(1)}(l)\right\rangle\right\}, \quad (10)$$

$$\frac{\partial^2\alpha_i^{(2)}(n)}{\partial t^2} = U_1\Delta_n\alpha_i^{(2)}(n) + \left\{\left\langle\epsilon_{n+1}\frac{\partial^2}{\partial t^2}\sum_{l=-\infty}^{n}\alpha_i^{(1)}(l)\right\rangle - \epsilon_{n+1}\frac{\partial^2}{\partial t^2}\sum_{l=-\infty}^{n}\alpha_i^{(1)}(l)\right\} - \left\{\left\langle\epsilon_n\frac{\partial^2}{\partial t^2}\sum_{l=-\infty}^{n-1}\alpha_i^{(1)}(l)\right\rangle - \epsilon_n\frac{\partial^2}{\partial t^2}\sum_{l=-\infty}^{n-1}\alpha_i^{(1)}(l)\right\}, \quad (11)$$

in the orders μ, $\mu^{3/2}$, μ^2, and μ^2, respectively. In Eqs. (8)–(11) the following designations are introduced:

$$\Delta_n\alpha(n) \equiv \alpha(n+1) - 2\alpha(n) + \alpha(n-1),$$

$$U_1 = \frac{1}{M}\frac{\partial^2U(q)}{\partial q^2}\bigg|_{q=0}, \quad U_2 = \frac{1}{2M}\frac{\partial^3U(q)}{\partial q^3}\bigg|_{q=0},$$

$$\epsilon_n = \frac{1}{\sqrt{\mu}}\frac{m_n}{M}.$$

Since the solution $q_n = q_{n+1}$ is considered as a state of the equilibrium we have that $U_1 > 0$. Here we also define the characteristics of the inhomogeneities. In the present statement

$$\langle\epsilon_n\rangle = 0, \quad \langle\epsilon_n\epsilon_k\rangle = B(|n-k|). \quad (12)$$

Let us start with the brief discussion of the solution of the linear equation (8), which can be represented in the form

$$\alpha_c^{(1)}(n) = \frac{1}{2\pi}\int_0^{2\pi} d\phi A(\phi)\cos\left[n\phi - 2\sqrt{U_1}\sin\left(\frac{\phi}{2}\right)t\right], \quad (13)$$

where $A(\phi)$ characterizes the spectrum of the wave packet. It displays the main peculiarity of the discrete problem. Namely, there are three physical reasons for the wave packet form to change. The first two are the nonlinearity and the scattering by the inhomogeneities.

The expansions (5a) and (5b) imply that the characteristic times of both processes are of the same order, which is taken into account by the dependence of the slow time τ. Another phenomenon we have to take into account is a dispersion caused by the discreteness. Consider two particular cases.

(a) First, let $A(\phi) \equiv 2A_0$ (A_0 being a constant). Then the unperturbed solution is

$$\alpha_c^{(1)}(n) = 2A_0 J_{2n}(2\sqrt{U_1}t), \tag{14}$$

where $J_l(t)$ is the Bessel function, and in terms of q_n it describes the evolution of a steplike (or "switching" solution) between the states $q_- = 0$ (at $n = -\infty$) and $q_+ = 2A_0$ (at $n = \infty$). Respectively, initially ($t = 0$) we have $q_n = 0$ at $n \leq 0$ and $q_n = 2A_0$ at $n > 0$.

(b) The second type of solution we can consider is an excitation localized in the space. So, the function

$$\alpha_c^{(1)}(n) = A_0[J_{2(n+1)}(2\sqrt{U_1}t) - J_{2n}(2\sqrt{U_1}t)] \tag{15}$$

describes evolution of the lattice if intially the only particle on the site $n = 0$ is excited with the amplitude of the deviation A_0.

Clearly, both types of the solution in terms of $\alpha_c^{(1)}(n)$ are wave packets localized in the space. Using the asymptotics [9] $J_x(x) = (2\pi)^{-1}6^{1/3}\Gamma(4/3)x^{-1/3} + O(x^{-2/3})$ at $x \to \infty$ we find that the amplitude of the wave packet decreases as $t^{-1/3}$ along the characteristics $n = \pm v_0 t$, where $v_0 = \sqrt{U_1}$ is the sound velocity, i.e., the group velocity of the long acoustic waves, and as $t^{-1/2}$ along other "directions" given by a fixed site number n and by the velocity less than v_0. All the trajectories characterized by the velocities higher than v_0 manifest faster decay.

Now we pass to the order $\mu^{3/2}$. Solving (9) subject to the zero initial condition and taking into account (8) we find

$$\alpha_i^{(1)}(n) = -\sqrt{U_1}\int_0^t dt' \sum_{m=-\infty}^{\infty} \epsilon_{m+1}[\alpha_c^{(1)}(m+1) - \alpha_c^{(1)}(m)]J_{2(n-m)-1}(2\sqrt{U_1}(t-t')). \tag{16}$$

Thus the leading order of the incoherent component $q_{i,n}$ is described by the formula

$$q_{i,n} = \sqrt{U_1}\int_0^t dt' \sum_{m=-\infty}^{\infty} \langle q_m\rangle \xi_{m,n}(t-t'), \tag{17}$$

where the designation

$$\xi_{m,n}(t) = \epsilon_m J_{2(n-m)-1}(2\sqrt{U_1}t) - \epsilon_{m-1}J_{2(n-m)+1}(2\sqrt{U_1}t) \tag{18}$$

is introduced. Some comments on the result (17) should be made here. First, even in the case of the absolutely random distribution of the masses the generated process $\xi_{m,n}(t)$ (and hence $q_{i,n}$) is correlated: so $\xi_{m\pm1,n} \neq 0$ even if $B(|n-k|) \sim \delta_{n,k}$. Second, the integral form of (17) stipulates the dependence of the process on the history (it has been mentioned in [7]). Third, the decay of the dispersion $\langle \xi_{m,n}^2(t)\rangle$ with time and the correlation function $\langle \xi_{m,n}(t)\xi_{m,n}(t+\tilde{t})\rangle$ with $\tilde{t}$ is governed by the power law (i.e., manifests the property of the fractal noise). At last, (17) and (18) give the linear transformation of the noise, while the nonlinear process is governed by Eq. (11) (the last, however, will not be discussed here).

The nonlinearity results in the slower decay of the addendum $\alpha_c^{(2)}(n)$ with time compared with $\alpha_c^{(1)}(n)$ (see Appendix A). Thus the dependence of the solution on the slow time can be found from the requirement for the right hand side of (10) to be zero [note also that under such a condition the mean field is obtained with the accuracy $O(\mu^3)$]. This means that we have to set

$$-2\frac{\partial}{\partial\tau}\frac{\partial}{\partial t}\alpha_c^{(1)}(n) + U_2\Delta_n[\alpha_c^{(1)}(n)]^2 - U_1\left\{\langle\epsilon_{n+1}[\alpha_i^{(1)}(n+1) - \alpha_i^{(1)}(n)]\rangle - \langle\epsilon_n[\alpha_i^{(1)}(n) - \alpha_i^{(1)}(n-1)]\rangle\right\} = 0. \tag{19}$$

It is obtained by direct calculations that

$$\frac{\partial}{\partial t}\left\{\langle\epsilon_{n+1}[\alpha_i^{(1)}(n+1) - \alpha_i^{(1)}(n)]\rangle - \langle\epsilon_n[\alpha_i^{(1)}(n) - \alpha_i^{(1)}(n-1)]\rangle\right\} = U_1\Delta_n \sum_{k=-\infty}^{\infty} B(|n-k|)\int_0^t dt'\alpha_c^{(1)}(k)\Delta_n J_{2(n-k)}(2\sqrt{U_1}(t-t')). \tag{20}$$

Now we differentiate (19) with respect to t and transform the first term taking into account (8). As a result we have the equation

$$\frac{\partial\alpha_c^{(1)}(n)}{\partial\tau} - U_2\alpha_c^{(1)}(n)\frac{\partial\alpha_c^{(1)}(n)}{\partial t} - \frac{U_1^2}{2}\sum_{k=-\infty}^{\infty} B(|n-k|) \times \int_0^t dt'\alpha_c^{(1)}(k)\Delta_n J_{2(n-k)}(2\sqrt{U_1}(t-t')) = 0, \tag{21}$$

which describes the evolution of the mean field in the lowest order. The second term in (21) is the contribution of the nonlinearity and the last one corresponds to "dissipative" losses caused by the transformation of the energy from the coherent component to the incoherent one due to the scattering by the inhomogeneities.

Equation (21) has to be considered in the quadrant $\tau > 0$; $t > 0$. The "initial" condition with respect to τ comes from the following idea. When $\mu = 0$ the function $\alpha_c^{(1)}(n)$ has to satisfy (8). Hence at $\tau = 0$ the function $\alpha_c^{(1)}(n)$ corresponds to the unperturbed motion. In other words, the expression in the right hand side of (13) serves as an initial condition for (21). The "boundary" conditions with respect to t follow from the natural requirement of decay of the mean field and its derivative when t tends to infinity.

III. LATTICE WITH SMALL SCALE INHOMOGENEITIES

As is known (see, e.g., [6]), it is the case of small scale inhomogeneities that provides the most favorable conditions for the application of the mean field method. In the case $B(|n-k|) = 2B_0\delta_{n,k}$ (B_0 being the constant), (21) reduces to

$$\frac{\partial\alpha_c^{(1)}(n,t)}{\partial\tau} - U_2\alpha_c^{(1)}(n,t)\frac{\partial\alpha_c^{(1)}(n,t)}{\partial t} - U_1^2 B_0\int_0^t dt'\alpha_c^{(1)}(n,t')\Delta_0 J_0(2\sqrt{U_1}(t-t')) = 0 \qquad (22)$$

(here we temporally use the explicit indication on the dependence of the function $\alpha_c^{(1)}$ on time).

The integral equation (22) obtained as a result of the direct multiscale expansion is hardly solvable in a generic case. Meantime we can expect its simplification at large times. Indeed, (22) describes evolution of the mean field and does not contain rapidly varying random coefficients. Any initial excitation of the lattice (even those having a characteristic scale one) will become flat with time due to the dispersion. Since all the parameters in (22) are of order one, the characteristic time of that process is of order one, as well. Hence at $t \gg 1$ the integral term should be reducible to a local one.

Let us investigate the last term in (22) in the limit $t \gg 1$ and concentrate on the solution in the neighborhood of the trajectory $n = v't$ where v' is from the interval $[0, v_0]$. To this end, using the properties of the Bessel functions we rewrite it in the form

$$-U_1 B_0\frac{\partial\alpha_c^{(1)}(n,t)}{\partial t} + U_1 B_0\frac{\partial^2}{\partial t^2}\int_0^t dt'\alpha_c^{(1)}(n,(t-t'))J_0(2\sqrt{U_1}t'). \qquad (23)$$

In order to estimate the last integral we note that due to the Bessel function the main contribution is given by the region where t' is of order one. On the other hand, as follows from the representation (13) the behavior of $\alpha_c^{(1)}(v't,t)$ at $v_0 t \gg 1$ is defined by the stationary point ϕ_{st} of the phase $s(\phi) = \frac{v'}{v_0}\frac{\phi}{2} - \left(1-\frac{t'}{t}\right)\sin\frac{\phi}{2}$. That point is found from the equation $\cos\frac{\phi_{st}}{2} = \frac{v'}{v_0}\left(1-\frac{t'}{t}\right)^{-1}$, which can be rewritten as follows: $\cos\frac{\phi_{st}}{2} = \left(1-\frac{\Delta v}{v_0}\right)\left(1+\frac{t'}{t}\right)$ (here $\Delta v = v_0 - v'$). If now we consider the trajectory corresponding to $\Delta v = \mu\tilde{v}$ ($\tilde{v}$ being the constant of order of v_0) the stationary phase is found to be $\phi_{st} = 2^{3/2}\left(\mu\frac{\tilde{v}}{v_0} - \frac{t'}{t}\right)^{1/2}$. Thus ϕ_{st} is a small quantity, which complicates the analysis of the temporal decay of $\alpha_c^{(1)}(v't,t)$ governed by (13) in a generic case. However, for our goals it is not necessary to investigate it. The only thing we want to show is that

$$\left|\frac{\partial\alpha_c^{(1)}(v't,t)}{\partial t}\right| \ll |\alpha_c^{(1)}(v't,t)|t^{-1/2} \qquad (24)$$

at $t \gg 1$ (here $t^{-1/2}$ describes decay of the envelope of the Bessel function). Differentiating (13) with respect to t and taking into account that according to the stationary phase method the prefactor which appeared has to be taken in the point ϕ_{st}, we come to the rough estimate

$$\left|\frac{\partial\alpha_c^{(1)}(v't,t-t')}{\partial t}\right| \sim \left(v'\phi_{st} - 2v_0\sin\frac{\phi_{st}}{2}\right)|\alpha_c^{(1)}(v't,t-t')| \sim 2^{3/2}\mu\tilde{v}\left(\mu\frac{\tilde{v}}{v_0} - \frac{t'}{t}\right)^{1/2}|\alpha_c^{(1)}(v't,t-t')|.$$

Since we are dealing only with the leading order of the expansion (5a) and required $\alpha_c^{(2)}(n)$ to be zero we have to restrict the consideration to the region $\mu^3 t \ll 1$ (or $\mu \ll t^{-1/3}$). Then the last estimate leads to (24). Consequently on the trajectory $n = v't$ the function $\alpha_c^{(1)}(n,t)$ can be considered as slowly varying compared with $J_0(2\sqrt{U_1}t)$ and the integral (23) can be approximated by $2^{-1}U_1^{-1/2}\alpha_c^{(1)}(n,t)$. Now by the use of new independent variables (τ, T), with $T = t - U_1 B_0\tau$, (22) is reduced to the Burgers equation

$$\frac{\partial\alpha_c^{(1)}(n)}{\partial\tau} - U_2\alpha_c^{(1)}(n)\frac{\partial\alpha_c^{(1)}(n)}{\partial T} - \frac{1}{2}U_1^{1/2}B_0\frac{\partial^2\alpha_c^{(1)}(n)}{\partial T^2} = 0. \qquad (25)$$

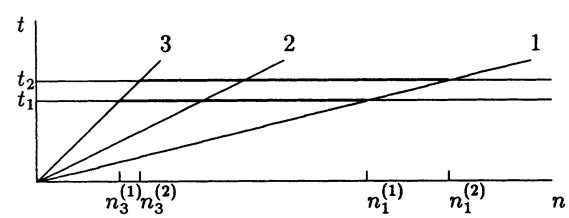

FIG. 1. The lattice segments (the thick lines), the dynamics of which is governed by the Burgers equation at different moments of time: the segment $[n_3^{(1)}, n_1^{(1)}]$ at t_1 and the segment $[n_3^{(2)}, n_1^{(2)}]$ at t_2 (here $t_2 > t_1$ and, respectively, the last interval is larger than the former one). These regions are determined by the cone between the trajectories (1) $n_1 = v_0 t$ and (3) $n_3 = v_0(1-\mu\tilde{v})t$ where $\tilde{v}$ is a constant of the order of v_0. Thus $n_3^{(i)} = v_0(1-\mu\tilde{v}')t_i$ and $n_1^{(i)} = v_0 t_i$. The trajectory 2 is given by $n = v_0(1-\mu v_0^2 B_0)t$ and corresponds to the region of maximum of the mean field [also the characteristic (2) corresponds to the moving frame in which the equation for the mean field is written in the continuum limit].

This equation is valid for the sites of the lattice in some cone near the trajectory $n = v_0(1 - \mu v_0^2 B_0)t$ (as Fig. 1 explains).

To analyze the decay of the wave packet amplitude we solve (25) by means of the Cole-Hopf transformation $[\alpha_c^{(1)}(n) = U_1^{1/2} U_2^{-1} B_0 (\partial/\partial T) \ln g(n,T)]$. Being interested in the region $T \gg 1$ we can approximate the solution by the expression

$$\alpha_c^{(1)}(n) = -\frac{1}{U_2 \tau} \frac{\int_0^\infty dt' g_0(n,t')(T-t') \exp\left[-\frac{(T-t')^2}{2U_1^{1/2} B_0 \tau}\right]}{\int_0^\infty dt' g_0(n,t') \exp\left[-\frac{(T-t')^2}{2U_1^{1/2} B_0 \tau}\right]}. \tag{26}$$

Here $g_0(n,T)$ has to be found from the effective initial conditions for the Burgers equation [generally speaking it differs from (13)].

It follows from the representation (26) that the main contribution to the integrals is made by the region $(T - \sqrt{2U_1^{1/2} B_0 \tau}, T + \sqrt{2U_1^{1/2} B_0 \tau})$. Hence the characteristic time of the diffusive process is $t \sim \mu^{-1}$ $(\tau \sim 1)$ and the asymptotics of $\alpha_c^{(1)}(n)$ is defined by the behavior of $g_0(n,t')$ in the region mentioned above. The function $g_0(n,t')$ cannot be obtained exactly [because it depends on the "history" governed by the nonlocal equation (22)]. Nevertheless (26) can be transformed using rather general ideas coming from the properties of the linear solution (13). Indeed, let us rewrite (26) in the form

$$\alpha_c^{(1)}(n) = \frac{T}{U_2 \tau} \frac{I_1(n,T,\tau)}{I_0(n,T,\tau)}, \tag{27}$$

where

$$I_p(n,T,\tau) = \int_0^\infty dx g_0(n,xT)(x-1)^p e^{-\lambda(x-1)^2}, \tag{28}$$

$\lambda = \frac{T^2}{2v_0 B_0 \tau}$, and $p = 0, 1$. We are going to estimate (28) in the limit $T \gg 1$, $\tau \geq 1$, which means that λ^{-1} is a small parameter and we can write

$$I_p(n,T,\tau) = \int_{-\infty}^\infty dy g_0(n,(1+y)T) y^p e^{-\lambda y^2} \tag{29}$$

(hereafter we hold only the leading orders of the asymptotic expansions). The general form of the Cole-Hopf transformation suggests that there exists a representation $g_0(n,t) \equiv \exp G(n,t)$, where $G(n,t)$ is a real function. The decay of the wave packet amplitude means that the derivative $\partial G(n,t)/\partial t$ goes to zero with t. However, here, taking into account that $G(n,t)$ is defined to within a constant, we assume that $G(n,t)$ itself is small enough at large t. Then $I_1(n,T,\tau)$ can be approximated by

$$I_1(n,T,\tau) \approx \int_{-\infty}^\infty dy G(n,(1+y)T) y e^{-\lambda y^2} \tag{30}$$

and $I_0(n,T,\tau) \approx \sqrt{\pi/\lambda}$. Thus there is the estimate

$$\alpha_c^{(1)}(n) \approx \frac{1}{U_2\sqrt{2v_0 B_0 \pi}} \frac{T^2}{\tau^{3/2}} I_1(n,T,\tau). \tag{31}$$

Now we recall two facts: first, $\alpha_c^{(1)}(n)$ is an auxiliary variable linked with the displacements of the masses by (2), second, it is a region near the trajectory $n = v_0 T$ (inside the cone between lines 1 and 3 in Fig. 1) that corresponds to the maximum of $\alpha_c^{(1)}(n)$. Thus the estimate for q_n can be written in the form

$$q_n \approx \frac{1}{U_2\sqrt{2v_0 B_0 \pi}} \frac{T^2}{\tau^{3/2}} \sum_{n=n_3}^{n_1} I_1(n,T,\tau) \tag{32}$$

(n_j being defined by Fig. 1).

For the next consideration it is necessary to concretize $G(n,t)$ since the integral $I_1(n,T,\tau)$ contains two functions with the large parameter T. While doing this we can restrict the analysis to the *asymptotic* form of $G(n,t)$. That is why the consideration will be carried out for a particular case when the initial condition for (25) coincides with (15). Thus it is taken that

$$g_0(n,t) = \exp\left\{\frac{U_2 A_0}{v_0^2 B_0} J_{2n-1}(2v_0 t)\right\}. \tag{33}$$

It seems that this choice reflects the general situation. This is, first, due to the completeness of the set of the Bessel function (see, e.g., [10]) and, second, due to the fact that for the large diversity of the functions $A(\phi)$ the asymptotic properties of the solution given by (13) are determined only by the phase of the expression in the integrand.

As is shown in Appendix B the function $g_0(n,t)$ given by (33) leads to the following asymptotic expression for the mean field amplitude $(n = v_0 T)$:

$$\langle q_n \rangle \approx \frac{A_0}{\pi 2^{1/4} v_0^{1/2} (\mu\tilde{v})^{1/4}} \frac{1}{T^{1/2}} \times \cos\left(-\frac{2^{5/2}}{3}(\mu\tilde{v})^{3/2} v_0 T + \frac{\pi}{4}\right). \tag{34}$$

This result can be rewritten in another form. To this end we recall that $\tilde{v} \sim v_0$, μ can be estimated as $\langle m_n^2\rangle/M^2$, and hence $\tau = \frac{\langle m_n^2\rangle}{M^2} T$. Then (34) takes the form

$$\langle q_n \rangle \approx \frac{\tilde{A}\langle m_n^2\rangle^{1/4}}{v_0^{3/4} M^{1/4}} \frac{1}{\tau^{1/2}} \cos\left(-\tilde{\alpha}\frac{v_0^{3/2}\sqrt{\langle m_n^2\rangle}}{M}\tau + \frac{\pi}{4}\right), \tag{35}$$

where $\tilde{A}$ and $\tilde{\alpha}$ are constants.

IV. DISCUSSION AND CONCLUSION

The result (35) valid in the time region $\mu^{-1} \ll t \ll \mu^{-3}$ has two new parameters, $\tilde{A}$ and $\tilde{\alpha}$, which must be considered as constants. In order to determine them we either have to solve (22) explicitly or to solve (25) at $t \gg 1$ using for the initial condition the results following from Eqs. (8)–(10) of the direct expansion. The last way is available if there exists a large enough overlapping region $1 \ll t \ll \mu^{-1}$, where the direct expansion is still valid $(t \ll \mu^{-1})$ and the transformation of the regular term in (22) already works $(t \gg 1)$. In the last case we have that $\tilde{A} \sim A_0$ where A_0 is the true amplitude of the initial excitation and hence the decay of the amplitude characterized by the quantity $\langle q_n \rangle / A_0$ does not depend on A_0 in the leading order. Though this conclusion differs from the observation reported in [7] it is not in contradiction with the result mentioned since we have dealt with the *weakly nonlinear* case. As it is expected at $t \sim \mu^{-1}$ the transition between laws $t^{-1/3}$ and $t^{-1/2}$ occurs.

The sum in (32) contains the terms with $n > n_3$. Evidently we can spread out the sum for all $n < n_3$ at least because the respective terms decay not slower than $t^{-1/2}$. Thus roughly the law of the amplitude decay can be referred to as $t^{-1/2}$, i.e., it coincides with the result obtained in [8] within the framework of the long wave approximation. In the meantime there are two distinctions. First, the above law is modulated by a cosine, which becomes important at $t \sim \mu^{-3/2}$. Second, and this is also related to a series of other previous results on the exactly integrable models [4, 5, 2], the exact stochastic one-soliton solution *does not* display the decreasing of the amplitude on each realization of the random mass distribution. The decay of the mean field is the result of the mathematical procedure, that is, averaging over *fluctuating velocities* and *phases* of the soliton. In contrast, the weakly nonlinear expansion given by (5a) and (5b) implies that the mean field, being a term of the leading order, is a quantity observable on one realization [in the case at hand the fluctuations (17) are the small addendum of a higher order].

The analysis provided above can be generalized to the pulses of the steplike form. For a representative of that class of solutions given by (14) we have to choose

$$g_0(n,t) = \exp\left\{\frac{U_2 A_0}{v_0^2 B_0}\int_0^{2v_0 t} dt' J_{2n}(t')\right\}. \tag{36}$$

To conclude we have obtained that the evolution of the mean field of the wave propagating along the disordered lattice is described by the Burgers equation if the scale of inhomogeneities is small enough. The law of the amplitude decay is $t^{-1/2}$. Also the dissipation leads to the slowing of the wave packet, which in the above situations is expressed by the renormalization of the velocity $v_0 \to v_0(1 - \mu v_0^2 B_0)$ of the maximum of the amplitude. Though to simplify consideration the particular examples of the lattice excitations were used, the asymptotics have very weak dependence on the type of initial condition and are valid for a rather general situation. Since we dealt with the discrete system it is natural that the equation for the amplitude has appeared in terms of the slow and fast *times* (rather than in terms of the traveling coordinate, as happens in the long wave approximation).

APPENDIX A: STRUCTURE OF SECULAR TERMS

Let us consider the equation

$$\frac{\partial^2 \alpha(n)}{\partial t^2} - U_1 \Delta_n \alpha(n) = U_2 \Delta_n [\alpha_c^{(1)}(n)]^2, \tag{A1}$$

where $\alpha_c^{(1)}(n)$ is a function given by (13) [evidently (A1) has the form of (10) if the dependence of τ is not introduced].

The solution of (A1) is represented as follows:

$$\alpha(n) = -\frac{1}{2\pi i}\frac{U_2}{U_1}\oint dz \int_0^t dt' \sum_{m=-\infty}^{\infty} z^{n-m-1}\Omega(z) \times \sin\left[\Omega(z)(t-t')\right][\alpha_c^{(1)}(m)]^2. \tag{A2}$$

Here $\Omega(z) = \sqrt{U_1(2 - z - z^{-1})}$ and the loop integral is around the unit circle.

Let us consider it in the point $n = N$ at the moment of time $t = N/v_0$ and investigate the limit $N \to \infty$. In other words, after the calculation of the loop integral we are interested in

$$I(N) = \frac{U_2}{U_1}\int_0^N dx \sum_{m=-\infty}^{\infty} \frac{\partial J_{2(N-m)}(2(N-x))}{\partial x} \times [\alpha_c^{(1)}(m,x)]^2 \tag{A3}$$

[here the temporal dependence of $\alpha_c^{(1)}(m)$ is indicated explicitly].

Both the Bessel function and $[\alpha_c^{(1)}(m)]^2$ display their maxima in the region where $x \sim m$. In order to use this fact we have to take into account the asymmetry of the decay of both functions. So the Bessel function decreases exponentially at $x > m$ while $[\alpha_c^{(1)}(m,x)]^2$ is exponentially small at $x < m$. Respectively from other sides, i.e., at $x < m$ and $x > m$, both functions have power decay (such behavior is a consequence of the existing limiting group velocity v_0). Thus in order to simplify (A3) further we have to split the integration of each term as follows: $\int_0^N \to \int_0^m + \int_m^N$, and replace x by m in the Bessel function for the first integral and in $[\alpha_c^{(1)}(m,x)]^2$ for the sceond one. After this approximation the integrals are evaluated exactly. Leaving the leading order we can write

$$I(N) \approx \frac{U_2}{U_1}\sum_{m=-\infty}^{\infty} J_{2(N-m)}(2(N-m)) \times \left\{[\alpha_c^{(1)}(m,m)]^2 - [\alpha_c^{(1)}(m,N)]^2\right\}. \tag{A4}$$

The above arguments allow us to restrict the sum to the terms with $0 < m < 2N$. Then we can exclude a finite number of terms in the neighborhood of N [where the difference in (A4) tends to zero] and neglect $[\alpha_c^{(1)}(m,N)]^2$ for the other ones characterized by $m \ll N$. Now assuming that $|\alpha_c^{(1)}(m,m)| = O(m^{-1/3})$, using the asymptotics of the Bessel function $J_{2(N-m)}(2(N-m)) \sim (N-m)^{-1/3} \sim N^{-1/3}$, and taking into account that the number $\tilde{N}$ of terms satisfying the condition $m \ll N$ grows linearly with N (say we can fix this growth by the law $\tilde{N} = \delta N$, δ being a small constant) we arrive at the estimate $I(N) = O((\tilde{N}/N)^{1/3})$. Thus $I(N)$ and hence $\alpha_c^{(2)}(n)$ decay much slower than $\alpha_c^{(1)}(n)$ on the trajectory $n = v_0 t$. This means that at $\mu t > 1$ the expansion (5a), (5b) fails unless the dependence on the slow time is introduced.

APPENDIX B: ASYMPTOTICS OF THE SOLUTION

In the case at hand $I_1(n,T,\tau)$ can be represented as

$$I_1(n,T,\tau) = 2\frac{\Lambda}{\pi}\int_0^\infty dy\, ye^{-\lambda y^2} \times \int_0^\pi d\phi \sin\{(2n-1)\phi - 2v_0 t\sin\phi\} \times \sin(2v_0 y \sin\phi), \quad \text{(B1)}$$

where $\Lambda = \frac{U_2 A_0}{v_0^4 B_0}$,

$$2v_0 t = 2v_0(1+\delta_+)T = \nu(1+\delta_+),$$

and $\delta_+ = \mu v_0^2 B_0$. The integral over y and then the sum over n are calculated explicitly to give

$$\sum_{n=n_3}^{n_1} I_1(n,T,\tau) = -\frac{\Lambda\nu}{4\sqrt{\pi}\lambda^{3/2}} \times \int_{-\pi}^{\pi} d\phi \exp(\phi - \sin\phi + i\gamma\sin^2\phi) \times \left(e^{-i\nu\mu\tilde{v}\phi} - e^{i\nu\delta_+\phi}\right). \quad \text{(B2)}$$

In this representation $\mu\tilde{v} \sim \delta_+ \sim \mu$ are the constants and $\nu \gg 1$ is a large parameter. This is a typical situation of the confluence of the critical points which requires rather delicate analysis. However, (B2) can be reduced to the known asymptotics taking into account that the main contribution to the integral is given by small ϕ, when the term with γ can be neglected compared with exponents containing $\tilde{v}$ and δ_+. In this way we come to the estimate

$$\sum_{n=n_3}^{n_1} I_1(n,T,\tau) \approx -\frac{\Lambda\nu}{4\sqrt{\pi}\lambda^{3/2}} \times \left[J_{\nu(1-\mu\tilde{v})}(\nu) - J_{\nu(1-\delta_+)}(\nu)\right]. \quad \text{(B3)}$$

At $\nu \gg 1$ the second Bessel function at $\mu \gg 1$ is estimated [9] by the Airy function decaying exponentially with ν, while for the first Bessel function we can use the asymptotics obtained by the stationary phase method [9]

$$J_\nu(\nu x) = \sqrt{\frac{2}{\pi\nu}}(x^2-1)^{-1/4} \times \cos\left[\nu\left(\arccos x^{-1} - \sqrt{x^2-1}\right) + \frac{\pi}{4}\right], \quad \text{(B4)}$$

with $x > 1$.

Now taking into account that $x = 1 + \mu\tilde{v}$ and holding the lowest order expansion in μ we arrive at (34).

It is worth pointing out here that the direct estimate of $\alpha_c^{(1)}(n)$ in the same way allows us to ensure that the solution indeed displays the property (24).

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