# Statistics of Stokes variables for correlated Gaussian fields

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The joint and marginal probability distribution functions of the Stokes variables are derived for correlated Gaussian fields [an extension of D. Eliyahu, Phys. Rev. E **47**, 2881 (1993)]. The statistics depend only on the first moment (averaged) Stokes variables and have a universal form for $S_1$, $S_2$, and $S_3$. The statistics of the variables describing the Cartesian coordinates of the Poincaré sphere are given also.

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## I. INTRODUCTION

In recent years, theoretical and experimental studies have been carried out extensively on multiple scattering of light in random media. Much effort has been devoted to the scalar wave model for the optical field, which was successfully used in many instances. However, it is becoming increasingly apparent that the vector nature of light plays an important, indeed sometimes dominant, role in many diverse phenomena [1–10]. Recently, the statistical properties of temporally [5, 11, 12] random (partially polarized) light and, also, of spatially random multiply scattered waves [6, 7, 13–19] from random media (speckle patterns) have attracted much theoretical and experimental interest. The statistics of the vector nature of light can be described by different groups of three or four variables: such as the field variables, the amplitudes and phases variables, or the ellipse variables which were lately investigated [5, 6, 17].

An alternative description of polarized light can be obtained from the Stokes variables. The marginal, individual probability density functions of the Stokes variables and their moments for correlated Gaussian fields were presented recently [17]. It is the purpose of the present paper to obtain the joint probabilities of all, and marginal probabilities of some, of the Stokes variables, in the general case for temporal or spatial randomness of correlated Gaussian fields.

## II. THE STOKES VARIABLES

Although the ellipsometric variables [17] completely specify the polarization state of waves and are readily visualized, they are not particularly conducive to understanding the transformations of polarized light. Moreover, they are difficult to measure directly and are not adaptable to a discussion of partially polarized light. In contrast to the ellipse variables, the real Stokes variables are an equivalent description of polarized light, but one of greater usefulness, particularly in scattering problems, and are easy to detect by only six intensity measurements at each speckle spot.

For simplicity, the four Stokes variables are denoted by $[I, S_1, S_2, S_3]$. For a plane wave propagating normal to the $x$-$y$ plane,

$$I = |E_x|^2 + |E_y|^2 = (A_x)^2 + (A_y)^2, \quad 0 \le I < \infty \ , \tag{1a}$$

$$S_1 = |E_x|^2 - |E_y|^2 = (A_x)^2 - (A_y)^2, \quad -\infty < S_1 < \infty \ , \tag{1b}$$

$$S_2 = E_x E_y^* + E_x^* E_y = 2A_x A_y \cos\delta, \quad -\infty < S_2 < \infty, \tag{1c}$$

$$S_3 = i(E_x E_y^* - E_x^* E_y) = 2A_x A_y \sin\delta, \quad -\infty < S_3 < \infty, \tag{1d}$$

where $E_x = E_x^r + iE_x^i$ and $E_y = E_y^r + iE_y^i$ ( $-\infty < E_{x,y}^{r,i} < \infty$) are the real and imaginary parts of the field variables with amplitudes $A_x, A_y$ and relative phases $\delta_x, \delta_y$ ($\delta = \delta_y - \delta_x$). These variables are time dependent for temporally random light, or coordinate dependent for spatial randomness.

## III. PROBABILITIES

Assuming Gaussian statistics, in the general case, as was shown by Goodman [20] and used successfully [5, 17], the field variables have probabilities which depend upon correlations among all the four components and may be written as

$$P(E_x^r, E_x^i, E_y^r, E_y^i) = \frac{1}{\pi^2\, d}\, \exp\left( -\frac{1}{d}[j_{22}|E_x|^2 + j_{11}|E_y|^2 - 2\mathrm{Re}(j_{12}E_x^* E_y)]\right) , \tag{2}$$

where $d = \det \tilde{\mathbf{J}}$, $\tilde{\mathbf{J}}$ is the $2 \times 2$ covariant Hermitian matrix ($j_{12} = j_{21}^* = j_{12}^r + ij_{12}^i$),

$$\tilde{\mathbf{J}} = \begin{pmatrix} \langle E_x|E_x\rangle & \langle E_x|E_y\rangle \\ \langle E_y|E_x\rangle & \langle E_y|E_y\rangle \end{pmatrix}, \tag{3}$$

and $\langle E_i|E_j\rangle$ denotes the ensemble average of $E_i^* E_j$ (time averaging for temporal randomness or spatial averaging in scattering problems). The assumptions in Eq. (2) are that $\langle|E_x^r|^2\rangle = \langle|E_x^i|^2\rangle$, $\langle|E_y^r|^2\rangle = \langle|E_y^i|^2\rangle$, $\langle E_x^r E_y^r\rangle = \langle E_x^i E_y^i\rangle = \frac{1}{2}j_{12}^r$, and $\langle E_x^r E_y^i\rangle = -\langle E_x^i E_y^r\rangle = \frac{1}{2}j_{12}^i$. These assumptions are justified for independent Feynman paths [17]. Instead of the coherency matrix $\tilde{\mathbf{J}}$, one may use the *average* Stokes variables: the *Stokes parameters*

$$\langle I\rangle = j_{11} + j_{22} \ , \tag{4a}$$

$$\langle S_1\rangle = j_{11} - j_{22} \ , \tag{4b}$$

$$\langle S_2\rangle = 2j_{12}^r \ , \tag{4c}$$

$$\langle S_3\rangle = 2j_{12}^i \ , \tag{4d}$$

$$d = \frac{1}{4}(\langle I\rangle^2 - \langle S_1\rangle^2 - \langle S_2\rangle^2 - \langle S_3\rangle^2) \ , \tag{4e}$$

and the degree of polarization $P$, given by

$$P = \sqrt{1 - \frac{4\det\tilde{\mathbf{J}}}{[\mathrm{tr}(\tilde{\mathbf{J}})]^2}} = \frac{\sqrt{\langle S_1\rangle^2 + \langle S_2\rangle^2 + \langle S_3\rangle^2}}{\langle I\rangle} \ , \tag{4f}$$

where $0 \le P \le 1$, meaning that the first moment of the Stokes variables obeys the inequality

$$\langle I\rangle^2 \ge \langle S_1\rangle^2 + \langle S_2\rangle^2 + \langle S_3\rangle^3 \ . \tag{5}$$

The equality holds if the light is fully polarized.

A simple model for the dependence of the elements of the covariant matrix $\tilde{\mathbf{J}}$ on the incident polarization state and the scattering medium with slab geometry for arbitrary incident and scattered wave-vector direction was given before [8, 17], for spatial randomness. Using Eq. (4) the dependence of the Stokes parameters on those system parameters is straightforward.

Fercher and Steeger [7, 15] initially calculated the statistics of the individual marginal Stokes variables $P(I), P(S_1), P(S_2)$, and $P(S_3)$ for independent fields where $A_x, A_y$, and $\delta$ were assumed to be independent. Later [16] they recalculated these statistics assuming correlations between $A_x$ and $A_y$ with the bivariant Rayleigh distribution but with $\delta$ independent. Barakat [11] gives these, individual marginal statistics, for the correlated Gaussian fields [Eq. (2)], assuming that $j_{12}^i = 0$, meaning $\langle S_3\rangle = 0$. In Ref. [17] the marginal probabilities of these variables for fully correlated fields and their moments were obtained. It was also found in [17] that the Goodman [20] type of multivariant complex Gaussian probability [Eq. (2)] has the significance (not as was claimed before [12]) needed in optics. In this paper we give the results for the *joint* probability distribution function (PDF) of the Stokes variables and their moments.

As mentioned before, in each speckle spot (or at a single time for temporal randomness) the light is fully polarized; therefore the four Stokes variables are connected by $I^2 = S_1^2 + S_2^2 + S_3^2$ for each point in space (or time). The reason is because of the infinite possibilities of fields with the same relative phase $\delta$ but with different $\delta_x$ [Eq. (1)] that give the same Stokes variables which, like the ellipse variables [17], do not depend on $\delta_x$. In that case, it is obvious that the Jacobian of that transformation is vanished $[\partial(I, S_1, S_2, S_3)/\partial(E_x^r, E_x^i, E_y^r, E_y^i) = 0]$.

Using the conditional probability of multivariant distribution one gets

$$P(I, S_1, S_2, S_3) = P(I|S_1, S_2, S_3)P(S_1, S_2, S_3) \ , \tag{6}$$

where the probability of finding $I$ knowing $S_1, S_2, S_3$, is

$$P(I|S_1, S_2, S_3) = \delta\left(I - \sqrt{S_1^2 + S_2^2 + S_3^2}\right) \ . \tag{7}$$

In order to find the joint probability of $S_1, S_2, S_3$ another variable is defined,

$$\alpha = E_y^r; \tag{8}$$

then, using Eq. (2) and the Jacobian

$$\frac{\partial(\alpha, S_1, S_2, S_3)}{\partial(E_x^r, E_x^i, E_y^r, E_y^i)} = \frac{8}{\sqrt{2}} I\sqrt{I - S_1 - 2\alpha^2} \ , \tag{9}$$

and integrating on $\alpha$ in the range $-\sqrt{\frac{1}{2}(I - S_1)}$ to $\sqrt{\frac{1}{2}(I - S_1)}$, one gets

$$P(S_1, S_2, S_3) = \frac{1}{4\pi d\sqrt{S_1^2 + S_2^2 + S_3^2}} \times \exp\Big[-\frac{1}{2d}\Big(\langle I\rangle\sqrt{S_1^2 + S_2^2 + S_3^2} - \langle S_1\rangle S_1 - \langle S_2\rangle S_2 - \langle S_3\rangle S_3\Big)\Big] \ , \tag{10}$$

where a degeneracy of 2 in the transformation is taken into account.

Using Eqs. (6), (7), and (10), the full joint PDF of the Stokes variables is given by

$$P(I, S_1, S_2, S_3) = \frac{\delta\left(I - \sqrt{S_1^2 + S_2^2 + S_3^2}\right)}{4\pi d I} \times \exp\Big[-\frac{1}{2d}\Big(\langle I\rangle I - \langle S_1\rangle S_1 - \langle S_2\rangle S_2 - \langle S_3\rangle S_3\Big)\Big] \ . \tag{11}$$

The Stokes variables are found to be dependent in any degree of polarization, as can be expected. The moments of the Stokes variables are easily calculated for unpolarized light ($P = 0$)

$$\langle I^{n_1} S_1^{n_2} S_2^{n_3} S_3^{n_4}\rangle = \begin{cases} (\langle I\rangle/2)^{n_1+n_2+n_3+n_4}\,(n_2-1)!!\,(n_3-1)!!\,(n_4-1)!!\,[(n_2+n_3+n_4+1)!!]^{-1} \\ \times\ (n_1+n_2+n_3+n_4+1)!, & n_2, n_3, n_4 \text{ even} \ , \\ 0, & \text{otherwise} \ , \end{cases} \tag{12}$$

where for $n$ odd $n!! = 1 \times 3 \times 5 \times \cdots \times n$, and $(-1)!! = 1$.

The probability for circular polarization ($S_1 = S_2 = 0$, $S_3 = \pm I$), as given from Eq. (11), is finite. This look like a contradiction to the prediction (Ref. [17]) of zero probability for that polarization. Actually, the probability depends on the measured quantity. The ellipticity PDF $\epsilon$ [17] depends on the value of $\sqrt{S_1^2 + S_2^2}$, where this square root is the reason for the zero probability for $\epsilon = 1$

($\sqrt{S_1^2+S_2^2}=0$) although this condition is the same as $S_1=S_2=0,\ S_3=\pm I$.

The $S_i$ Stokes variables are in the same range [Eq. (1)] and have the same form in the PDF. Integration gives the marginal PDF of some of the Stokes variables. First integration on one of the $S_k$ variables gives

$$P(I,S_i,S_j)=\frac{1}{2\pi d\sqrt{I^2-S_i^2-S_j^2}}\times\exp\left[-\frac{1}{2d}\left(\langle I\rangle I-\langle S_i\rangle S_i-\langle S_j\rangle S_j\right)\right]\times\cosh\left(\frac{1}{2d}\langle S_k\rangle\sqrt{I^2-S_i^2-S_j^2}\right)\times\Theta(I^2-S_i^2-S_j^2)\ , \qquad (13)$$

where $\Theta(\cdots)$ is the step function, $i,j,k=1,2,3$, and $i\neq j\neq k$. Then integration on the total intensity $I$ gives

$$P(S_i,S_j)=\frac{1}{2\pi d}\exp\left[\frac{1}{2d}\left(\langle S_i\rangle S_i+\langle S_j\rangle S_j\right)\right]\times K_0\left(\frac{1}{2d}\sqrt{\langle I\rangle^2-\langle S_k\rangle^2}\sqrt{S_i^2+S_j^2}\right)\ , \qquad (14)$$

where $K_0(\cdots)$ is the modified Bessel function of the second kind of order zero [21]. This PDF is divergent at $S_i,S_j=0$ and decays as $P(S_i\to 0,S_j\to 0)\sim -\ln(S_i^2+S_j^2)$.

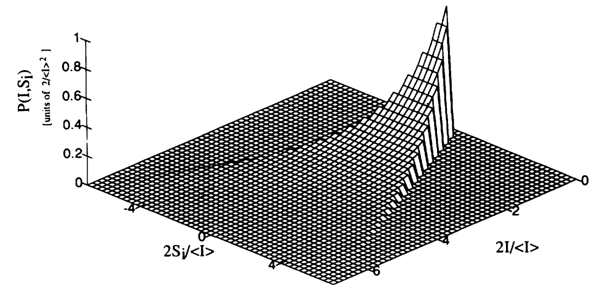


FIG. 1. The PDF of $I,S_i$ for unpolarized light ($P=0$).

On the other hand, integrating Eq. (13) on $S_j$ gives

$$P(I,S_i)=\frac{1}{2d}\exp\left[-\frac{1}{2d}\left(\langle I\rangle I-\langle S_i\rangle S_i\right)\right]\times I_0\left(\frac{1}{2d}\sqrt{\langle S_j\rangle^2+\langle S_k\rangle^2}\sqrt{I^2-S_i^2}\right)\times\Theta(I-|S_i|), \qquad (15)$$

where $I_0(\cdots)$ is the modified Bessel function of the first kind of order zero [21]. This PDF is plotted in Fig. 1 for unpolarized light ($P=0$).

The correlation coefficient between the Stokes variables is

$$\langle yS_i^n\rangle=\frac{2d\langle y\rangle\langle S_i^n\rangle}{\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2}+\frac{\langle y\rangle\langle S_i^{n+1}\rangle}{\sqrt{\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2}}+\frac{2(-1)^n(2d)^{n+2}(n+1)!\langle y\rangle}{(\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2)\left[\sqrt{(\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2)}+\langle S_i\rangle\right]^{n+2}}, \qquad (16)$$

where $y$ is one of the Stokes variables $[I,S_1,S_2,S_3]$, $y\neq S_i$, and $\langle S_i^n\rangle$ is given below in Eq. (19). For the lower powers of $S_i$

$$\langle yS_i\rangle=2\langle y\rangle\langle S_i\rangle\ , \qquad (17a)$$

$$\langle yS_i^2\rangle=\langle y\rangle(\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2+5\langle S_i\rangle^2)\ , \qquad (17b)$$

$$\langle yS_i^3\rangle=3\langle y\rangle\langle S_i\rangle(3\langle I\rangle^2-3\langle S_j\rangle^2-3\langle S_k\rangle^2+5\langle S_i\rangle^2)\ , \qquad (17c)$$

The individual marginal PDF is

$$P(S_i)=\frac{\exp\left(\frac{1}{2d}\left[\langle S_i\rangle S_i-|S_i|\sqrt{\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2}\right]\right)}{\sqrt{\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2}}\ , \qquad (18)$$

with maxima at $S_i=0$ and moments

$$\langle S_i^n\rangle=\frac{n!}{2^{n+1}\sqrt{\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2}}\times\left[\left(\sqrt{\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2}+\langle S_i\rangle\right)^{n+1}+(-1)^n\left(\sqrt{\langle I\rangle^2-\langle S_j\rangle^2-\langle S_k\rangle^2}-\langle S_i\rangle\right)^{n+1}\right]\ , \qquad (19)$$

in full agreement with our previous result [17].

The general correlation of two Stokes variables is given by

$$\langle y^mS_i^n\rangle=(2d/\langle y\rangle)^m\left[\frac{\partial^m\langle S_i^n(x,y)\rangle}{\partial x^m}\right]_{x=1}\times\begin{cases}(-1)^m & \text{if } y=I\\ 1 & \text{if } y=S_j \text{ or } S_k,\end{cases} \qquad (20)$$

where $\langle S_i^n(x,y)\rangle$ is given by multiplying $\langle y\rangle^2$, in the square brackets in Eq. (19), by $x^2$.

Alternatively, the vector statistics can be described by $\Sigma$, the Poincaré sphere. It provides a convenient way of representing polarized light, predicting how any given retarder will change the polarization form, and is very useful in crystal optics. The method is essentially one of mapping; each point on the sphere represents a different polarization state, and conversely. A general point on the surface of the (unit radius) Poincaré sphere is specified in terms of the longitude ($2\psi_+$) and the latitude ($2\omega$) [17], where $-\pi \leq 2\psi_+ \leq \pi$ and $-\pi/2 \leq 2\omega \leq \pi/2$. The significance of those angles is easily stated. They represent a completely polarized beam whose ellipse has azimuth $\psi_+$, ellipticity $\varepsilon = |\tan\omega|$, and a handedness that is left or right according to whether the point lies in the upper or lower hemisphere. Therefore the spherical coordinates of the Poincaré sphere represent the ellipse variable. Alternatively, each point on $\Sigma$ may be specified by means of Cartesian coordinates $\xi_1$, $\xi_2$, and $\xi_3$, which are referred to the $S_i$ Stokes variables normalized by the intensity $I$ ($\xi_i = S_i/I$). Their joint PDF is calculated from Eq. (11) and given by

$$P(\xi_1,\xi_2,\xi_3) = \frac{1}{4\pi}\delta\left(1-\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}\right) \times \frac{1-\eta_1^2-\eta_2^2-\eta_3^2}{(1-\eta_1\xi_1-\eta_2\xi_2-\eta_3\xi_3)^2}\,, \tag{21}$$

where $\eta_i = \langle S_i\rangle/\langle I\rangle$.

Similarly to the Stokes variables, one can seek for the marginal PDF of two Poincaré Cartesian variables

$$P(\xi_i,\xi_j) = \frac{(1-\eta_1^2-\eta_2^2-\eta_3^2)}{4\pi\sqrt{1-\xi_i^2-\xi_j^2}}\Theta(1-\xi_i^2-\xi_j^2) \times \left[\frac{1}{\left(1-\eta_i\xi_i-\eta_j\xi_j-\eta_k\sqrt{1-\xi_i^2-\xi_j^2}\right)^2}+\frac{1}{\left(1-\eta_i\xi_i-\eta_j\xi_j+\eta_k\sqrt{1-\xi_i^2-\xi_j^2}\right)^2}\right]\,. \tag{22}$$

or the individual marginal PDF

$$P(\xi_i) = \frac{(1-\eta_i\xi_i)(1-\eta_1^2-\eta_2^2-\eta_3^2)}{2\left[(1-\eta_i\xi_i)^2-(\eta_j^2+\eta_k^2)(1-\xi_i^2)\right]^{3/2}} \times\Theta(1-|\xi_i|)\,. \tag{23}$$

## IV. SUMMARY

In this paper the PDF of the Stokes variables of partially polarized light was investigated assuming Gaussian correlated fields. New statistics were found for the joint and marginal probabilities, which depend only on the average Stokes variables (the Stokes parameters). The statistics were found to have a universal form for the three $S_i$ variables. Each Stokes parameter $\langle S_i\rangle$ is responsible for breaking the symmetry of the joint and marginal PDF's around $S_i = 0$. The moments of two Stokes variables were also given. The statistics of the Cartesian variables of the unit radius Poincaré sphere ($\Sigma$) were also found.

These statistics offer new and simple possibilities for investigating polarization processes.


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