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Effect of a large-amplitude wave on the one-dimensional velocity distribution of particles in a linearized Fokker-Planck collisional plasma

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The evolution of a one-dimensional velocity distribution is studied in the presence of a monochromatic large-amplitude periodic force which is turned on adiabatically. The periodic Vlasov-Poisson equations are solved in the presence of a linearized Fokker-Planck collision term. For a constant driving force, the system is found to approach, after transient oscillations, a steady state which is maintained by one wave at the driving frequency. This is in contrast to the result in the absence of collisions where the steady state tends to be supported by several waves. An analytical solution for the steady-state distribution function in the presence of a driven large-amplitude wave is obtained by a Hamiltonian approach. The distribution function is expanded in powers of a small parameter Γ proportional to the collision strength. From the expansion, the zeroth order term is shown to give the space-averaged distribution function correct to first order in Γ. Comparison with the results of the simulations and of the harmonics expansion method shows that the solution estimates the distribution with good accuracy. The plateau in the wave trapping regime is analyzed, and the current driven by the large-amplitude traveling wave is determined.

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I. INTRODUCTION

Modification of a particle distribution by a periodic force is of great importance in plasma physics, and appears as a central problem in a number of statistical systems. Large-amplitude plasma waves can be driven in plasma by electrostatic probes and exciters [1, 2], and by several processes such as parametric instabilities and beat-frequency mixing of laser or rf beams. These processes have applications in plasma heating and current drive [3, 4], plasma diagnostics [5], ionospheric sounding [6], and plasma lasers [7]. In many of these applications, the particle velocity distribution is strongly modified, and nonthermal particles and particle diffusion are generated, the modeling of which is of contemporary interest in physics. In various contexts, the plasma wave is driven by an external force which sustains the oscillations against various damping mechanisms [8]. In such a system, collisions cause effects which are also found to be important in the theory of confinement of particles by magnetic mirrors and in the neoclassical theory of plasma diffusion in magnetic confinement systems [9].

Detailed studies of the evolution of a Vlasov equation in the presence of monochromatic nonlinear oscillations have been conducted by several authors [4, 10–13]. Much of the work has concentrated on the studies of dispersion, damping, particle acceleration, and wave coupling of a nonlinear wave. By using perturbation theory, so-called generalized quasilinear equations [11] have been derived for studies of the evolution of the particle velocity distribution and the wave electric field. Recently, the existence of stable time-asymptotic states has been proved and their dependence on initial and boundary conditions has been analyzed for slowly evolving systems by using kinetic theory based on adiabatic invariants of the equation [4]. Particle and Vlasov-Poisson simulations of the evolution of a large-amplitude wave [14, 15] have indicated the formation of a bump in the distribution at velocities above the phase velocity with a consequent sideband instability [16] leading to the generation of a multitude of modes causing the flattening of the bump.

In the presence of collisions, the steady-state velocity distribution of particles has been solved analytically from the perturbed Vlasov equation in the limit of weak collisions and weak oscillation amplitude for special types of Fokker-Planck collision terms [9, 17, 18]. Using the so-called Lenard-Bernstein collision operator [19], Zakharov and Karpman [17] solved the perturbed Vlasov equation, and calculated the wave damping decrement by a Hamiltonian approach for the deeply trapped and untrapped particles. They assumed that v_b/v_p, v_b/v_e, v_e/v_p, and ν_e/ω are much smaller than unity. Here, $\sqrt{2}v_b$ is the trapping width of the particles in the wave potential, $v_p = \omega/k$ is the wave phase velocity, v_e is the electron thermal velocity, ν_e is the collision frequency, and ω and k are the wave angular frequency and wave vector, respectively. In the opposite linear limit of weak oscillation amplitude or strong collisions, i.e., when $\nu_b^6/\omega^4\nu_e^2$ is small, the distribution function and damping have been solved by perturbation theory based on small wave am-

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plitude [18, 19]. Here, ν_b is the bounce frequency of the particles in the wave potential. By numerical integration of a Fokker-Planck equation, evidence for the similarity between the Vlasov and Fokker-Planck solutions at the early phase of evolution has been obtained [20]. However, the computer runs were too short to reach the Fokker-Planck equilibrium, and no definite conclusions were drawn on the existence of the large-amplitude time-asymptotic states [4, 11, 21] predicted by the Vlasov-Poisson equations.

In the present paper, the evolution of a one-dimensional velocity distribution and the wave is investigated in the presence of collisions and a large-amplitude periodic force by solving the Vlasov-Poisson equations with a Fokker-Planck collision term. The differences in the time-asymptotic solutions with and without collisions are shown to ensue from the different time-asymptotic spectra of waves developed in these two systems, and from the diffusion of electrons from the bulk distribution to the wave region in the presence of collisions. In order to compare the results with an analytical theory, the Vlasov equation with a Fokker-Planck collision term is solved in closed form for the steady-state distribution in a periodic configuration in the limit of weak collisions. The solution extends the result of Zakharov and Karpman to large wave amplitudes, and makes a comparison with simulations possible.

In order to obtain the solution, the Fourier expansion $\Sigma_n f_n(H)e^{iny}$ is applied for the distribution having the dependence $f(H,y)$, where H is the Hamiltonian of the particles and y is the periodic coordinate. Since the constant H trajectories of the trapped particles are closed in phase space, $f(H,y)$ has to be defined in a nonstandard way outside of this trajectory to accomplish the Fourier expansion. We find that the solution obtained agrees with the solution found by the harmonics expansion of the Boltzmann equation and reduces in an appropriate limit to that obtained with the Lenard-Bernstein collision term in Ref. [17]. The solution is found to be valid for small ν_e/ν_b and $v_b^2/v_p v_e$. No separate restrictions on v_b/v_e are assumed.

In the limit of vanishing collisionality, the distribution only depends on H, and is obtained from the lowest order expansion. However, as will be demonstrated in the present paper, this distribution is not the same as the one obtained in the absence of collisions. It will be shown that, in an asymptotic expansion in the collision parameter Γ, the space-averaged distribution has no $O(\Gamma)$ corrections. Consequently, as will be confirmed by our simulations, the steady-state distribution is modeled by the lowest order solution with good accuracy in practical applications. It is shown that for weak collisionality the solutions only depend on the two parameters v_p/v_b and v_e/v_b. This makes it feasible to efficiently survey the set of possible nonlinear states and the driven current in a large parameter range.

In Sec. II, we first present the analytic result for the time-asymptotic state in the presence of a nonlinear wave. In Sec. III, the simulations with the Vlasov-Poisson system in the presence of a Fokker-Planck collision term are presented and discussed. Here, the comparisons of the numerical results with the analytical solution for the distribution function, driven current, and electric field are presented. The conclusions are given in Sec. IV.

II. TIME-ASYMPTOTIC SOLUTION

For driven modes in the presence of collisions, the steady-state distribution function can be obtained from the perturbed Vlasov equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{eE}{m_e}\cos(kx-\omega t)\frac{\partial f}{\partial v} = \Gamma C(f), \tag{1}$$

where $f(t,x,v)$ is the distribution function of electrons with mass m_e and charge $-e$ in the phase space (x,v) at time t. E is the constant amplitude of the total electric field which is sustained by an external field, and which oscillates with the wave number k and angular frequency ω. On the right-hand side, $\Gamma C(f)$ gives the effect of collisions on the conservation of f in the phase space. $C(f)$ is assumed to have the form

$$C(f) = \frac{\partial}{\partial v}\left[\frac{\partial f/\partial v + (v/v_e^2)f}{\tau(|v|)}\right], \tag{2}$$

which with $\tau(|v|)^{-1} = v_p^3/(|v|^3 + v_e^3)$ reduces to the linearized one-dimensional Fokker-Planck collision model [22] in the high velocity ($|v| \gg v_e$) and low velocity ($|v| \ll v_e$) regimes. Here, $v_e = \sqrt{T_e/m_e}$ is the electron thermal velocity, and $\Gamma = (Z_i+2)\nu_e v_e^5/v_p^3$ gives the strength of collisions. Here, Z_i is the average charge of plasma ions which are assumed to form a stationary background, $\nu_e = n_e e^4 \ln\Lambda/(4\pi\epsilon_0^2 m_e^2 v_e^3)$ is the collision frequency, and $v_p = \omega/k$ is the wave phase velocity.

A. Fourier expansion

By introducing the new variables $u = v - v_p$ and $y = kx - \omega t$, Eq. (1) can be written as

$$\frac{\partial f}{\partial t} + ku\frac{\partial f}{\partial y} - \frac{eE}{m_e}\cos(y)\frac{\partial f}{\partial u} = \Gamma C(f). \tag{3}$$

Using y and the energy $H = m_e u^2/2 + (eE/k)\mathrm{sin}y$ as independent variables, and following the derivation in Ref. [17], the steady-state equation reduces to

$$\frac{\partial f^{\pm}(H,y)}{\partial y} = \frac{\Gamma m_e^2}{k}\frac{\partial}{\partial H}\left[\frac{u^{\pm}(H,y)}{\tau^{\pm}(H,y)}\left(\frac{\partial f^{\pm}(H,y)}{\partial H} + \frac{f^{\pm}(H,y)}{T}\right) + \frac{\omega/k}{\tau^{\pm}(H,y)T_e}f^{\pm}(H,y)\right], \tag{4}$$

where $\partial/\partial v = mu^{\pm}(H,y)\partial/\partial H$. The superscript $\pm$ indicates whether one should choose the positive or the negative sign in the definition of the relative velocity $u = \pm\sqrt{(2/m_e)[H-(eE/k)\mathrm{sin}y]}$. We first note that in the limit $\Gamma \to 0$ it follows from Eq. (4) that $\partial f^{\pm}(H,y)/\partial y \to 0$, that is, $f^{\pm} \to g^{\pm}(H)$. This is as expected because in the limit of weak collisions the Hamiltonian trajectories of the particles are weakly perturbed, and the distribution function only depends on H, when $\Gamma \to 0$.

We define the following Fourier expansions:

$$f^{\pm}(H,y)=\sum_{\ell} g_{\ell}^{\pm}(H)e^{i\ell y}, \tag{5}$$

$$u^{\pm}(H,y)/\tau^{\pm}(H,y)=\sum_{\ell} a_{\ell}^{\pm}(H)e^{i\ell y}, \tag{6}$$

$$v_p/\tau^{\pm}(H,y)=\sum_{\ell} b_{\ell}^{\pm}(H)e^{i\ell y} \tag{7}$$

by assuming the system to be periodic in y with the period 2π. For the untrapped particles, the constant H trajectories in the phase space (u,y) are continuous and periodic in y, while for the trapped particles they form separate closed curves appearing periodically in y. In order to be more exact in our definition, we generalize any function $\vartheta(H,y)$ so that $\vartheta(H,y)=0$ if y is outside of the closed curves. When this generalization for the functions $f^{\pm}(H,y)$, $u^{\pm}(H,y)/\tau^{\pm}(H,y)$, and $v_p/\tau^{\pm}(H,y)$ is adopted in Eqs. (5)–(7), the Fourier transforms $g_{\ell}^{\pm}(H)$, $a_{\ell}^{\pm}(H)$, and $b_{\ell}^{\pm}(H)$ are defined in a unique way.

With the help of the expansions, we solve Eq. (4) to first order in Γ:

$$g_0^{\pm}(H)=C\exp\left\{-\frac{1}{T_e}\int^{H}\left[1+\frac{\oint\frac{\omega/k}{\tau(H,y)}dy}{\oint\frac{u(H,y)}{\tau(H,y)}dy}\right]dH\right\}, \tag{8}$$

$$\begin{aligned} g_{\ell\neq 0}^{\pm}(H) &= \frac{\Gamma m_e^2}{ki\ell}\frac{\partial}{\partial H}\left\{\left[\frac{\partial g_0^{\pm}}{\partial H}+\frac{g_0^{\pm}}{T_e}\right]a_{\ell}^{\pm}(H)+\frac{g_0}{T_e}b_{\ell}^{\pm}(H)\right\} \\ &\equiv \frac{\Gamma m_e^2}{ki\ell}A_{\ell}^{\pm}(H). \end{aligned} \tag{9}$$

Here, the integrals $\oint\cdots dy$ are performed over one period of the constant H trajectories in the phase space (u,y). We have for the untrapped particles

$$\oint\frac{\omega/k}{\tau(H,y)}dy=\int_0^{2\pi}\frac{\omega/k}{\tau^{\pm}(H,y)}dy\equiv 2\pi b_0^{\pm}(H),$$

$$\oint\frac{u(H,y)}{\tau(H,y)}dy=\int_0^{2\pi}\frac{u^{\pm}(H,y)}{\tau^{\pm}(H,y)}dy\equiv 2\pi a_0^{\pm}(H),$$

and for the trapped particles

$$\oint\frac{\omega/k}{\tau(H,y)}dy=2\pi[b_0^{+}(H)-b_0^{-}(H)],$$

$$\oint\frac{u(H,y)}{\tau(H,y)}dy=2\pi[a_0^{+}(H)-a_0^{-}(H)],$$

respectively.

The function $g_0^{\pm}(H)$ was obtained from the lowest order expansion of Eq. (4)

$$\frac{\partial}{\partial H}\left[a_0^{\pm}(H)\left(\frac{\partial g_0^{\pm}}{\partial H}+\frac{g_0^{\pm}}{T_e}\right)+b_0^{\pm}(H)\frac{g_0^{\pm}}{T_e}\right]=0, \tag{10}$$

which gives

$$a_0^{\pm}(H)\left(\frac{\partial g_0^{\pm}}{\partial H}+\frac{g_0^{\pm}}{T_e}\right)+b_0^{\pm}(H)\frac{g_0^{\pm}}{T_e}=D, \tag{11}$$

where D is constant. For the untrapped particles, D has to vanish so that the condition $g_0^{\pm}\to 0$ for $H\to\infty$ is fulfilled. For the trapped particles, we use the fact that $g_0^{+}(H)=g_0^{-}(H)$, which is a consequence of the requirement that the trapped particle distribution function should be continuous at $u=0$ in the limit $\Gamma\to 0$. Consequently, we find the solution in Eq. (8) with the given definitions for the integrals over one oscillation period. When τ is constant and equal to unity in the definition of the collision term in Eq. (2) and v_b/v_e is taken to approach zero, the zeroth order solution (8) approaches the zeroth order solution in Eqs. (23) and (25) of Ref. [17] for the untrapped and trapped particles, respectively. This result was obtained with the Lenard-Bernstein collision term in the small-amplitude limit.

The space-averaged distribution function $f_0^{\pm}=\frac{1}{2\pi}\int_0^{2\pi}f^{\pm}dy$ obtained from Eqs. (8) and (9) is found to equal $\frac{1}{2\pi}\int_0^{2\pi}g_0^{\pm}(H)dy$ to first order in Γ. In order to see this, we first note that

$$\int_0^{2\pi}[e^{-imy}-(-1)^m e^{imy}]h[\sin(y)]dy=0 \tag{12}$$

holds for any integrable function h and for any integer m. If we also note the corollaries of Eq. (12):

$$\oint\frac{u(H,y)}{\tau(H,y)}e^{-i\ell y}dy=(-1)^{\ell}\oint\frac{u(H,y)}{\tau(H,y)}e^{i\ell y}dy,$$

$$\oint\frac{\omega/k}{\tau(H,y)}e^{-i\ell y}dy=(-1)^{\ell}\oint\frac{\omega/k}{\tau(H,y)}e^{i\ell y}dy,$$

we find

$$\begin{aligned} &\int_0^{2\pi}\sum_{\ell\neq 0}g_{\ell}^{\pm}(H)e^{i\ell y}dy \\ &\quad=\int_0^{2\pi}\sum_{p\geq 1}[e^{ipy}-(-1)^p e^{-ipy}]g_p^{\pm}(H)dy=0. \end{aligned} \tag{13}$$

Therefore the corrections to $f_0^{\pm}=\frac{1}{2\pi}\int_0^{2\pi}g_0^{\pm}(H)dy$ for finite Γ have to be at least of second order in Γ.

B. Harmonics expansion

The result is in agreement with the one obtained from the harmonics expansion of Eq. (1),

$$in(kv-\omega)p_n+\frac{F}{2m}\frac{\partial}{\partial v}(p_{n-1}+p_{n+1})=\Gamma C(q_n),\quad n\geq 0, \tag{14}$$

$$in(kv-\omega)q_n+\frac{F}{2m}\frac{\partial}{\partial v}(q_{n-1}+q_{n+1})=\Gamma C(p_n),\quad n\geq 1, \tag{15}$$

which divides Eq. (1) into two sets which are only coupled through the perturbation terms on the right-hand side. We have expanded $f=\sum_{n=-\infty}^{\infty}f_n(v)e^{in(kx-\omega t)}$ and defined $f_n-(-1)^n f_{-n}=p_n$ and $f_n+(-1)^n f_{-n}=q_n$. Setting $\Gamma=0$ in Eq. (15) is equivalent to expanding Eqs. (14) and (15) to first order in Γ and making the system (14) and (15) independent of Γ.

The coefficients in the expansion $f=\sum_n f_n(v)e^{iny}$ can now be calculated from the solution in Eqs. (8) and (9) as $f_n=\frac{1}{2\pi}\int_0^{2\pi}\sum_{\ell}g_{\ell}(H)e^{i\ell y}e^{-iny}dy$, where we have omitted the superscripts $\pm$ in order to shorten the notation. We find for the functions p_n and q_n

$$f_n \pm (-1)^n f_{-n} = \frac{1}{2\pi}\int_0^{2\pi}[e^{-iny} \pm (-1)^n e^{iny}]\sum_{\ell\neq 0}\frac{\Gamma m^2}{ki\ell}A_\ell(H)e^{i\ell y}dy + \frac{1}{2\pi}\int_0^{2\pi}[e^{-iny} \pm (-1)^n e^{iny}]g_0(H)dy. \quad (16)$$

By direct substitution, we find that the solution in Eq. (16) with $g_0(H)$ given in Eq. (8) satisfies Eqs. (14) and (15) to first order in Γ. The details of the proof are deferred to the Appendix. We also note that according to Eq. (12) $\frac{1}{2\pi}\int_0^{2\pi}[e^{-iny} - (-1)^n e^{iny}]g_0(H)dy = 0$ so that p_n has to be first order in Γ, and has no zeroth order part. Correspondingly, the first order part of $f_n + (-1)^n f_{-n}$ in Eq. (16) vanishes, and therefore q_n has to be zeroth order in Γ and has no first order part. This proves that the solution in Eq. (16) is an exact solution of Eqs. (14) and (15), provided one sets $\Gamma = 0$ in Eq. (15). The distribution function $q_0/2$ obtained from these equations equals $(1/2\pi)\int_0^{2\pi} g_0(H)dy$.

The method based on solving Eqs. (14) and (15) can be applied to solving Eq. (1) for different collision operators and for any collision strength. It also proves that the error in obtaining the space-averaged distribution function from the analytical solution in Eq. (8), obtained in the limit $\Gamma \to 0$, is $O(\Gamma^2)$. When Eqs. (14) and (15) are solved numerically, the harmonic series has to be truncated by assuming $f_n = 0$ for $n > M$, and only those equations in (14) and (15) for which $n \leq M$ are taken into account. Because of the truncation, the solution of Eqs. (14) and (15) can become more inaccurate for increasing amplitude of the wave.

III. SIMULATION OF TIME BEHAVIOR

To show that the steady-state distributions obtained from Eqs. (8) and (9) are the time-asymptotic states, we solve the evolution of the electron distribution from the one-dimensional Vlasov-Poisson system in the presence of a Fokker-Planck collision term in periodic configuration space:

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - e[E(x,t) + E_{ext}(t)\cos(kx-\omega t)]\frac{\partial f}{\partial v} = \Gamma C(f), \quad (17)$$

$$\frac{\partial E(x,t)}{\partial x} = -\frac{en_i}{\epsilon_0}\left[\int f dv - 1\right]. \quad (18)$$

Here, ϵ_0 is the vacuum permittivity, and n_i is the neutralizing background ion density. The electrostatic waves are adiabatically excited by an external field $E_{ext}(t)\cos(kx-\omega t)$, where $E_{ext}(t)$ grows from zero to a maximum value by $\omega_p t = 2500$. We use the linearized Fokker-Planck collision operator [22] which in the high and low velocity limits agrees with Eq. (2). The one-dimensional operator, obtained after integration over the perpendicular velocities, has the form

$$\Gamma C(f) = \frac{\partial}{\partial v_\parallel}\left\{\int_0^\infty 2\pi v_\perp D_\parallel G(v_\perp)dv_\perp \frac{\partial f(v_\parallel)}{\partial v_\parallel} + \int_0^{2\pi} 2\pi v_\perp\left[D_\times\frac{\partial G(v_\perp)}{\partial v_\perp} - A_\parallel G(v_\perp)\right]dv_\perp \times f(v_\parallel)\right\}. \quad (19)$$

In the above equation, $v_\perp$ and $v_\parallel$ refer to velocities perpendicular and parallel to the electric field vector of the wave, respectively. The terms $D_\times$, $D_\parallel$, and $A_\parallel$ are the elements of the diffusion tensor and the friction vector. In defining Eq. (19), we have temporarily used the notation $v_\parallel$ for v, and have reserved the notation v for $v^2 = v_\perp^2 + v_\parallel^2$.

The two-dimensional electron distribution has the form

$$\hat{f}(v_\parallel, v_\perp) = f(v_\parallel)G(v_\perp) = f(v_\parallel)\frac{1}{2\pi m_e^2 v_e^2}e^{-v_\perp^2/2m_e^2 v_e^2}, \quad (20)$$

where $G(v_\perp)$ is the virtual Maxwellian electron distribution in the perpendicular direction. The perpendicular background distribution is constant in time.

For the collision operator in Eq. (19), we have defined

$$D_\parallel = \frac{1}{2}\sum_k \gamma_k n_k\{[\Phi(a_k v) - \Psi(a_k v)](1 - v_\parallel^2/v^2) + 2\Psi(a_k v)v_\parallel^2/v^2\}\frac{1}{v}, \quad (21)$$

$$D_\times = \frac{1}{2}\sum_k \gamma_k n_k[3\Psi(a_k v) - \Phi(a_k v)]\frac{v_\parallel v_\perp}{v^2}\frac{1}{v}, \quad (22)$$

$$A_\parallel = -\sum_k \frac{\gamma_k n_k}{m_k}2a_k^2 v^2\Psi(a_k v)\frac{v_\parallel}{v^3}. \quad (23)$$

In the above equations, the sums $(\sum_k)$ are over different particle species forming the background plasma. The factor γ_k is the collision constant, and n_k is the density of species k. The factor a_k is related to the inverse thermal velocity of group k ($a_k^2 = m_k/2T_k$). The functions Φ and Ψ are defined as

$$\Phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy \quad (24)$$

and

$$\Psi(x) = \frac{\Phi(x) - x\Phi'(x)}{2x^2}. \quad (25)$$

The collision constant is

$$\gamma_k = \frac{e^2 q_k^2 \ln\Lambda_{ek}}{4\pi\epsilon_0^2}, \quad (26)$$

where $\ln\Lambda_{ek}$ is the Coulomb logarithm and q_k is the charge of species k. In our simulations we have used the value 15 for the Coulomb logarithm.

Equations (17) and (18) are solved by the splitting scheme [23] with an additional step for the integration of the collision operator (cf. [24]). At the beginning of the simulation, the electron distribution is Maxwellian.

Figure 1 shows the space-averaged distribution $f_0 = (1/L)\int_0^L f dx$ obtained from Eqs. (17) and (18) for $n_e = n_i = 2 \times 10^{19}$ m^{-3}, $T_e = 100$ eV, $\ln\Lambda = 15$, $k\lambda_D = 0.35$, $\omega = 1.22\omega_p$, and $L = 2\pi/k$, where T_e is the electron temperature, n_e is the electron density, λ_D is the electron Debye length, L is the length of the periodic box, and ω_p is the electron plasma frequency. The angular frequency ω of the external wave has been obtained from the linear dispersion relation of the plasma waves. With the present parameters we find $\nu_e/\omega = 1.1 \times 10^{-5} \ll 1$ for the collision frequency. To speed up computations, Γ was therefore multiplied by an acceleration factor $\kappa \gg 1$.

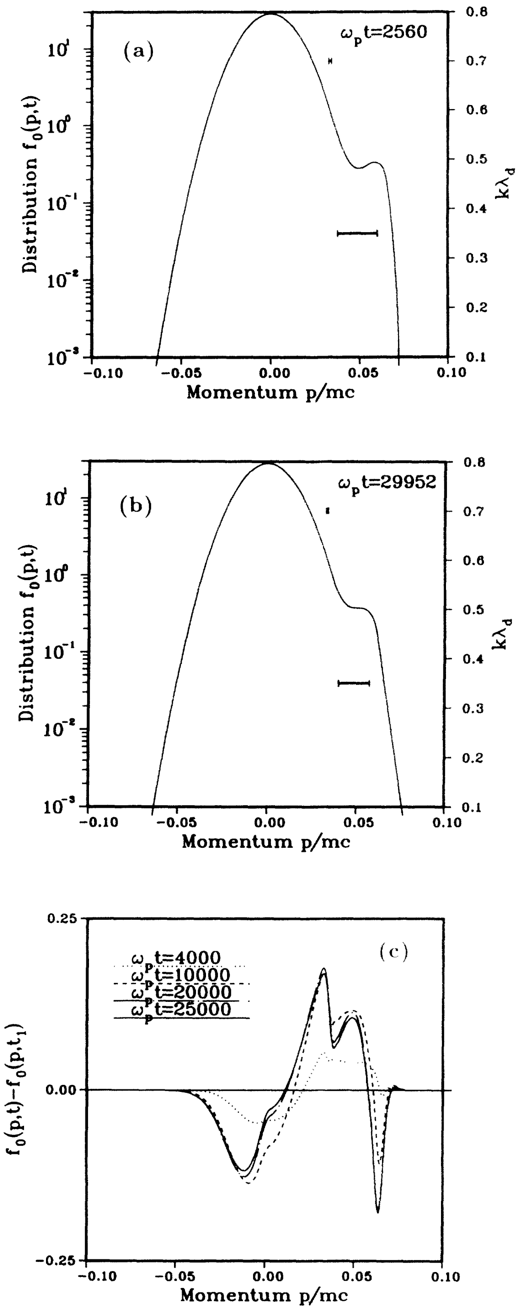

FIG. 1. The space-averaged velocity distribution in arbitrary units obtained from the simulation with $\kappa = 80$ at two time instants (a) $t_1 = 2560\omega_p^{-1}$ and (b) $t_2 = 29\,952\omega_p^{-1}$. The trapping regions $|v - v_p| < \sqrt{2}v_b$ of the modes with different wave numbers are indicated by lines (see the right-hand scale). (c) Also shown is the difference of the distribution at various intermediate time instants from the distribution at t_1.

The velocity distribution function obtained from simulations with $\kappa = 80$ is shown at two time instants, $t_1 = 2560\omega_p^{-1}$ and $t_2 = 29952\omega_p^{-1}$. Also shown is the difference of the distribution at various intermediate time instants from the distribution at t_1. When comparing the distributions at $t > 2t_2/3$ and at t_2, no observable changes can be found. Consequently, the distribution at $t = t_2$ represents the time-asymptotic distribution with a good accuracy. At early times, the distribution has a positive slope in the resonant region which indicates acceleration of electrons from velocities below the phase velocity to velocities above the phase velocity during the ramp-up of the external electric field. In runs with zero collisionality, the electrons are similarly accelerated but the positive slope is found to rapidly flatten through the consequent excitation of other modes with phase velocities around v_p by the sideband instability [15]. In the presence of collisions, the extra modes are found to be damped and the slope has a tendency to slowly flatten through the collisional diffusion of the electrons from the bulk distribution to the resonant region. The results shown in Fig. 1 were found with a box length L equal to the wavelength of the external wave. In simulations with 3 or 12 times longer box length, the same time-asymptotic distribution and similar damping of the excited additional modes was observed although the spectrum of the excited modes at the initial phase was more dense.

By changing the acceleration factor κ, no significant change in the time asymptotic state was observed as long as $\kappa\nu_e\tau_b \ll 1$, where $\tau_b^{-1} \equiv \nu_b = \sqrt{eEk/m_e}$ is the inverse bounce time of electrons in the wave potential. At the steady state shown in Fig. 1, $\kappa\nu_e\tau_b = 0.007$ was found. The increase in κ has the tendency of decreasing $\partial f/\partial v$ in the trapping region. For very large values of κ, i.e., $\kappa\nu_e\tau_b > 1$, the distribution approaches Maxwellian. The observation that the time-asymptotic distribution function has a negligible dependence on κ for $\kappa\nu_e\tau_b \ll 1$ is in accordance with the analytical result obtained in the previous section. As is shown in Fig. 2, the time-asymptotic space-averaged distribution function obtained from the simulation agrees well with the distribution $(1/2\pi)\int_0^{2\pi} g_0(H)dy$ obtained from Eq. (8). For smaller κ, the agreement is found to be even better. In calculating the distribution from Eq. (8), we have used the electric field amplitude obtained by spatial Fourier analysis of the total self-consistent field of the simulation. The agreement with the harmonics expansion solution from Eqs. (14) and (15), where harmonics up to $n = 5$ were considered, is not so perfect, which indicates the need to take into account fairly many harmonics in the expansion when the wave has a large amplitude.

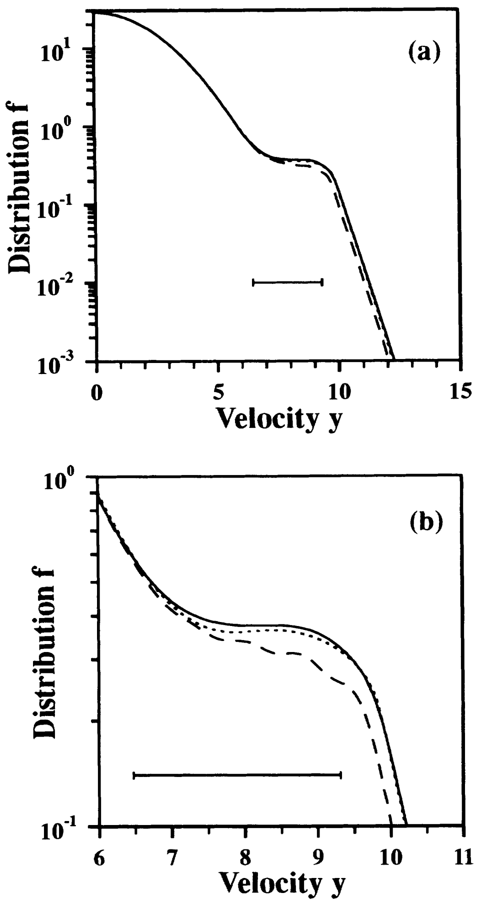

FIG. 2. The average distribution $f_0(y)$ with $y = v/v_b$ obtained from Eqs. (8), (14), and (15) for $v_p/v_b = 7.88$ and $v_e/v_b = 2.257$, corresponding to the time-asymptotic state obtained in Fig. 1. Here $v_b \equiv \tau_b^{-1}/k = \sqrt{eE/m_e k}$. The solution by using series expansion (14) and (15) is shown for $M = 5$ (dashed line). The analytical solution (8) is shown by a dotted line. The time-asymptotic distribution ($\omega_p t = 29\,952$) solved from Eqs. (17) and (18) is shown for comparison (solid line). The trapping region $|v - v_p| < \sqrt{2} v_b$ is indicated by a

However, in the parameter regime $v_b^2/v_p v_e < O(0.1)$ the calculations with the number of harmonics limited by M=5–8 were found to give the shape of the distribution and the current (see below) with a satisfactory accuracy.

Figures 3 and 4 show the time evolution of the electron current and the Fourier component $k\lambda_D = 0.35$ of the electric field. The approach to the steady state can be seen. In comparison to the case with no collisions, the obtained current is larger because of the diffusion of the electrons from the bulk part of the distribution to the resonant region. The time needed to achieve the steady state described by Eq. (8) is found to be inversely proportional to Γ. Consequently, in the absence of collisions a different steady state is found which cannot be obtained from Eq. (8). For decreasing collisionality, the oscillations in the current and in the amplitudes of the excited waves seen in the early phase in Figs. 3 and 4

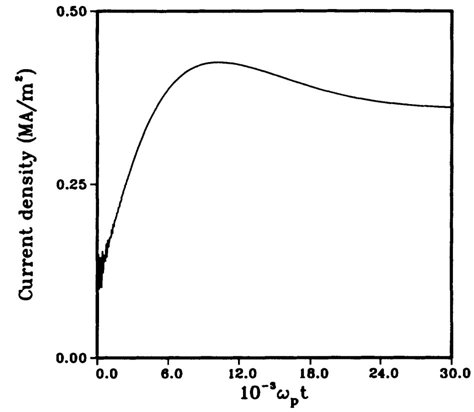

FIG. 3. Time evolution of the electron current density in the simulation shown in Fig. 1.

persist longer and the system evolves for a longer time in the manner of the collisionless case. In accordance with the theory of sideband instability for collisionless plasmas [16, 15], several waves around the phase velocity of the main wave are excited which exchange energy, causing the amplitude oscillations.

For very weak Γ, it has not been possible to reach the steady state and to distinguish between the collisionless and collisional cases because of limited computing time in our numerical simulations. On the other hand, for finite Γ the system has always been found to evolve to the steady state which is described by Eq. (8) for weak collisions or by Eqs. (14) and (15) also for strong collisions. It is of interest to note the weak generation of higher harmonics (or other Fourier components) of the electric field in contrast to the large number of harmonics f_n needed to model the distribution function.

The distribution in the absence of collisions with the same driving force as in the previous example, but with

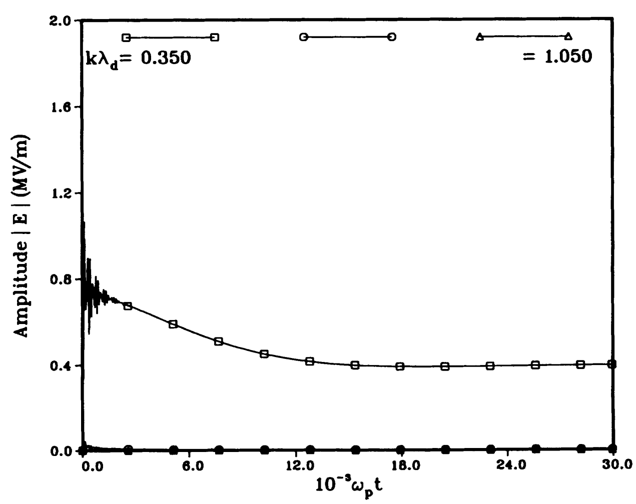

FIG. 4. Time evolution of the Fourier component $k\lambda_D = 0.35$ of the electric field in the simulation shown in Fig. 1. The second and third harmonics are found to have small amplitudes.

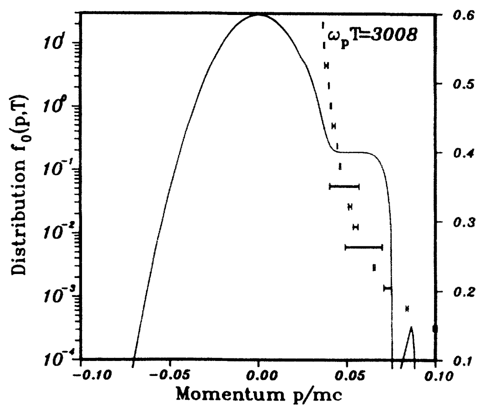

FIG. 5. The average distribution function solved from Eqs. (17) and (18) in the absence of collisions ($\Gamma = 0$) for the monochromatic driving force used in the case of Fig. 1. The trapping regions of the excited waves are indicated by lines. The length of the simulation box was chosen to be $L = 12 \times 2\pi/0.35$.

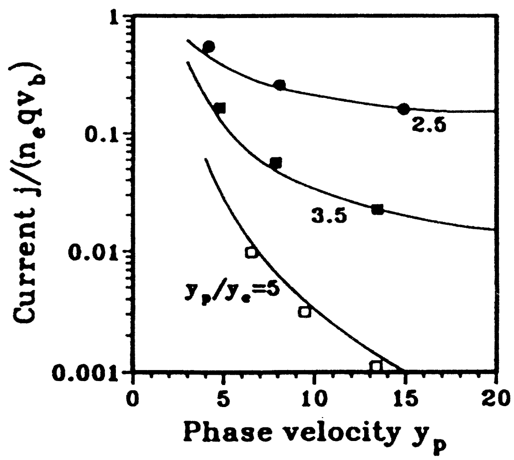

FIG. 6. The normalized total current density of the distribution obtained from Eq. (8) versus the normalized phase velocity $y_p = v_p/v_b$ (solid lines). The corresponding results obtained from Eqs. (17) and (18) are shown for $y_p/y_e = 2.5$ (●), 3.5 (■), and 5 (□), where $y_e = v_e/v_b$.

$L = 12 \times 2\pi/k$, is depicted in Fig. 5 at time $t = 3008\omega_p^{-1}$. The space-averaged distribution in this case reaches an apparent steady state through oscillations fairly quickly, in accordance with earlier results from particle and Vlasov-Poisson simulations [13, 15]. A noteworthy feature in the present results is a flat and broad plateau. As shown by the Fourier analysis of the electric field, the field structure is more complex than in the steady state with collisions. Several Fourier modes are excited, which play a role in the formation of the plateau. As in the collisional case, the waves are excited by the sideband instability after electron acceleration in the early phase of the wave excitation. In contrast to the case with dissipation, the waves with wave numbers (and frequencies) different from that of the driving force are not damped away in the collisionless case, and appear to persist. On the basis of the present simulation, it is not possible to say whether the distribution in Fig. 5 represents a true steady state or not. Much longer simulations would be needed to verify that, and the accuracy is not guaranteed with the present method in the collisionless case.

The steady-state current density $[qn_e \int vg_0(H)dv]/[\int g_0(H)dv]$ obtained from the analytical solution in Eq. (8) is shown in Fig. 6 as a function of $y_p = v_p/v_b$ and $y_e = v_e/v_b$. In the shown region, the current density obtained from Eqs. (14) and (15) with $M = 8$ was found to agree with the result in Fig. 6 with an accuracy better than 1%. As shown in Fig. 6, the current density obtained from the solution in Eq. (8) is found to deviate less than 10% from the solutions of Eqs. (17) and (18) in the parameter range shown. The largest deviations are found for small y_p, where the difference between the collision operators becomes more pronounced. At very large amplitudes when the wave energy density exceeds or is of the same order as the thermal energy density of the plasma, i.e., when $(y_p/y_e)^2/y_p^4 \geq 1$, we have found solutions with complex time behavior showing transient double-humped distribution functions with similar features to those obtained by Krapchev and Ram [4] for a collisionless plasma. In this regime, the distribution perhaps cannot be modeled with the linearized collision model (2), and no comparison with the solution of Eqs. (17) and (18) is shown in Fig. 2.

IV. CONCLUSIONS

The effect of a nonlinear periodic force on particle distribution in a linearized Fokker-Planck collisional plasma has been investigated. The problem has been reduced in the case of weak collisions to the calculation of the approximate distribution function in Eqs. (8) and (9), where a linearized Fokker-Planck type of collision operator term has been considered. The method is applied to a study of the effects of a plasma wave with a modest energy on the particle distribution. It is expected that the solution in Eqs. (8) and (9) can be further extended to describe waves whose energy may be comparable to or greater than the thermal energy of the particles [4, 21], or the particle distribution in a magnetic well with a wide trapped velocity cone, for studies of particle and energy confinement by magnetic mirrors and of neoclassical theory of plasma diffusion in toroidal magnetic confinement systems [9]. Here, a more general Fokker-Planck collision term has to be considered.

In the case of a driven large-amplitude plasma wave and in the presence of collisions, the time-asymptotic distribution is found to be maintained by one wave only, in contrast to the case with no collisions, where several waves appear to maintain the distribution in the steady state. The present results indicate that the time-asymptotic distribution is insensitive to the collision rate as long as the collision time is much larger than the bounce time $\sqrt{m_e/eEk}$ of electrons in the wave potential. This is demonstrated by the expansion of Eqs. (14) and (15) to first order in Γ, where the dependence on

Γ vanishes, and also by our simulations. The analytical solution in Eq. (8) obtained in the limit $\Gamma \to 0$ can thus be used with good accuracy to describe the distribution for finite Γ. No assumption concerning the ratio v_b/v_e is required, which makes this solution useful for problems concerned with large-amplitude nonlinear waves. By comparison with the simulation results, the $\Gamma \to 0$ solution has been found to be valid in the range $\nu_e/\nu_b \ll 1$ and $v_b^2/v_e v_p < 1$. For very weak collisions, it has not been possible in the numerical simulations to reach the steady state and to distinguish between the collisionless and collisional cases because of limited computing time. However, for very slow collisional diffusion it can be true that the collisional steady state described by Eq. (8) for a single wave in a periodic box is never reached in a realistic physical system because of nonideal conditions (see below).

For strong collisionality, i.e., when the collision time approaches or becomes smaller than the trapping time, the solution to Eq. (1) has to be found from the numerical solution of Eqs. (14) and (15). According to our simulations, the high collisionality is found to steepen the distribution in the resonance region in agreement with the results in Ref. [18]. Consequently, the asymmetry of the distribution or the driven current created by the wave is reduced.

The collisional effects as they are presented in this paper become important when the physical system is long enough or periodic so that $L/v > \nu_e^{-1}$, where v is the typical velocity of the particle, L characterizes the length of the system, and ν_e^{-1} denotes the typical collision time. In systems which are short compared to the collision length or strongly inhomogeneous, boundary and gradient effects dominate and more detailed studies of the distribution are needed. An additional ingredient which may also alter the present results is the onset of stochasticity in the particle orbits on a time scale smaller than the collision time. The stochasticity can arise because of additional overlapping of trapping regions of electrostatic modes in velocity space or because of nonideal geometry of the system [25]. These phenomena may cause the breakdown of the invariants of the trapped particle motion thus leading to time-asymptotic distributions different from the present theory. It is important to note that, in the periodic simulations of the present paper, such effects have not been found in the presence of collisions. In magnetized plasmas, the electron plasma waves are usually driven along the field lines, in which case the present unmagnetized theory should be sufficient. It has also been shown in the case of current drive in magnetized plasmas [26] that the one-dimensional description of the wave-particle interaction captures the salient features of the higher dimensional solutions. For accurate calculation of $f(v)$, a fully two-dimensional or three-dimensional numerical treatment would be necessary.

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APPENDIX

In order to show that $f_n \pm (-1)^n f_{-n}$ in Eq. (16) satisfy Eqs. (14) and (15) in our paper, we find for the left-hand sides (LHS) of these equations

$$\begin{aligned}(\mathrm{LHS}) = {} & ink(v-\omega/k)\frac{1}{2\pi}\int_0^{2\pi}[e^{-iny}\pm(-1)^n e^{iny}]\left[\sum_{\ell\neq 0}\frac{\Gamma m^2}{ki\ell}A_\ell(H)e^{i\ell y}+g_0(H)\right]dy \\ & +\frac{F}{2m}\frac{1}{2\pi}\int_0^{2\pi}[e^{-i(n-1)y}+e^{-i(n+1)y}\pm(-1)^{n-1}e^{i(n-1)y}\pm(-1)^{n+1}e^{i(n+1)y}] \\ & \times m(v-\omega/k)\frac{\partial}{\partial H}\left[\sum_{\ell\neq 0}\frac{\Gamma m^2}{ki\ell}A_\ell(H)e^{i\ell y}+g_0(H)\right]dy. \end{aligned} \tag{A1}$$

The last term in Eq. (A1) can be partially integrated by noting that $[e^{-i(n-1)y}+e^{-i(n+1)y}\pm(-1)^{n-1}e^{i(n-1)y}\pm(-1)^{n+1}e^{i(n+1)y}]=2\cos(y)[e^{-iny}\mp(-1)^n e^{iny}]$. This term then reads [note that $-\frac{F}{k}\cos(y)\frac{\partial}{\partial H}A_\ell(H)=\frac{\partial}{\partial y}A_\ell(H)$]

$$\begin{aligned} & k(v-\omega/k)\frac{1}{2\pi}\int_0^{2\pi}\sum_{\ell\neq 0}\frac{\Gamma m^2}{ki\ell}[i(\ell-n)e^{-iny}\mp i(\ell+n)(-1)^n e^{iny}]A_\ell(H)e^{i\ell y}dy \\ & \qquad +k(v-\omega/k)\frac{1}{2\pi}\int_0^{2\pi}[-ine^{-iny}\mp in(-1)^n e^{iny}]g_0(H)dy. \end{aligned} \tag{A2}$$

Consequently, the zeroth order (in Γ) parts of the LHS in Eq. (A1) sum up to zero. From this we can deduce that the zeroth order part in Eq. (16) satisfies the homogeneous parts of Eqs. (14) and (15). The terms to first order in Γ in Eq. (A1) sum up to give

$$(\mathrm{LHS}) = \frac{1}{2\pi}\int_0^{2\pi}\sum_{\ell\neq 0}\Gamma m^2(v-\omega/k)[e^{-iny}\mp(-1)^n e^{iny}]A_\ell(H)e^{i\ell y}dy. \tag{A3}$$

The right-hand side (RHS) of Eqs. (14) and (15) in the main text can be found to first order in Γ by substituting $g_0(H)$ for f. This gives

$$\begin{aligned}(\mathrm{RHS}) &= \Gamma m^2(v-\omega/k)\frac{\partial}{\partial H}\left\{\left[\frac{v-\omega/k}{\tau(H,y)}\left(\frac{\partial}{\partial H}+\frac{1}{T}\right)+\frac{\omega/k}{\tau(H,y)T}\right]\frac{1}{2\pi}\int_0^{2\pi}[e^{-iny}\mp(-1)^n e^{iny}]g_0(H)dy\right\}\\ &= \Gamma m^2(v-\omega/k)\frac{1}{2\pi}\int_0^{2\pi}[e^{-iny}\mp(-1)^n e^{iny}]\frac{\partial}{\partial H}\left\{\left[\frac{\partial g_0}{\partial H}+\frac{g_0}{T}\right]\sum_\ell \frac{e^{i\ell y}}{L(H)}\oint e^{-i\ell y}\frac{u(H,y)}{\tau(H,y)}dy\right.\\ &\quad\left.+\frac{g_0}{T}\sum_\ell\frac{e^{i\ell y}}{L(H)}\oint e^{-i\ell y}\frac{\omega/k}{\tau(H,y)}dy\right\}dy. \end{aligned} \tag{A4}$$

By noting the definition of $A_\ell(H)$ in Eq. (9), we see that the sum of the terms with $\ell \neq 0$ in the RHS equals the LHS given in Eq. (A3). Therefore, the solutions for p_n and q_n in Eq. (16) satisfy Eqs. (14) and (15), provided the term with $\ell = 0$ in Eq. (A4) vanishes for all integers n. This takes place when $g_0(H)$ is given by Eq. (8), which closes our proof.

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