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Two-dimensional stimulated Brillouin scattering

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The evolution of the stimulated Brillouin scattering (SBS) instability in time and two spatial dimensions is studied analytically. An exact solution of the linearized equations governing SBS in a finite homogeneous plasma shows that this two-dimensional instability usually saturates because of the convection of the ion-acoustic wave, regardless of whether the associated one-dimensional interaction is convectively or absolutely unstable. The steady-state intensity profile of the Stokes light wave is often highly two dimensional.

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Stimulated Brillouin scattering (SBS) in a plasma is the decay of an incident light wave (0), also called a pump wave, into a Stokes light wave (1) and an ion-acoustic wave (2). The conservation of energy and momentum in this process is reflected in the frequency and wave-vector matching conditions

$$\omega_0=\omega_1+\omega_2\ , \quad \mathbf{k}_0=\mathbf{k}_1+\mathbf{k}_2\ , \tag{1}$$

the second of which is illustrated in Fig. 1(a). SBS is important in the field of inertial confinement fusion (ICF) [1] because it can scatter the incident light away from the target, thereby reducing the amount of energy available to drive the compressive heating of the nuclear fuel. The inhomogeneities of a typical ICF plasma complicate the analysis of SBS considerably. However, as the main effect of these inhomogeneities is to limit the region over which SBS occurs, it is not unreasonable to model the interaction region as a homogeneous slab whose dimensions are chosen with the true plasma inhomogeneities in mind.

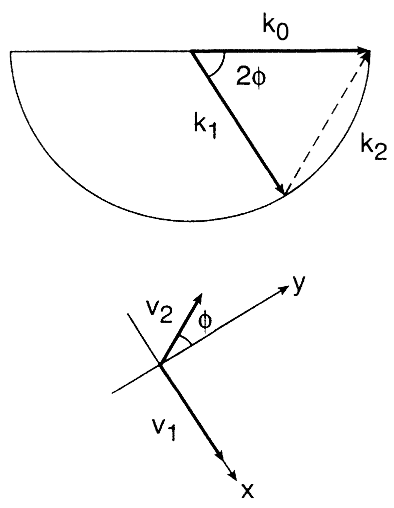

FIG. 1. Interaction geometry for SBS. The angular dependence of γ_0, the temporal growth rate of SBS, stems from the fact that $k_2=2k_0\sin\phi$. The x component of $\mathbf{v}_2$ is always negative.

This simplification facilitates the study of two-dimensional effects, about which little is known [2].

The initial evolution of the SBS instability in a homogeneous plasma is governed by the linearized equations

$$\begin{aligned} &v_{1x}\partial_x A_1=-i\gamma_0 A_2^*\ ,\\ &(\partial_t-v_{2x}\partial_x+v_{2y}\partial_y+\nu_2)A_2^*=i\gamma_0 A_1\ , \end{aligned} \tag{2}$$

where A_1 represents the vector potential associated with the Stokes wave, A_2 represents the plasma density fluctuation associated with the ion-acoustic wave, γ_0 is the temporal growth rate of SBS in an infinite plasma, and ν_2 is the temporal decay rate of the ion-acoustic wave due to Landau damping. Explicit formulas for these two rates are well known [3,4]. In the context of this paper, it is more useful to exhibit their angular dependence explicitly by writing $\gamma_0=\gamma_b(\sin\phi)^{1/2}$ and $\nu_2=\nu_b\sin\phi$, where the subscript b refers to backward SBS. It follows from Fig. 1(b) that $v_{1x}=v_1$, $v_{2x}=v_2\sin\phi$, and $v_{2y}=v_2\cos\phi$. There are two time scales associated with SBS in a finite plasma: the (short) transit time of the Stokes wave and the (long) transit time of the ion-acoustic wave. Since the transition of SBS from an instability characterized by spatiotemporal growth to one characterized by a convective steady state or by absolute temporal growth occurs on the latter time scale, the t derivative was omitted from the first of Eqs. (2).

Although the origin of Eqs. (2) was described for SBS in a plasma, the equations themselves can be used to model stimulated Raman scattering in a plasma and parametric instabilities in other nonlinear media, many of which are truly homogeneous. Thus the results of this paper are also relevant to current research in basic plasma physics, parametric electronics, nonlinear optics, and fluid dynamics.

The solution of Eqs. (2) is facilitated by changing variables according to $x/l_x\to x$, $v_{2x}y/v_{2y}l_x\to y$, $v_{2x}t/l_x\to t$, $\gamma_0 l_x/(v_{1x}v_{2x})^{1/2}\to\gamma$, $\nu_2 l_x/v_{2x}\to\alpha$, $v_{1x}^{1/2}A_1\to A_1$, and $-iv_{2x}^{1/2}A_2^*\to A_2$. Using the new variables, Eqs. (2) can be rewritten as

$$\partial_x A_1=\gamma A_2\ , \quad (\partial_t-\partial_x+\partial_y+\alpha)A_2=\gamma A_1\ . \tag{3}$$

Equations (3) can be solved analytically, regardless of what initial and boundary conditions are imposed on the wave amplitudes. However, the mathematical details of this paper are minimized by the conditions

$$A_1(0,x,y)=1\ ,\quad A_1(t,0,y)=1\ , \tag{4}$$

and

$$A_2(0,x,y)=0\ ,\quad A_2(t,x,0)=0\ ,\quad A_2(t,1,y)=0\ , \tag{5}$$

which are associated with the amplification of an externally generated Stokes wave in a rectangular plasma that is completely illuminated by the pump wave.

Even though one could argue that Eqs. (3)–(5) describe the simplest two-dimensional problem relevant to SBS, their solution is neither easy to obtain nor easy to understand. Thus it is advantageous to consider first the model equations

$$\partial_x A_1=\gamma A_2\ ,\quad (\partial_t+\partial_y)A_2=\gamma A_1\ , \tag{6}$$

together with conditions (4) and the first two of conditions (5). This model problem contains the essential feature of two-dimensional SBS, namely the evolution of coupled waves as they propagate in different directions.

By using the characteristic variables $\tau=t$, $\xi=x$, and $\eta=y-t$, the model equations can be rewritten as

$$\partial_\xi A_1=\gamma A_2\ ,\quad \partial_\tau A_2=\gamma A_1\ . \tag{7}$$

It follows immediately that

$$\partial^2_{\tau\xi} A_1=\gamma^2 A_1\ . \tag{8}$$

Equation (8) is a standard equation of mathematical physics. The domain for which its solution is sought is illustrated in Fig. 2. It is evident from Eq. (8) that A_1 depends only parametrically on the variable η. Physically, this means that the evolution of A_1 on each characteristic plane, labeled by its associated value of η, is independent of the evolution of A_1 on the neighboring planes. This property of A_1 and the characteristic lines associated with the evolution of A_1 in τ and ξ are illustrated in Fig. 3(a). Because Eq. (8) is invariant under time translations, its solution can be written in terms of the elapsed time $\tau-\tau_0(\eta)$, where $\tau_0=0$ for $\eta\geq 0$ and $\tau_0=-\eta$ for $\eta<0$. Thus a particular solution of Eq. (8) is

$$A_1(\tau,\xi,\eta)=I_0\{2\gamma[(\tau-\tau_0)\xi]^{1/2}\}\ , \tag{9}$$

where I_0 is the modified Bessel function of the first kind of order 0. By a fortunate coincidence, this solution has the desired property that $A_1=1$ on the bottom, rear, and side boundaries of Fig. 2. In terms of the variables used in Eq. (6), the elapsed time is $t-t_0(t,y)$, where $t_0=0$ for $y\geq t$ and $t_0=t-y$ for $y<t$. Thus the solution of the model problem is

$$\begin{aligned} A_1(t,x,y)&=I_0[2\gamma(tx)^{1/2}]H(y-t)+t\leftrightarrow y\ ,\\ A_2(t,x,y)&=(t/x)^{1/2}I_1[2\gamma(tx)^{1/2}]H(y-t)+t\leftrightarrow y\ , \end{aligned} \tag{10}$$

where H is the Heaviside function.

Snapshots of A_1 are displayed in Fig. 4 for $t=0.25$ and $t=0.75$. In Fig. 4(a) most of the wave has yet to feel the effect of the boundary at $y=0$ and, consequently, is still growing in time. The wave evolution is almost one dimensional. In Fig. 4(b) most of the wave has already felt the effect of the boundary and, consequently, has stopped growing in time. The steady-state amplitude profile of the wave is highly two dimensional.

The dependence of solution (10) on the elapsed time and the fact that the growth of waves 1 and 2 at a particular location stops as soon as the information that a boundary exists at $y=0$ reaches that location are both due to the propagation of wave 2 in the y direction. To understand these phenomena, consider the wave evolution in the neighborhood of an interior point (x,y). Initially, $A_1=1$ and $A_2=0$. After a time interval dt, $A_2=\gamma dt$. Even though the portion of wave 2 that was originally at (x,y), with zero amplitude is now at

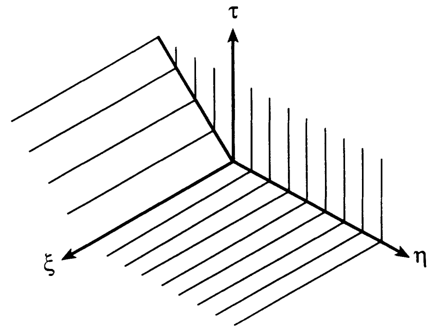

FIG. 2. Shape of the interaction region associated with the characteristic variables τ, ξ, and η.

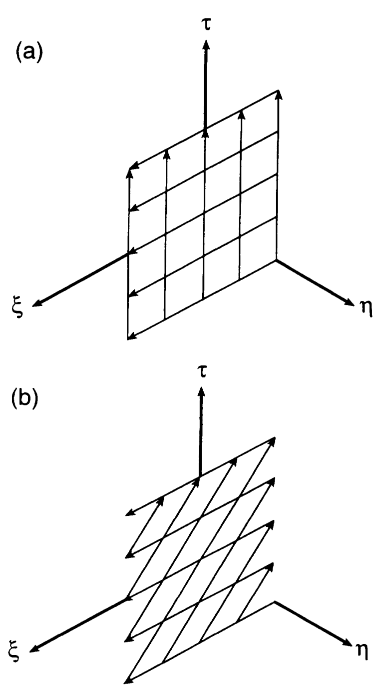

FIG. 3. Characteristic lines associated with the evolution of A_1 and A_2 are illustrated for (a) the model equations (7) and (b) the SBS equations (11).

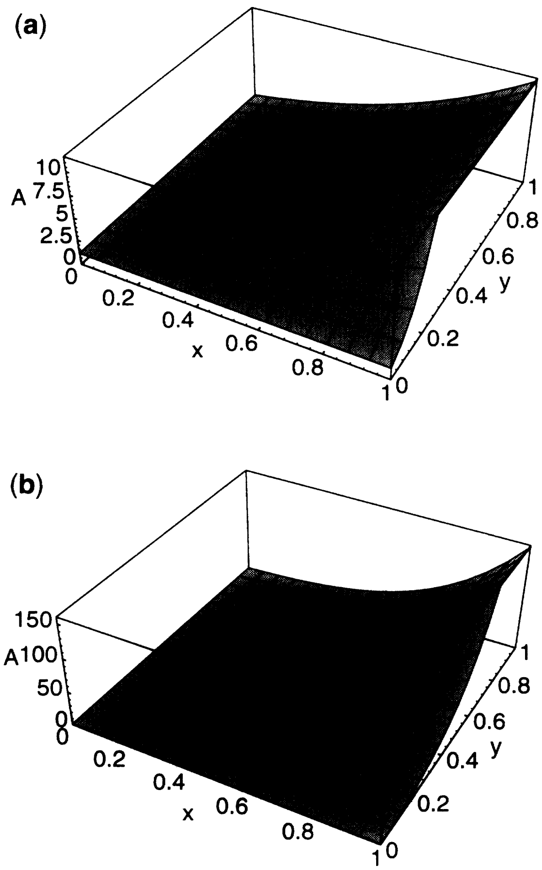

FIG. 4. Snapshots of A_1, obtained from the first of Eqs. (10), are displayed for (a) $t=0.25$ and (b) $t=0.75$. In both cases, $\gamma=4$.

$(x,y+dy)$, it has been replaced by the portion of wave 2 that was originally at $(x,y-dy)$. This replacement continues, as does the temporal growth of the portions of waves 1 and 2 at (x,y), until $t=y$. At this time, the portion of wave 2 at (x,y) has attained its maximal amplitude, which is limited by the time available for it to grow as it propagates inward from the boundary at $y=0$. This growth time is none other than the elapsed time that appears in Eqs. (9) and (10). The preceding remarks are valid for arbitrary x. When $t>y$, the growth of wave 1 as it propagates in the x direction is driven by wave 2, which is in steady state. Since the amplitude of wave 1 at the boundary $x=0$ is independent of time, wave 1 must also be in steady state.

The extension of the preceding analysis to the SBS equations (3) is straightforward. By using the characteristic variables defined above, Eqs. (3) can be rewritten as

$$\partial_\xi A_1=\gamma A_2\ , \quad (\partial_\tau-\partial_\xi+\alpha)A_2=\gamma A_1\ . \tag{11}$$

The domain for which the solution is sought is identical to that shown in Fig. 2. As before, the evolution of A_1 and A_2 on each characteristic plane, labeled by its associated value of η, is independent of the evolution of A_1 and A_2 on the neighboring planes, and the solution of Eqs. (11) depends on the elapsed time $\tau-\tau_0(\eta)$. Thus, at any interior point (x,y) the waves grow in time until $t=y$, at which time their growth stops. Equations (11) differ from the model equation in that the ion-acoustic wave propagates in the negative x direction. Despite this difference, which is illustrated in Fig. 3(b), Eqs. (11) can be solved analytically. The general version of the one-dimensional coupled-wave equations, in which v_{1x} and v_{2x} are arbitrary, has been solved by Bobroff and Haus [5], and, more recently, by Williams and McGowan [6]. Equations (11), which were simplified by the assumption that $v_{2x} \ll v_{1x}$, have been solved by McKinstrie and co-workers [7,8]. [Although the solution of the latter equations is a special case of the solution of the former, it is noteworthy in its own right because it is easier to use and to interpret physically.] For the initial and boundary conditions (4) and (5), the Stokes amplitude is given by

$$A_1(t,x,y)=H(y-t)\int_0^t G_{11}(x,t')dt'+t\leftrightarrow y\ , \tag{12}$$

where

$$\begin{aligned} G_{11}(t,x)=\gamma\sum_{n=-\infty}^{\infty}\Bigg\{&\left[\frac{t-n+x}{t+n}\right]^{n+1/2} I_{2n+1}\{2\gamma[(t+n)(t-n+x)]^{1/2}\} \\ &-\left[\frac{t-n+x}{t+n}\right]^{n-1/2} I_{2n-1}\{2\gamma[(t+n)(t-n+x)]^{1/2}\}\Bigg\} \\ &\times H(t+n)H(t-n+x)\exp(-\alpha t)+\delta(t)H(t+x) \end{aligned} \tag{13}$$

is the Green function that describes the effect on the Stokes wave of an impulse applied to itself. Similarly,

$$A_2(t,x,y)=H(y-t)\int_0^t G_{21}(x,t')dt'+t\leftrightarrow y\ , \tag{14}$$

where

$$\begin{aligned} G_{21}(t,x)=\gamma\sum_{n=-\infty}^{\infty}\Bigg\{&\left[\frac{t-n+x}{t+n}\right]^{n} I_{2n}\{2\gamma[(t+n)(t-n+x)]^{1/2}\} \\ &-\left[\frac{t-n+x}{t+n}\right]^{n-1} I_{2n-2}\{2\gamma[(t+n)(t-n+x)]^{1/2}\}\Bigg\}H(t+n)H(t-n+x)\exp(-\alpha t) \end{aligned} \tag{15}$$

is the Green function that describes the effect on the ion-acoustic wave of an impulse applied to the Stokes wave. No snapshots of the wave amplitudes are displayed because they are qualitatively similar to those of Fig. 4. In the following discussion, the properties of the wave amplitudes associated with Eqs. (3) are described in terms of the dimensional quantities associated with Eqs. (2).

Before one can fully understand the evolution of two-dimensional SBS, one must first understand the evolution of the reduced one-dimensional interaction with which it is associated. This one-dimensional interaction is characterized by the Green functions (13) and (15). Initially, $t \ll l_x/v_2 \sin\phi$ and the Stokes output $A_1(t,l_x)$ grows as $\exp[2\gamma_b(l_x t \sin\phi/v_1)^{1/2} - \nu_b t \sin\phi]$. What happens subsequently depends on how strongly the interaction is driven. When $\gamma_b/(v_1 v_2)^{1/2} < \nu_b/2v_2$, the Stokes output eventually saturates due to convection in the x direction. The saturation time is given by [7,8]

$$t_s \approx (a/b - 1) l_x / v_2 \sin\phi \ , \tag{16}$$

where

$$a = \frac{l_x \nu_b}{2v_2} \ , \quad b = \frac{l_x}{2} \left[\frac{\nu_b^2}{v_2^2} - \frac{4\gamma_b^2}{v_1 v_2} \right]^{1/2} \tag{17}$$

and the steady-state Stokes output is given by [7,8]

$$\lim_{w \to \infty} A_1(t,l_x) \approx \frac{2b \exp(a-b)}{a+b} \ . \tag{18}$$

When $\gamma_b/(v_1 v_2)^{1/2} \ll \nu_b/2v_2$, $t_s \ll l_x/v_2 \sin\phi$. The saturation time and Stokes output are almost independent of the ion-acoustic speed because the damping term dominates the ion-acoustic equation and the amplitude of the ion-acoustic wave is slaved to that of the Stokes wave. In contrast, when $\gamma_b/(v_1 v_2)^{1/2} \approx \nu_b/2v_2$, $t_s \sim l_x/v_2 \sin\phi$. Because the convection term in the ion-acoustic equation is comparable to the damping term, the saturation time and Stokes output depend sensitively on the ion-acoustic speed [7]. In both cases, the Stokes output is independent of the scattering angle [8]. When $\gamma_b/(v_1 v_2)^{1/2} > \nu_b/2v_2$, the interaction is absolutely unstable and the Stokes output continues to grow in time with a growth rate that depends on the scattering angle. Eventually, $t \gg l_x/v_2 \sin\phi$ and the Stokes output grows as $\exp\{[2\gamma_b(v_2/v_1)^{1/2} - \nu_b]t \sin\phi\}$.

It is clear from the discussion following Eqs. (11) that the growth time of two-dimensional SBS cannot exceed the time taken for the ion-acoustic wave to convect across the plasma in the y direction, regardless of whether the associated one-dimensional interaction is convectively or absolutely unstable [9]. This convention time is given by

$$t_c = l_y / v_2 \cos\phi \ . \tag{19}$$

It follows from Eqs. (16), (17), and (19) that the growth of near-forward SBS in a square plasma is limited by convection in the y direction and is inherently two dimensional. The stokes output is much less than that predicted by the reduced one-dimensional model described above. When $\gamma_b/(v_1 v_2)^{1/2} < \nu_b/2v_2$, the growth of sideward and near-backward SBS is limited by convection in the x direction. Thus, the evolution of sideward and near-backward SBS is almost one dimensional and the Stokes output can be calculated using Eq. (18). When $\gamma_b/(v_1 v_2)^{1/2} > \nu_b/2v_2$, the growth of sideward and near-backward SBS is limited by convection in the y direction and is inherently two dimensional. With the exception of backward scattering, SBS is convectively saturated before the Stokes and ion-acoustic waves can grow exponentially in time with the absolute instability growth rate. The preceding discussion is based on the assumption that t_l, the temporal pulse width of the pump wave, is infinite. When t_l is finite, the growth of SBS is limited by the smallest of t_c, t_l, and t_s.

Finally, it should be mentioned that numerical simulations of two-dimensional SBS have been made by Amin *et al.* [10] for the complimentary case in which the ion-acoustic wave is subject to viscous damping.

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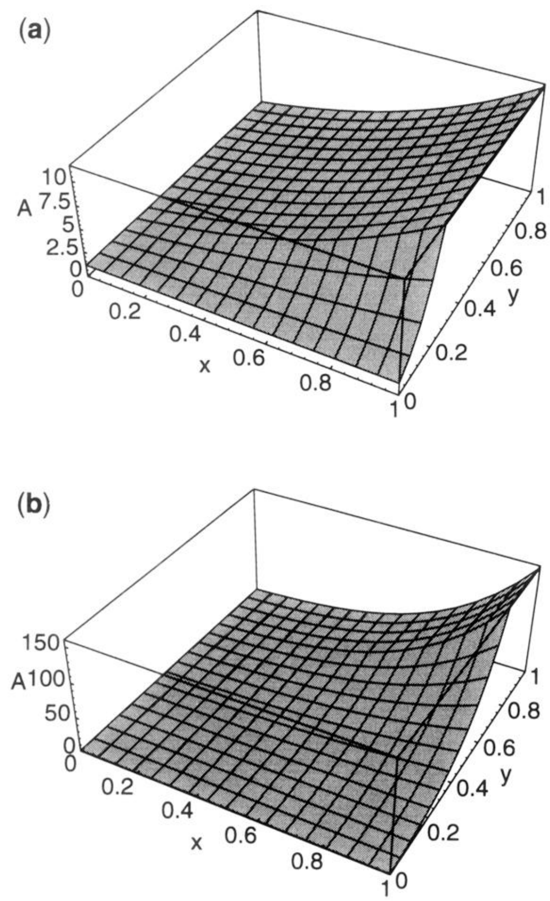

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