Non-neutral dynamics of splay states in Josephson-junction arrays

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We study the effect of the phase dependent conductance on the splay state dynamics of series Josephson arrays. In most cases, the splay states are destabilized, and the array evolves into a more coherent state, e.g., the in-phase state. A smaller parameter range exists where the splay state becomes attracting.

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I. INTRODUCTION

Recent numerical simulations have uncovered a peculiar puzzle concerning the nonlinear dynamics of one dimensional (i.e., series) Josephson-junction arrays. When dc biased, these arrays can exhibit periodic oscillations of the type variously called "antiphase" or "splay state" behavior: each of the N oscillators has the same wave form, yet no two oscillators are in phase [1–4]. The peculiarity is that under a wide variety of circumstances the splay states are neither stable nor unstable [5–7]; in typical dynamical systems, neutral stability occurs only at special parameter values, i.e., bifurcation points. But these arrays exhibit neutrally stable splay states over a wide range of parameters. The puzzle is why these arrays exhibit this nongeneric behavior at all. Similar results hold for splay states of globally coupled solid state laser arrays [8,9].

Important steps toward solving this puzzle were made by several authors for the case where the junction capacitance C_J is negligible [10–13], i.e., using the so-called resistively shunted junction model, corresponding to point contact type junctions [14]. Indeed, Watanabe and Strogatz [13] have shown rigorously that for $C_J = 0$ the circuit equations exhibit an extraordinary structure of the type normally associated with Hamiltonian systems, even though the Josephson array is manifestly dissipative. Physically, this structure shows itself as constants of the motion. While this implies directly the neutral stability of the splay states, it is not the whole story. Existing numerical work [7] suggests that the nongeneric neutral stability of splay states also holds in cases where the junction capacitance is important, and no analytic progress has been made in this direction.

It has also been shown that, when $C_J = 0$, breaking the symmetry fundamentally changes the stability properties of the splay state [10,12]. However, this was done in an *ad hoc* way, modifying the governing dynamical equations in a way unmotivated by physical considerations. The purpose of this paper is to see if a physically relevant effect, typically ignored in theoretical studies of the dynamics of Josephson-junction arrays [15], removes the neutral stability of the splay state. Namely, we consider the role of the phase dependent conductance [16], sometimes called the $\epsilon\cos\phi$ term. We present the results of numerical simulations for both $C_J = 0$ and $C_J \neq 0$ cases, for arrays of size $N = 5$. For $C_J = 0$, we gain insight into the behavior of very large arrays by extending the analytic work of Strogatz and Mirollo [11], valid in the $N \to \infty$ limit. We find that under most circumstances the splay states lose their neutral stability. For very small ϵ the numerical results are inconclusive, though consistent with the splay states being neutrally stable. Depending on the value of the conductance constant ϵ, the splay state becomes either stable or unstable; in the latter case the system is attracted to a more coherent dynamical state, either a cluster state or the in-phase state. Our analytic results for $N \to \infty$ show that for small ϵ, one can make the splay state attracting or repelling by controlling the sign of ϵ.

II. BACKGROUND

The equations of motion for a Josephson-junction array can be derived easily from the Josephson relations and simple circuit analysis [2,14]. The equations corresponding to the N junction array shunted by a resistor, inductor, and capacitor shown in Fig. 1 are given by

$$\beta\ddot{\varphi}_k + (1 + \epsilon\cos\varphi_k)\dot{\varphi}_k + \sin\varphi_k + \dot{Q} = I_{dc}\,, \qquad (2.1)$$

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \frac{1}{N}\sum_{j=1}^{N}\dot{\varphi}_k\,, \qquad (2.2)$$

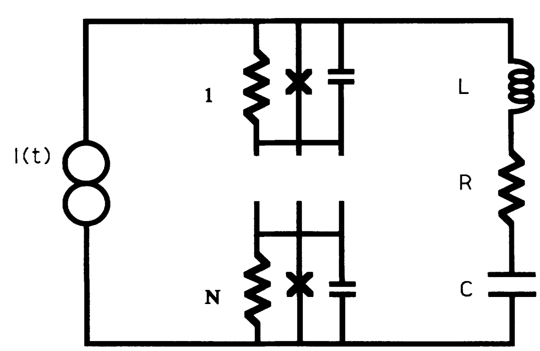

FIG. 1. Circuit schematic for an RLC shunted Josephson-junction array.

where $k = 1, 2, \ldots, N$; I_{dc} is the drive current; Q and $\dot{Q}$ are the charge and current in the RLC branch of the circuit; φ_k is the quantum phase difference across the kth Josephson junction, R, L, and C are the resistance, inductance, and capacitance of the RLC branch; ϵ is a parameter associated with the phase dependent conductance of the junction; N is the number of junctions in the array; and β is a parameter that depends on the physical properties of the junction and is proportional to its capacitance C_J. Time and current have been rescaled to make the equations dimensionless. We consider the RLC parallel load because it is the most general [2,11]. Note that since ϵ and β do not carry the subscript k, we are implicitly assuming that all of the junctions are identical.

Previous studies of the stability of the splay states treated the conductance as being constant, i.e., taking $\epsilon = 0$ in Eq. (2.1). However, the conductance actually depends on the phase of the junction because the quasiparticle tunneling current depends on the quantum states of the superconductors [16], which leads to the $(1 + \epsilon \cos \phi_k)$ conductance factor in Eq. (2.1).

We ask the following question: what effect does the $\epsilon \cos \varphi$ term have on the neutral stability of the splay states of a Josephson-junction array? In particular, is the neutral stability an artifact of taking $\epsilon = 0$ or does it persist when the phase dependent conductance effect is included? For the $\beta = 0$ case, we find that the $\epsilon \cos \varphi$ term always breaks the neutral stability, but that the splay state can be either linearly stable or linearly unstable, depending on the sign of ϵ. For the $\beta \neq 0$ case, the neutral stability seems to be preserved for small values of ϵ.

In the next section we consider the $\beta = 0$ case: we extend a calculation of Strogatz and Mirollo in order to predict the effect of the $\epsilon \cos \phi$ term on the stability of the splay state and compare the predictions with numerical analysis. In Sec. IV we turn to the $\beta \neq 0$ case, where we have only numerical work to guide us at present. We summarize our results in Sec. V.

III. THE $\beta = 0$ CASE

For junctions with negligible capacitance, we set $\beta = 0$ in Eqs. (2.1) and (2.2), yielding the $(N+2)$-order system of equations

$$(1 + \epsilon \cos \varphi_k)\dot{\varphi}_k + \sin \varphi_k + \dot{Q} = I_{dc}\,, \tag{3.1}$$

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \frac{1}{N}\sum_{j=1}^{N} \dot{\varphi}_k\,. \tag{3.2}$$

Following Ref. [11], in the $N \to \infty$ limit we can define $\rho(\varphi, t)$ to be the density of oscillators at angle φ at time t. The summation in Eq. (3.2) becomes

$$\frac{1}{N}\sum_{j=1}^{N} \dot{\varphi}_j = \int_{\varphi=0}^{2\pi} \dot{\varphi}\rho(\varphi, t)d\varphi\,. \tag{3.3}$$

From Eq. (3.1) the velocity field $\dot{\varphi}$ is given by

$$\dot{\varphi} = \frac{I_{dc} - \dot{Q} - \sin \varphi}{1 + \epsilon \cos \varphi}\,.$$

Substituting this and Eq. (3.3) into Eq. (3.2) gives

$$L\ddot{Q} + (R+1)\dot{Q} + \frac{1}{C}Q = \int_{\varphi=0}^{2\pi} \frac{I_{dc} - \dot{Q} - \sin \varphi}{1 + \epsilon \cos \varphi}\rho(\varphi, t)d\varphi\,. \tag{3.4}$$

We also have a continuity equation

$$\frac{\partial}{\partial t}\rho(\varphi, t) + \frac{\partial}{\partial \varphi}\left(\rho(\varphi, t)\left[\frac{I_{dc} - \dot{Q} - \sin \varphi}{1 + \epsilon \cos \varphi}\right]\right) = 0\,, \tag{3.5}$$

which just says that the number of oscillators is conserved locally (with respect to φ). We are interested in the splay states of the system. For finite N, these are solutions in which all of the oscillators have the same wave form, but are shifted in time from each other by increments of kT/N, where k is an integer, T is the period of the oscillation, and N is the total number of oscillators. One such splay state is defined by

$$\varphi_k(t) = \varphi_{k+1}\left(t + \frac{T}{N}\right)\,. \tag{3.6}$$

Because of the permutation symmetry of the oscillator array equations, the existence of one splay state implies the existence of $(N-1)!$ distinct splay phase states.

In the limit $N \to \infty$, Strogatz and Mirollo [11] identified the splay state with the stationary solution to Eqs. (3.4) and (3.5). To see why this is reasonable, note that the splay state condition (3.6) implies that $Q(t)$ has harmonic content only at frequencies 0, $\frac{2\pi}{T}N$, $\frac{4\pi}{T}N$, etc. [1]. As $N \to \infty$, $Q(t)$ becomes constant, so $\dot{Q} = \ddot{Q} = 0$. Moreover, condition (3.6) implies that the time interval between successive crossings of a particular value φ_* are equal and independent of φ_*; in the continuum limit this means that the current J is a constant. From the continuity equation (3.5),

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial \varphi}J = 0\,.$$

$J =$ const implies that $\partial_t \rho = 0$.

Substituting the conditions $\frac{\partial}{\partial t}\rho(\varphi, t) = 0$, $\dot{Q} = 0$, and $\ddot{Q} = 0$ into Eq. (3.5) and integrating with respect to φ gives

$$\rho_0(\varphi; \epsilon)\left[\frac{I_{dc} - \sin \varphi}{1 + \epsilon \cos \varphi}\right] = \nu\,, \tag{3.7}$$

where $\rho_0(\varphi; \epsilon)$ is the stationary distribution of the splay state and ν is a constant. We can fix the value of ν by normalizing the density $\rho_0(\varphi; \epsilon)$

$$\int_0^{2\pi} \rho(\varphi, t)d\varphi = 1 \tag{3.8}$$

so that integrating Eq. (3.7) implies

$$\rho_0(\varphi;\epsilon)=\frac{\omega}{2\pi}\frac{1+\epsilon\cos\varphi}{I_{dc}-\sin\varphi}, \tag{3.9}$$

where $\omega\equiv\sqrt{I_{dc}^2-1}$ and we have assumed that $I_{dc}>1$. Meanwhile, substituting the splay state conditions $\dot{Q}=\ddot{Q}=0$ into Eq. (3.4), we find

$$\frac{1}{C}Q_0=\int_{\varphi=0}^{2\pi}\frac{\omega}{2\pi}d\varphi, \tag{3.10}$$

so that $Q_0=C\omega$.

The next step is to test the stability of the splay state by letting $Q=Q_0+q$ and $\rho(\varphi,t)=\rho_0(\varphi;\epsilon)+\eta(\varphi,t)$, where q and η are infinitesimal. Substituting these relations into Eqs. (3.4) and (3.5) and throwing out all nonlinear terms we find

$$L\ddot{q}+(R+1)\dot{q}+\frac{1}{C}q=\int_{\varphi=0}^{2\pi}\frac{I_{dc}-\sin\varphi}{1+\epsilon\cos\varphi}\eta(\varphi,t)d\varphi\,, \tag{3.11}$$

$$\frac{\partial}{\partial t}\eta-\frac{\omega}{2\pi}\dot{q}\frac{\partial}{\partial\varphi}\left(\frac{1}{I_{dc}-\sin\varphi}\right)+\frac{\partial}{\partial\varphi}\left(\eta(\varphi,t)\left[\frac{I_{dc}-\sin\varphi}{1+\epsilon\cos\varphi}\right]\right)=0\,. \tag{3.12}$$

We introduce the auxillary variable [10,11]

$$G(\varphi;\epsilon)=\frac{\omega}{2\pi}\int_{\varphi'=0}^{\varphi}\rho_0(\varphi';\epsilon)d\varphi'\,.$$

Using Eq. (3.9) and evaluating the integral, G is

$$G(\varphi;\epsilon)=\frac{1}{\pi}\tan^{-1}\left(\frac{-I_{dc}(1+\cos\varphi)+\sin\varphi}{\omega\sin\varphi}\right)+\frac{1}{2}+\epsilon\frac{\omega}{2\pi}\ln\left(\frac{I_{dc}}{I_{dc}-\sin\varphi}\right)\,.$$

We now expand $\eta(\varphi,t)$ as a Fourier series in G

$$\eta(\varphi,t)=\rho_0(\varphi;\epsilon)\sum_{m=-\infty}^{\infty}a_m(t)e^{2\pi imG(\varphi;\epsilon)}\,.$$

Factoring $\rho_0(\varphi;\epsilon)$ out of the Fourier coefficients will be useful later. Substituting this expansion into Eqs. (3.11) and (3.12) and using Eq. (3.9) we get

$$L\ddot{q}+(R+1)\dot{q}+\frac{1}{C}q=\frac{\omega}{2\pi}\sum_{m=-\infty}^{\infty}a_m(t)\int_{\varphi=0}^{2\pi}e^{2\pi imG(\varphi;\epsilon)}d\varphi\,, \tag{3.13}$$

$$\rho_0(\varphi;\epsilon)\sum_{m=-\infty}^{\infty}\dot{a}_m(t)e^{2\pi imG(\varphi;\epsilon)}-\frac{\omega}{2\pi}\dot{q}\frac{\partial}{\partial\varphi}\left(\frac{1}{I_{dc}-\sin\varphi}\right)+\omega\sum_{m=-\infty}^{\infty}a_m(t)ime^{2\pi imG(\varphi;\epsilon)}G'(\varphi;\epsilon)=0\,. \tag{3.14}$$

This expansion has the orthogonality property

$$\int_{\varphi=0}^{2\pi}e^{2\pi ikG(\varphi;\epsilon)}\rho_0(\varphi;\epsilon)d\varphi=\delta_{k,0}\,, \tag{3.15}$$

where $\delta_{k,0}$ is the Kronecker delta. We multiply Eq. (3.14) by $\exp[-2\pi inG(\varphi;\epsilon)]$ and integrate from 0 to 2π

$$\sum_{m=-\infty}^{\infty}\dot{a}_m(t)\int_{\varphi=0}^{2\pi}e^{2\pi i(m-n)G(\varphi;\epsilon)}\rho_0(\varphi;\epsilon)d\varphi-\frac{\omega}{2\pi}\dot{q}\int_{\varphi=0}^{2\pi}e^{-2\pi inG(\varphi;\epsilon)}\frac{\cos\varphi}{(I_{dc}-\sin\varphi)^2}d\varphi+i\omega\sum_{m=-\infty}^{\infty}ma_m(t)\int_{\varphi=0}^{2\pi}e^{2\pi i(m-n)G(\varphi;\epsilon)}\rho_0(\varphi;\epsilon)d\varphi=0\,.$$

Using the orthogonality property defined in Eq. (3.15) to evaluate the integrals, we get

$$\dot{a}_n=\left(\frac{\omega}{2\pi}\int_{\varphi=0}^{2\pi}e^{-2\pi inG(\varphi;\epsilon)}\frac{\cos\varphi}{(I_{dc}-\sin\varphi)^2}d\varphi\right)\dot{q}-i\omega na_n. \tag{3.16}$$

If we make the definitions

$$b_n(\epsilon)\equiv\frac{\omega}{2\pi}\int_{\varphi=0}^{2\pi}e^{-2\pi inG(\varphi;\epsilon)}\frac{\cos\varphi}{(I_{dc}-\sin\varphi)^2}d\varphi$$

and

$$c_n(\epsilon)\equiv\frac{1}{L}\frac{\omega}{2\pi}\int_{\varphi=0}^{2\pi}e^{2\pi inG(\varphi;\epsilon)}d\varphi$$

and let $\dot{q}\equiv p$, then we can write Eqs. (3.13) and (3.16) as the (infinite) system of first order equations

$$\dot{q} = p\,, \tag{3.17}$$

$$\dot{p} = \frac{-1}{LC}q - \frac{R+1}{L}p + \sum_{m=-\infty}^{\infty} c_m(\epsilon) a_m\,, \tag{3.18}$$

$$\dot{a}_n = b_n(\epsilon) p - i\omega n a_n\,. \tag{3.19}$$

In the $\epsilon = 0$ case it turns out that $b_n(0) \equiv 0$ for all $|n| \neq 1$ [10], which radically simplifies the problem. We have

$$\dot{q} = p, \quad \dot{p} = -\frac{1}{LC}q - \frac{R+1}{L}p + \sum_{m=-\infty}^{\infty} c_m(0) a_m\,,$$

$$\dot{a}_{-1} = b_{-1}(0) + i\omega a_{-1}\,,$$

$$\dot{a}_1 = b_1(0) - i\omega a_1\,,$$

$$\dot{a}_n = -i\omega n a_n,\; n \neq \pm 1$$

and all but four of the infinite set of eigenvalues can be found by inspection [11]. The corresponding eigenvectors have a simple form; for any $k \neq \pm 1$, we set $a_k = 1$ and $a_n = 0$ for all other $n \neq \pm 1$. We see immediately that the corresponding eigenvalue is $\lambda_k^{(0)} = -i\omega k$: there is thus an infinite set of pure imaginary eigenvalues, hence the massive neutral stability. In this basis, it is natural to label the neutral eigenvalues by the nonzero component a_k where $k \neq \pm 1$. (The four other eigenvectors have $a_n = 0$ for all $n \neq \pm 1$.)

Things are not so simple when we consider the case $\epsilon \neq 0$, but we can still make progress by using perturbation theory, treating the $\epsilon = 0$ case as the unperturbed problem. We can write Eqs. (3.17)–(3.19) in matrix form as

$$\frac{d}{dt}\vec{x} = [\mathbf{H} + \tau \mathbf{V}(\epsilon)]\vec{x}\,,$$

where $\mathbf{H}$ is the unperturbed matrix defined by Eqs. (3.17)–(3.19) with $\epsilon = 0$, $\mathbf{V}(\epsilon)$ is the matrix defined by the differences between the unperturbed case and the perturbed case, and τ is a formal expansion parameter that we will set to unity at the end. Of course, these matrices are infinite dimensional, and in practice we have to truncate at some value of n. In the Fourier basis, this corresponds to including only a finite number of modes, commonly known as a Galerkin truncation [17].

Standard perturbation theory tells us that the lowest order correction to the eigenvalue corresponding to the variable a_n is given by

$$\lambda_n^{(1)} = \frac{\langle \Psi_n^{(0)} | \mathbf{V}(\epsilon) | \Phi_n^{(0)} \rangle}{\langle \Psi_n^{(0)} | \Phi_n^{(0)} \rangle}\,,$$

where $\Psi_n^{(0)}$ is the nth left eigenvector of the unperturbed matrix and $\Phi_n^{(0)}$ is the nth right eigenvector, i.e., $(\mathbf{H} - \lambda_n^{(0)}\mathbf{1})|\Phi_n^{(0)}\rangle = 0$ and $\langle \Psi_n^{(0)}|(\mathbf{H} - \lambda_n^{(0)}\mathbf{1}) = 0$.

As mentioned above, these eigenvectors have a particularly simple form, with the result that if we define γ_p as the entry in right eigenvector corresponding to the p variable and γ_m as the entry corresponding to the a_m variable, then the first order correction to the mth eigenvalue is given by $(\gamma_p\ \Delta b_m)/\gamma_m$, where

$$\Delta b_n(\epsilon) \equiv b_n(\epsilon) - b_n(0).$$

In terms of the unperturbed matrix coefficients c_n and b_n, this is

$$\lambda_n^{(1)} = \frac{-c_n\ i\omega n}{\frac{-1}{LC} - \frac{R+1}{L} i\omega n + \omega^2 n^2 + \frac{c_{-1}b_{-1}}{1-n} - \frac{c_1 b_1}{1+n}}\ \Delta b_n(\epsilon)\,. \tag{3.20}$$

We can evaluate c_n, c_{-1}, c_1, b_{-1}, and b_1 in closed form (see Appendix B of Ref. [10] for more details). We have been unable to integrate $\Delta b_n(\epsilon)$ in closed form, but it can be evaluated numerically and tabulated (note that it depends only on the single parameter I_{dc}).

In Fig. 2 we plot the real parts of the eigenvalues corresponding to a_2, a_3, and a_4 as a function of ϵ, for some typical set of system parameters. Based on just these few eigenvalues, whose real parts are all zero in the unperturbed case, we conclude that the splay state is linearly stable over some range of $\epsilon < 0$ and otherwise linearly unstable. The nongeneric neutral stability seems to be broken by the presence of the phase dependent conductance term, though the magnitude of the effect is rather small (that is, the eigenvalues have rather small real parts).

Our next step is to compare these predictions based on Eq. (3.20) with direct numerical integration of the circuit equations for an array of five junctions. A detailed quantitative comparison of the eigenvalues, however, is not possible: those based on the $n = 5$ truncation in a Fourier basis of the inifinite N problem have no one-to-one correspondence with the eigenvalues of the (exact) $N = 5$ array equations. This stands in contrast with the $\epsilon = 0$ problem, where a direct comparison is possible, and the agreement is excellent [7,11]. The difference is that, in the latter case, no truncation of the infinite-N limit was needed: the eigenvalues divide neatly into two sets, with four eigenvalues having (generically) a nonzero real part and all other eigenvalues being pure imaginary, which is precisely what one finds numerically for the finite-N problem. In the $\epsilon \neq 0$ case, we expect that the four perturbed non-neutral eigenvalues are the same for both the calculation and the numerics. We can qualitatively check the other $N-2$ eigenvalues by checking if the asymptotic dynamical state of the numerically integrated differential equations is consistent with the stability prediction of the analytic calculation. For example, for small positive ϵ the splay state is predicted to be linearly unstable, so after a long enough time we expect to find the system in some other type of (attracting) state.

Starting with φ_k's spaced at intervals of $2\pi/N$, $\dot{\varphi}_k$'s equal to I_{dc}, and $\epsilon = 0$, we integrate Eqs. (3.1) and (3.2) using a fourth order Runge-Kutta scheme until the system relaxes to a periodic orbit. Operationally, we are picking an initial condition in our $N + 2$ dimensional phase space near the $N-2$ dimensional neutrally stable manifold; the neutrally stable manifold is an *attractor* in the full phase space, but motion on the manifold is *neutrally stable*. We then turn ϵ on and again integrate for a long time. By computing the Euclidean distance between points in phase space of successive zero crossings of one of the φ's (recall that the φ's are angles defined on a circle) we can get some measure of whether the system is on a periodic orbit and how long we have to integrate to eliminate transients. To determine whether the periodic orbit is a splay state, we also look at the time between successive zero crossings Δt_k of any of the phase variables: a splay phase state has Δt_k's (very nearly) equal, an in-phase state has Δt_k's approximately zero, and a cluster state has some Δt_k's equal and some zero.

To characterize the stability of the periodic orbit, we use Floquet theory. First, we linearize Eqs. (3.1) and (3.2) about some assumed periodic solution $\{Q^*, \dot{Q}^*, \varphi_1^*, \ldots, \varphi_N^*\}$. Letting $\varphi_k = \varphi_k^* + \eta_k$ and $Q = Q^* + q$, the linearized equations are

$$\dot{\eta}_k = \left(\frac{-\cos\varphi_k^* - \epsilon + I_{dc}\epsilon \sin\varphi_k^* - \epsilon \dot{Q}^* \sin\varphi_k^*}{(1+\epsilon\cos\varphi_k^*)^2} \right) \eta_k + \left(\frac{-1}{1+\epsilon\cos\varphi_k^*} \right) \dot{q} \, , \tag{3.21}$$

$$L\ddot{q} + R\dot{q} + \frac{1}{C} q = \frac{1}{N} \sum_{j=1}^{N} \dot{\eta}_j \, . \tag{3.22}$$

Floquet's theorem says that we can write the fundamental matrix $\Phi(t)$ as the product of a periodic matrix $\mathbf{P}(t)$ and a matrix $\exp(\mathbf{B}t)$, where $\mathbf{B}$ is a constant matrix [18]. The eigenvalues of $\mathbf{B}$ determine the linear stability

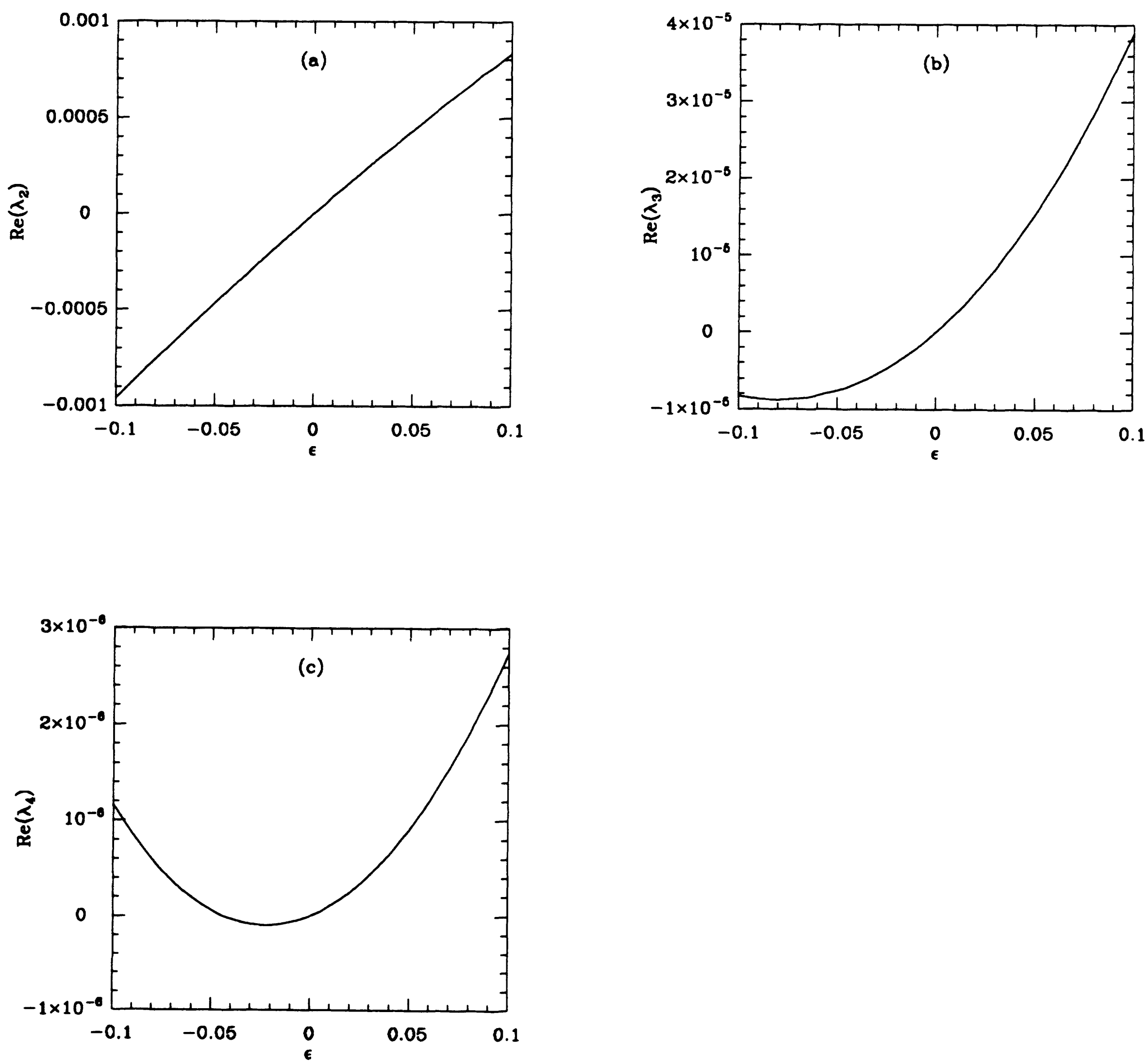

FIG. 2. Plot of the real part of the eigenvalues corresponding to the a_2, a_3, and a_4 Fourier coefficients as a function of ϵ. These are exactly 0 for $\epsilon = 0$.

of the periodic orbit. One way to compute $\mathbf{B}$ is to choose $\Phi(t_0)$ to be the identity matrix and numerically integrate the equations for one period T to determine $\Phi(t_0+T)$ [2]. The eigenvalues of the resulting matrix, $\exp(\mathbf{B}T)$, are the Floquet multipliers which determine the stability of the underlying periodic orbit: a multiplier with magnitude precisely equal to 1 implies neutral stability.

Obviously, numerical errors prevent us from seeing eigenvalues that are exactly 1. However, we know rigorously that the eigenvalue corresponding to the tangent of the orbit *is* 1, so by looking at the actual value of the tangent eigenvalue we can get a measure of what error bar to associate with the numerically detemined multipliers. We measure eigenvalues from 25 consecutive orbits and look at the mean of the eigenvalues $|\bar{\mu}_k|$ and the variance of the eigenvalues with respect to 1 defined by

$$\sigma_k = \frac{1}{M}\sum_{j=1}^{M}(|\mu_k^{(j)}| - 1)^2 .$$

In Table I we list the Floquet multipliers near 1 and corresponding variances σ for given values of ϵ. There are typically three; one corresponds to the tangent of the orbit. Using the $\epsilon = 0$ case we infer that for a variance with respect to $1.0\,\sigma$ less than about 10^{-6} the corresponding multiplier can be taken as exactly 1.0. For $\epsilon = 0.01$, σ is very close to our cutoff, but the average value of the corresponding multiplier is a little greater than 1.0 and we conclude that the splay state is unstable, albeit just barely: physically, the system evolves away from the splay state only very slowly. Using the same reasoning, we conclude that for $\epsilon = -0.01$ the splay state is asymptotically stable, in agreement with the earlier prediction.

For the case of $\epsilon = 0.1$, we find that the final dynamical state is a stable $1{\times}2{\times}2$ cluster state [i.e., $\varphi_1(t) \neq \varphi_2(t) = \varphi_3(t) \neq \varphi_4(t) = \varphi_5(t)$]. For the case of $\epsilon = -0.1$, the final state is the splay state, but now it is clearly linearly stable. For $\epsilon = 0.5$ the $1{\times}2{\times}2$ cluster state is the final dynamical state, and even more stable, and for $\epsilon = -0.5$ the splay state has apparently lost stability, and the system is attracted to the inphase state.

TABLE I. Near critical Floquet multipliers for the $\beta = 0$ case, with parameter values of $R = 0.1$, $L = 0.5$, $C = 0.5$, and $I_{dc} = 1.9$.

ϵ	$\lvert\bar{\mu}\rvert$	σ	Dynamical state
0.0	1.000899	8.1×10^{-7}	splay
	0.999903	1.0×10^{-8}	(neutral)
	0.999156	7.1×10^{-7}	
0.01	1.001381	1.91×10^{-6}	splay
	1.000064	$< 10^{-9}$	(unstable)
	1.000036	$< 10^{-9}$	
-0.01	1.000208	4.0×10^{-8}	splay
	0.999535	2.2×10^{-7}	(stable)
	0.998845	1.33×10^{-6}	
0.1	1.000089	$< 10^{-9}$	$1 \times 2 \times 2$ cluster
	0.992344	5.861×10^{-5}	(stable)
	0.989103	1.1874×10^{-4}	
-0.1	1.000022	$< 10^{-9}$	splay
	0.997078	8.54×10^{-6}	(stable)
	0.997078	8.54×10^{-6}	
0.5	1.089430	7.99775×10^{-3}	$1 \times 2 \times 2$ cluster
	0.999871	2.0×10^{-8}	(stable)
-0.5	0.999663	1.1×10^{-7}	inphase (stable)

The conclusion from Table I is that for positive ϵ the splay state is always unstable, though for small ϵ the dynamics are very slow and it might take a long time for the system to leave the splay orbit. The splay state is linearly stable over some range $0 > \epsilon > -0.1$ for these parameters, but for larger negative ϵ the splay state eventually loses stability. We have not made an extensive study as a function of the other circuit parameters; nevertheless, we can use these data to test whether the analytic calculation is reliable.

In Fig. 2 we have plotted the results based on the perturbation analysis using a truncation at $n = 5$, for the same parameters as those used in the numerical simulations. Plotted is the real part of each of the near-neutral eigenvalues versus ϵ. Since all it takes is one positive eigenvalue to make the splay state unstable, the range over which the splay state is stabilized is just the intersection of the individual ranges. From the figure, we see that the eigenvalue for the coefficient a_2 is approximately proportional to ϵ; the corresponding modal perturbation to the splay state grows for $\epsilon > 0$ and decays for $\epsilon < 0$. Similar behavior is seen for the eigenvalue corresponding to the coefficient a_3, though the plot indicates that perhaps for $\epsilon < -0.1$ the curve might turn up and cross the 0 axis again, rendering the splay state unstable. For the eigenvalue of the coefficient a_4, we see this happen at $\epsilon \approx -0.05$. Considering just the three eigenvalues in Fig. 2, the splay state is predicted to be stable for the range $-0.05 < \epsilon < 0$; comparing this with numerical results in Table I gives us a measure of the range of validity of the $O(\epsilon)$ perturbation theory.

One limitation of this approach is that there are in principle an infinite number of coefficients to be checked. On the other hand, we find that the size of the real part of the eigenvalues diminishes rapidly for successively higher coefficients. For instance, $\mathrm{Re}(a_4)$ has magnitude of order 10^{-6}. This trend continues: for the much higher order coefficients the real part of the corresponding eigenvalue is effectively zero.

IV. THE $\beta \neq 0$ CASE

The analytic calculation of the preceding section does not readily generalize to the case where $\beta \neq 0$: we can still pass over to a continuum limit, but we can no longer solve explicitly for the splay state in closed form, much less perform the corresponding stability analysis. Thus, in this section we rely solely on numerical methods to study Eqs. (2.1) and (2.2), using the same techniques as described in the preceding section.

Our results are summarized in Table II. The $\epsilon = 0$ case gives us some idea of error bars for the numerically determined Floquet multipliers, for our particular choice of time step and integration time. From this, we deduce that values of σ less than about 10^{-6} are consistent with

TABLE II. Near critical Floquet multipliers for the $\beta \neq 0$ case, with parameter values of $\beta = 1.0$, $R = 0.1$, $L = 0.5$, $C = 0.5$, and $I_{dc} = 1.9$.

ϵ	$\lvert\bar{\mu}\rvert$	σ	Dynamical state
0.0	1.000697	4.9×10^{-7}	splay
	0.999611	1.5×10^{-7}	(neutral)
	0.997970	4.12×10^{-6}	
0.01	1.000225	5.0×10^{-8}	splay
	0.999671	1.1×10^{-7}	(neutral)
	0.999073	8.6×10^{-7}	
-0.01	1.000655	4.3×10^{-7}	splay
	0.999709	9.0×10^{-8}	(neutral)
	0.998350	2.72×10^{-6}	
0.1	1.002025	4.1×10^{-6}	inphase
	0.974491	6.5069×10^{-4}	(stable)
	0.974491	6.5071×10^{-4}	
-0.1	1.000541	2.9×10^{-7}	splay
	0.999590	1.7×10^{-7}	(neutral)
	0.997665	5.45×10^{-6}	
0.5	0.999073	8.6×10^{-7}	inphase
			(stable)
-0.5	1.000134	2.0×10^{-8}	splay
	0.998927	1.15×10^{-6}	(neutral)
	0.997392	6.8×10^{-6}	

neutral stability of the splay states. As a separate check, we also look at the time series of the aymptotic dynamical state to see if it is still approximately splay (i.e., $\Delta t_k \approx \Delta t_j$) or if the system has wandered away from the splay state into some other periodic orbit.

From the data, we see that for the cases of $\epsilon = 0.01$ and $\epsilon = -0.01$ the final dynamical states are still the splay state and the σ's are consistent with neutral stability. As we increase ϵ to 0.1, the splay state loses stability to a periodic in-phase state which is asymptotically stable. For ϵ equal to -0.1, the splay state seems to be neutrally stable or possibly (weakly) linearly stable. At $\epsilon = 0.5$ the in-phase state is still the final dynamical state and has become more stable. For $\epsilon = -0.5$, the system again ends up in a splay state which is either neutrally stable or possibly linearly stable.

Unlike the results for $\beta = 0$, it appears that the neutral stability of the splay state is preserved for some range of ϵ, which is somewhat surprising, and is otherwise not neutrally stable. One possible explanation is that the splay state undergoes a bifurcation at some critical value ϵ_c between 0.01 and 0.1. Another possibility is that for any $\epsilon \neq 0$ the phase dependent conductance term always breaks the neutral stability, but by a very small amount as measured by the Floquet multipliers for small ϵ. In this scenario, it would appear that making $\epsilon < 0$ tends to nudge the Floquet multipliers to values less than 1, making the splay state asymptotically stable, while putting $\epsilon > 0$ tends to make the Floquet multipliers greater than 1, in the latter case causing the splay state to lose stability. If the multipliers were just barely larger than 1, the dynamics could be so slow that our integration time is not long enough to see the system leave the vicinity of the unstable splay state, giving the appearance of neutral stability.

V. DISCUSSION

In this paper, we have studied the effect of the phase dependent conductance on the splay state dynamics, both for junctions having negligible capacitance ($\beta = 0$) and for junctions with substantial capacitance ($\beta > 0$). For the case $\beta = 0$, we have found good agreement between our perturbation theory calculation and numerical simulations: the presence of the the $\epsilon \cos \phi$ term always destroys the neutral stability of the splay state. For sufficiently small values of $|\epsilon|$, the sign of ϵ determines whether the splay state is stable or unstable. In this regime, even when the splay state is rendered unstable the effect is so weak that it takes a very long time—longer than our simulations were run—for the system to settle down to an attracting state. For somewhat larger values of $|\epsilon|$ ($|\epsilon| > 0.1$), we find that the splay state is always unstable, and the array settles down to some more coherent dynamical state, either the in-phase state or a cluster state.

For $\beta \neq 0$ our study was restricted to numerics. For the parameter values considered, there is a range of ϵ for which the neutral stability of the splay state is apparently preserved. For larger positive values of ϵ, the splay state becomes unstable and the system ends up in the in-phase state. For large negative values of ϵ, the results are ambiguous: the splay state is either neutrally stable or weakly linearly unstable.

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