Mean exit times for free inertial stochastic processes

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We study the mean exit time of a free inertial random process from a region in space. The acceleration alternatively takes the values $+a$ and $-a$ for random periods of time governed by a common distribution $\psi(t)$. The mean exit time satisfies an integral equation that reduces to a partial differential equation if the random acceleration is Markovian. Some qualitative features of the behavior of the system are discussed and checked by simulations. Among these features, the most striking is the discontinuity of the mean exit time as a function of the initial conditions.

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I. INTRODUCTION

Many stochastic models in the literature are high-damping models, that is, they represent systems for which inertial effects can be neglected. This simplification allows one to deal only with position variables $X(t)$ without having to worry about independent associated velocity variables $\dot{X}(t)$. The dynamical equations in these models therefore contain only the velocity $\dot{X}(t)$ but not the acceleration $\ddot{X}(t)$ [1,2]. However, in practice the high-damping approximation is clearly not always appropriate. There has therefore been a long-standing and continuing (but not broadly successful) interest in developing methods to deal with inertial processes [3–6,8,9]. This paper deals with our own continuing efforts in this direction.

The simplest inertial random process is described by an evolution equation of the form

$$\ddot{X}(t) = F(t), \tag{1.1}$$

where $F(t)$ is a noise whose statistics must be specified. Our work has dealt principally with systems driven by dichomotous processes, that is, ones in which $F(t)$ is dichotomous noise that takes on the values $\pm a$ with a given switching probability density $\psi(t)$ [3–6]. In recent work, we focused on the joint probability density $p(x,y,t)$, for the probability that the position $X(t)$ lies between x and $x+dx$ and that the velocity $Y(t)\equiv \dot{X}(t)$ lies between y and $y+dy$. We also obtained exact equations for the marginal probability density $p(y,t)$ of the velocity and the marginal probability density $p(x,t)$ of the position. These densities were considered for the process (1.1) [3] and also for one in which there is an additional damping term proportional to $\dot{X}(t)$ [4,6]. In this paper, we consider the exit time of the process (1.1) out of the interval $0\le x\le L$.

From a dynamical point of view, the time evolution of process (1.1) with a dichotomous driving force is determined by two different dynamics, $x^{\pm}(t;x_0,y_0)$, where $x^{+}(t;x_0,y_0)$ and $x^{-}(t;x_0,y_0)$ are, respectively, the solutions to Eq. (1.1) with $F(t)=a$ and $-a$, and with initial conditions $x_0 = X(0)$ for the position and $y_0=\dot{X}(0)$ for the velocity of the process. The system randomly switches between these two dynamics at random times governed by the switching probability density $\psi(t)$. The explicit expressions for $x^{\pm}(t;x_0,y_0)$ are

$$x^{\pm}(t;x_0,y_0) = x_0 + y_0 t \pm \frac{1}{2}at^2. \tag{1.2}$$

Similarly, the random process

$$Y(t) = \dot{X}(t) \tag{1.3}$$

representing the velocity also has two different dynamics:

$$y^{\pm}(t;y_0) = y_0 \pm at. \tag{1.4}$$

The times to reach the boundaries 0 and L when the system evolves deterministically under positive or under negative acceleration are the ballistic times. We denote them by $\tau_0^{\pm}(x_0,y_0)$ and $\tau_L^{\pm}(x_0,y_0)$, where, for example, $\tau_0^{+}(x_0,y_0)$ is the time to first reach the boundary $x=0$ if $F(t)=a$ without ever having reached the boundary $x=L$; similarly, for example, $\tau_L^{-}(x_0,y_0)$ is the time to first reach the boundary $x=L$ if $F(t)=-a$ without ever having reached the boundary $x=0$.

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Note that the boundary that is crossed first under ballistic motion depends on the initial condition, which in turn implies that the forms of $\tau_0^{\pm}(x_0,y_0)$ and of $\tau_L^{\pm}(x_0,y_0)$ depend on the initial conditions. In particular, the following situations may occur (see Fig. 1):

Positive acceleration. Here $F(t)=a$. If

$$y_0+\sqrt{2ax_0}>0, \tag{1.5}$$

the system first reaches the upper boundary $x=L$, and does so at time

$$\tau_L^+(x_0,y_0)=\frac{-y_0+\sqrt{y_0^2+2a(L-x_0)}}{a}. \tag{1.6}$$

If instead

$$y_0+\sqrt{2ax_0}<0, \tag{1.7}$$

then the system first reaches the lower boundary $x=0$ and does so at time

$$\tau_0^+(x_0,y_0)=\frac{-y_0-\sqrt{y_0^2-2ax_0}}{a}. \tag{1.8}$$

Negative acceleration. Here $F(t)=-a$. If

$$y_0-\sqrt{2a(L-x_0)}>0, \tag{1.9}$$

the system first reaches the upper boundary $x=L$, and does so at time

$$\tau_L^-(x_0,y_0)=\frac{y_0-\sqrt{y_0^2-2a(L-x_0)}}{a}. \tag{1.10}$$

If instead

$$y_0-\sqrt{2a(L-x_0)}<0, \tag{1.11}$$

then the system first reaches the lower boundary $x=0$ and does so at time

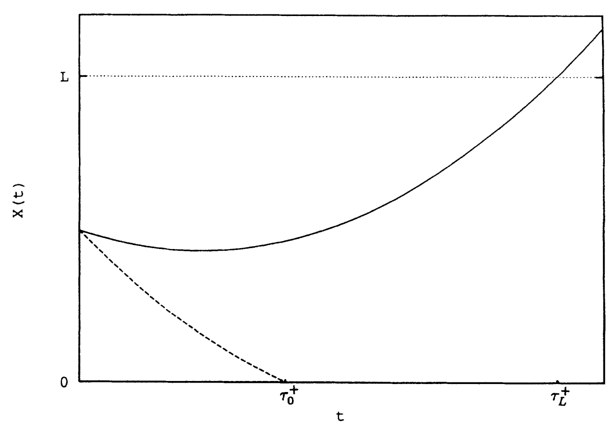

FIG. 1. Solid curve: ballistic trajectory $X(t)$ as given in Eq. (1.2), that is, $F(t)=a$ and $y_0+(2ax_0)^{1/2}>0$. The system first crosses the boundary L and does so at time τ_L^+. Dashed curve: ballistic trajectory when $F(t)=a$ and $y_0+(2ax_0)^{1/2}<0$. The system first crosses the boundary 0 and does so at time τ_0^+. A similar figure can be drawn for $F(t)=-a$.

$$\tau_0^-(x_0,y_0)=\frac{y_0+\sqrt{y_0^2+2ax_0}}{a}. \tag{1.12}$$

As noted above, in this paper we consider the exit time of the process (1.1) out of the interval $0\le x\le L$. Note that the translational invariance of (1.1) implies that there is no loss of generality in choosing these specific boundary intervals: the exit time out of an interval $z_1\le x\le z_2$ from an initial point x_0 is the same as the exit time out of the interval $0\le x\le L$ from an initial location x_0-z_1, with $L\equiv z_2-z_1$. Our principal formal interest is that of most studies of exit time or first passage time analyses, namely, in obtaining integral equations and, if possible, differential equations for the mean exit time and related quantities [7]. We succeed in this attempt: we find integral equations for the mean exit time for arbitrary dichotomous noise, and we find equivalent partial differential equations for this time when the dichotomous noise is Markovian. However, we are not able to find analytic solutions to these equations, and therefore the interest in these expressions at this point is primarily formal and serves mainly to confirm that it is possible to find such equations. In the process of this derivation and with the help of the equations, however, we have been able to arrive at a number of interesting qualitative and semiquantitative conclusions that we have confirmed or complemented with direct numerical simulations. These will hopefully guide future searches for analytic solutions to this and to more complicated problems. Thus, after first laying out the formal analysis in Secs. II, III, and IV we present our qualitative and numerical results and discuss the behavior of the mean exit time in Sec. V. We end with a brief summary in Sec. VI.

II. INTEGRAL EQUATIONS FOR THE MEAN EXIT TIME

We define the first-exit-time probability density out of the interval $0\le x\le L$ as follows: Given $X(0)=x_0$ and $\dot{X}(0)=y_0$, let $f(t;x_0,y_0)dt$ be the probability that the process $X(t)$ exits the interval $0\le x\le L$ within the time interval $(t,t+dt)$ without ever having left this interval during the time span $[0,t]$. The fact that $F(t)$ has two realizations prompts us to define the two auxiliary probability densities

$$f^{\pm}(t;x_0,y_0)\equiv f(t;x_0,y_0|F(0)=\pm a). \tag{2.1}$$

If we assume that the initial values $F(0)=\pm a$ occur with equal probability, then

$$f(t;x_0,y_0)=\frac{1}{2}\left[f^+(t;x_0,y_0)+f^-(t;x_0,y_0)\right]. \tag{2.2}$$

The auxiliary probability density $f^+(t;x_0,y_0)$ satisfies the following integral equation:

$$\begin{aligned} f^+(t;x_0,y_0) &= \Psi(t)\delta(t-\tau^+(x_0,y_0)) \\ &\quad + \int_0^{\tau^+(x_0,y_0)} dt'\,\psi(t') \int_0^L dx \int_{-\infty}^{\infty} \\ &\quad \times dy\, f^-(t-t';x,y)h^+(x,y;t'|x_0,y_0)\ , \end{aligned} \tag{2.3}$$

where

$$\Psi(t) = \int_t^\infty dt'\,\psi(t') \tag{2.4}$$

and

$$h^+(x,y;t'|x_0,y_0) = \delta(x - x^+(t';x_0,y_0))\delta(y - y^+(t';y_0)) \ , \tag{2.5}$$

and where $x^+(t';x_0,y_0)$ and $y^+(t';y_0)$ are, respectively, given in Eqs. (1.2) and (1.4). The quantity $\tau^+(x_0,y_0)$ is the ballistic time to first escape through either of the boundaries at $x = 0$ and L under the dynamics $F(t) = +a$. In other words, depending on the initial condition $\tau^+(x_0,y_0)$ is either $\tau_L^+(x_0,y_0)$ or $\tau_0^+(x_0,y_0)$.

Equation (2.3) is easily derived with the following reasoning. The first boundary crossing event occurs either before or after the first time that $F(t)$ switches from one value to the other. The first term on the right hand side of (2.3) accounts for the probability that the escape occurs at time $t = \tau^+(x_0,y_0)$ *before* the first switch in $F(t)$, while the second term on the right hand side of (2.3) arises from the fact that if the escape has *not* occured during the first time interval, then a switch must have taken place at time t' (which must be smaller than the ballistic time to reach a boundary). At time t' the system is at position $x = x^+(t';x_0,y_0)$ with velocity $y = y^+(t';y_0)$. Analogous reasoning immediately leads to the corresponding equation for $f^-(t;x_0,y_0)$:

$$f^-(t;x_0,y_0) = \Psi(t)\delta(t-\tau^-(x_0,y_0)) + \int_0^{\tau^-(x_0,y_0)} dt'\,\psi(t')\int_0^L dx\int_{-\infty}^\infty dy\, f^+(t-t';x,y)h^-(x,y;t'|x_0,y_0) \ , \tag{2.6}$$

where now $\tau^-(x_0,y_0)$ is the ballistic time to escape through either boundary if $F(t) = -a$, and

$$h^-(x,y;t'|x_0,y_0) = \delta(x - x^-(t';x_0,y_0))\,\delta(y - y^-(t';y_0)) \ , \tag{2.7}$$

The set of coupled integral equations (2.3) and (2.6) is somewhat simplified by taking the Laplace transform to obtain

$$\hat{f}^\pm(s;x_0,y_0) = e^{-s\tau^\pm(x_0,y_0)}\Psi(\tau^\pm(x_0,y_0)) + \int_0^{\tau^\pm(x_0,y_0)} dt'\,e^{-st'}\psi(t')\int_0^L dx\int_{-\infty}^\infty dy\,\hat{f}^\mp(s;x,y)h^\pm(x,y;t|x_0,y_0) \ . \tag{2.8}$$

We define $T^+(x_0,y_0)$ and $T^-(x_0,y_0)$ to be the mean exit times out of the region $0 < x_0 < L$, $-\infty < y_0 < \infty$, if initially $F(0) = a$ and $F(0) = -a$, respectively. We note that due to the symmetry of the inertial process (1.1) these two times are related to one another:

$$T^-(x_0,y_0) = T^+(L - x_0, -y_0). \tag{2.9}$$

Since

$$T^\pm(x_0,y_0) = -\frac{\partial}{\partial s}\hat{f}^\pm(s;x_0,y_0)\bigg|_{s=0} , \tag{2.10}$$

we can use Eq. (2.8) to find a set of coupled integral equations of the mean exit times. Straightforward manipulation gives

$$T^+(x_0,y_0) = G^+(x_0,y_0) + \int_0^{\tau^+(x_0,y_0)} dt\,\psi(t)\int_0^L dx\int_{-\infty}^\infty dy\,T^-(x,y)h^+(x,y;t|x_0,y_0), \tag{2.11}$$

$$T^-(x_0,y_0) = G^-(x_0,y_0) + \int_0^{\tau^-(x_0,y_0)} dt\,\psi(t)\int_0^L dx\int_{-\infty}^\infty dy\,T^+(x,y)h^-(x,y;t|x_0,y_0), \tag{2.12}$$

where

$$G^\pm(x_0,y_0) = \int_0^{\tau^\pm(x_0,y_0)} dt\,\Psi(t). \tag{2.13}$$

Our next step is to rewrite Eqs. (2.11) and (2.12) in a more explicit form that is more convenient for further manipulation. The δ function expressions (2.5) and (2.7) allow us to carry out the integrals on the right hand sides of (2.11) and (2.12) explicitly, with the results

$$\begin{aligned}\int_0^{\tau^\pm(x_0,y_0)} & dt\,\psi(t)h^\pm(x,h;t|x_0,y_0) \\ &= \int_0^{\tau^\pm(x_0,y_0)} dt\,\psi(t)\delta(x - x^\pm(t;x_0,y_0))\delta(y - y^\pm(t;x_0,y_0)) \\ &= \frac{1}{a}\psi\left[\pm\left(\frac{y-y_0}{a}\right)\right]\Theta\left(\pm\left(\frac{y-y_0}{a}\right)\right)\Theta\left(\tau^\pm(x_0,y_0)\mp\left(\frac{y-y_0}{a}\right)\right) \\ &\quad\times\delta\left(x - x_0 \mp\left(\frac{y^2-y_0^2}{2a}\right)\right) ,\end{aligned} \tag{2.14}$$

where $\Theta(x)$ is the Heaviside step function. Substitution into (2.11) and (2.12) yields

$$T^+(x_0,y_0) = G^+(x_0,y_0) + \frac{1}{a}\int_0^L dx \int_{y_0}^{y_0+a\tau^+(x_0,y_0)} \times dy\, T^-(x,y)\psi\left(\frac{y-y_0}{a}\right) \times\delta\left(x-x_0-\frac{y^2-y_0^2}{2a}\right), \tag{2.15}$$

$$T^-(x_0,y_0) = G^-(x_0,y_0) + \frac{1}{a}\int_0^L dx \int_{y_0-a\tau^-(x_0,y_0)}^{y_0} \times dy\, T^+(x,y)\psi\left(\frac{y_0-y}{a}\right) \times\delta\left(x-x_0-\frac{y_0^2-y^2}{2a}\right). \tag{2.16}$$

Equations (2.15) and (2.16) constitute a formal solution to the exit time problem. They are valid for any form of the switching probability density $\psi(t)$, and can be used as a convenient starting point for numerical analysis when no further simplification is possible.

III. DIFFERENTIAL EQUATION FOR THE MEAN EXIT TIME

To proceed further we must specify the form of the switching probability density. We choose $F(t)$ to be dichotomous Markov noise, for which the switching probability density is exponential,

$$\psi(t) = \lambda e^{-\lambda t} \qquad (\lambda > 0). \tag{3.1}$$

Substituting this interval density into the integral equations (2.15) and (2.16) we find

$$T^+(x_0,y_0) = G^+(x_0,y_0) + \frac{\lambda}{a}e^{\lambda y_0/a}\int_{y_0}^{y_0+a\tau^+(x_0,y_0)} \times dy \int_0^L dx\, T^-(x,y)e^{-\lambda y/a} \times\delta\left(x-x_0-\frac{y^2-y_0^2}{2a}\right), \tag{3.2}$$

$$T^-(x_0,y_0) = G^-(x_0,y_0) + \frac{\lambda}{a}e^{-\lambda y_0/a}\int_{y_0-a\tau^-(x_0,y_0)}^{y_0} \times dy \int_0^L dx\, T^+(x,y)e^{\lambda y/a} \times\delta\left(x-x_0-\frac{y^2-y_0^2}{2a}\right). \tag{3.3}$$

Taking the y_0-derivative of Eq. (3.2) and reorganizing terms leads to

$$\left(\frac{\partial}{\partial y_0}+\frac{\lambda}{a}\right)[T^+(x_0,y_0)-G^+(x_0,y_0)] + \frac{\lambda}{a}T^-(x_0,y_0) = \frac{\lambda}{a}\left[\frac{\partial}{\partial y_0}(y_0+a\tau^+)\right]\int_0^L dx\, T^-(x,y_0+\tau^+)e^{-\lambda\tau^+}\delta(x-\zeta) + \frac{\lambda}{a}e^{\lambda y_0/a}\int_{y_0}^{y_0+a\tau^+}\int_0^L dx\, T^-(x,y)e^{-\lambda y/a}\frac{\partial}{\partial y_0}\left[\delta\left(x-x_0-\frac{y^2-y_0^2}{2a}\right)\right], \tag{3.4}$$

where the dependence of τ^+ on x_0 and y_0 is understood and where ζ as well as the missing subscript on τ^+ is either 0 or L depending on the initial position and velocity. On the other hand, the x_0 derivative of (3.2) yields

$$\frac{\partial}{\partial x_0}\left[T^+(x_0,y_0)-G^+(x_0,y_0)\right] = \frac{\lambda}{a}\left[\frac{\partial}{\partial x_0}(y_0+a\tau^+)\right]\int_0^L dx\, T^-(x,y_0+\tau^+)e^{-\lambda\tau^+}\delta(x-\zeta) + \frac{\lambda}{a}e^{\lambda y_0/a}\int_{y_0}^{y_0+a\tau^+}\int_0^L dx\, T_-(x,y)e^{-\lambda y/a}\frac{\partial}{\partial x_0}\left[\delta\left(x-x_0-\frac{y^2-y_0^2}{2a}\right)\right]. \tag{3.5}$$

If $y_0 + \sqrt{2ax_0} > 0$, then $\tau^+ = \tau_L^+(x_0, y_0)$, $\zeta = L$, and

$$y_0 + a\tau^+(x_0, y_0) = \sqrt{y_0^2 + 2a(L - x_0)}. \qquad (3.6)$$

Likewise, when $y_0 + \sqrt{2ax_0} < 0$ then $\tau^+ = \tau_0^+(x_0, y_0)$, $\zeta = 0$, and

$$y_0 + a\tau^+(x_0, y_0) = -\sqrt{y_0^2 - 2ax_0}. \qquad (3.7)$$

Note that in both cases one has

$$\frac{\partial}{\partial y_0}[y_0 + a\tau^+(x_0, y_0)] = -\frac{y_0}{a}\frac{\partial}{\partial x_0}[y_0 + a\tau^+(x_0, y_0)]. \qquad (3.8)$$

Thus, comparing Eq. (3.4) with Eq. (3.5) we obtain

$$\left(a\frac{\partial}{\partial y_0} + y_0 + \frac{\partial}{\partial x_0} - \lambda\right) T^+(x_0, y_0) + \lambda T^-(x_0, y_0) = \left(a\frac{\partial}{\partial y_0} + y_0 + \frac{\partial}{\partial x_0} - \lambda\right) G^+(x_0, y_0). \qquad (3.9)$$

Analogous reasoning leads to the following differential equation for $T^-(x_0, y_0)$:

$$\left(-a\frac{\partial}{\partial y_0} + y_0 + \frac{\partial}{\partial x_0} - \lambda\right) T^-(x_0, y_0) + \lambda T^+(x_0, y_0) = \left(-a\frac{\partial}{\partial y_0} + y_0 + \frac{\partial}{\partial x_0} - \lambda\right) G^-(x_0, y_0). \qquad (3.10)$$

Using (2.13) one can immediately see that the inhomogeneous term in both of these equations is

$$\left(\pm a\frac{\partial}{\partial y_0} + y_0 + \frac{\partial}{\partial x_0} - \lambda\right) G^\pm(x_0, y_0) = -1. \qquad (3.11)$$

Therefore the mean exit times $T^\pm(x_0, y_0)$ obey the following set of first-order coupled partial differential equations:

$$a\frac{\partial T^+}{\partial y_0} + y_0\frac{\partial T^+}{\partial x_0} - \lambda(T^+ - T^-) = -1, \qquad (3.12)$$

$$-a\frac{\partial T^-}{\partial y_0} + y_0\frac{\partial T^-}{\partial x_0} - \lambda(T^- - T^+) = -1. \qquad (3.13)$$

The left hand side of this system of equations is the adjoint of the system satisfied by the complete probability density function $p(\pm a, x, y; t)$ [3] because the full system $(F(t), x, y)$ is Markovian when $\psi(t)$ is exponential.

The set of Eqs. (3.12) and (3.13) can be combined to yield separate second-order partial differential equations for T^+ and T^-. One easily obtains

$$a^2\frac{\partial^2 T^+}{\partial y_0^2} - y_0^2\frac{\partial^2 T^+}{\partial x_0^2} + (2\lambda y_0 + a)\frac{\partial T^+}{\partial x_0} = -2\lambda, \qquad (3.14)$$

$$a^2\frac{\partial^2 T^-}{\partial y_0^2} - y_0^2\frac{\partial^2 T^-}{\partial x_0^2} + (2\lambda y_0 - a)\frac{\partial T^-}{\partial x_0} = -2\lambda. \qquad (3.15)$$

Finally we note that the quantity of physical interest is frequently the mean exit time regardless of the initial value of the noise [cf. Eq. (2.2)],

$$T(x_0, y_0) = \frac{1}{2}\left[T^+(x_0, y_0) + T^-(x_0, y_0)\right]. \qquad (3.16)$$

From (3.14) and (3.15) one finds the fourth-order partial differential equation

$$a^4\frac{\partial^4 T}{\partial y_0^4} + y_0^4\frac{\partial^4 T}{\partial x_0^4} - 2a^2y_0^2\frac{\partial^4 T}{\partial y_0^2\partial x_0^2} - 4\lambda y_0^3\frac{\partial^3 T}{\partial x_0^3} + 4\lambda a^2 y_0\frac{\partial^3 T}{\partial y_0^2\partial x_0} - 4a^2y_0\frac{\partial^3 T}{\partial y_0\partial x_0^2} + 4\lambda a^2\frac{\partial^2 T}{\partial y_0\partial x_0} + (4\lambda^2y_0^2 - 3a^2)\frac{\partial^2 T}{\partial x_0^2} = 0. \qquad (3.17)$$

To complete the story it is finally necessary to obtain the boundary conditions that accompany the equations for T^+ and T^-. From Eqs. (2.13) and (2.15) it follows immediately that $T^+(x_0, y_0) = 0$ if the ballistic time to reach the appropriate boundary is also zero, i.e., if $\tau^+(x_0, y_0) = 0$. Similarly, $T^-(x_0, y_0)$ vanishes if $\tau^-(x_0, y_0)$ does. On the other hand, it can easily be seen that $\tau^\pm(L, y_0) = 0$ if and only if $y_0 > 0$, and that $\tau^\pm(0, y_0) = 0$ if and only if $y_0 < 0$. Therefore the appropriate boundary conditions for the escape problem are

$$T^\pm(L, y_0) = 0 \quad \text{if} \quad y_0 > 0, \qquad (3.18)$$

$$T^\pm(0, y_0) = 0 \quad \text{if} \quad y_0 < 0. \qquad (3.19)$$

The average mean exit time, Eq. (3.16), satisfies the same boundary conditions.

IV. FREE INERTIAL PROCESSES DRIVEN BY GAUSSIAN WHITE NOISE

The partial differential equation for the mean exit time simplifies considerably when the inertial process is driven by Gaussian white noise. The equation can be derived in a more straightforward way than by our procedure [8], but alternative derivations are restricted to Gaussian white noise whereas the strength of our approach is its much greater generality. In any case, as is well known, in the limit

$$a \to \infty, \qquad \lambda \to \infty, \qquad \frac{a^2}{\lambda} \equiv D < \infty \qquad (4.1)$$

the dichotomous Markov noise $F(t)$ becomes Gaussian white noise. In other words, in this limit the process given by Eq. (1.1) goes to the process

$$\ddot{X}(t) = \eta(t)\ , \qquad (4.2)$$

where $\eta(t)$ is Gaussian and δ-correlated noise,

$$\langle\eta(t)\eta(t')\rangle = D\delta(t - t''). \qquad (4.3)$$

In the Gaussian white-noise limit (4.1) we see from

Eq. (3.16) and Eqs. (3.14) and (3.15) that the equation for the mean exit time $T(x_0, y_0)$ reads

$$\frac{1}{2}D\frac{\partial^2 T}{\partial y_0^2} + y_0\frac{\partial T}{\partial x_0} = -1, \tag{4.4}$$

with boundary conditions

$$T(L, y_0) = 0 \qquad \text{if} \quad y_0 > 0, \tag{4.5}$$

$$T(0, y_0) = 0 \qquad \text{if} \quad y_0 < 0. \tag{4.6}$$

In fact, using alternative approaches one can obtain a more general result in this case. Thus, let $\mathbf{Z}(t) = (Z_1(t), \ldots, Z_n(t))$ be an n-dimensional random process whose dynamical evolution is given by the set of Langevin equations

$$\dot{Z}_i(t) = f_i[\mathbf{Z}(t)] + \sum_{i,j} G_{ij}[\mathbf{Z}(t)]\eta_j(t), \tag{4.7}$$

where $\eta_j(t)$, $j = 1, 2, \ldots, n$ are Gaussian mutually independent δ-correlated random variables. The mean exit time $T(\mathbf{z}_0)$ out of a region $\mathcal{R}$ surrounded by boundary $\mathcal{S}$ satisfies the partial differential equation [1]

$$\frac{1}{2}D\sum_{i,j} G_{ij}(\mathbf{z}_0)\frac{\partial^2 T(\mathbf{z}_0)}{\partial z_{0,i}\partial z_{0,j}} + \sum_i f_i(\mathbf{z}_0)\frac{\partial T(\mathbf{z}_0)}{\partial z_{0,i}} = -1 \tag{4.8}$$

with the boundary condition

$$T(\mathbf{z}_0) = 0, \qquad \mathbf{z}_0 \in \mathcal{S}. \tag{4.9}$$

The process (4.2) corresponds to the choices $n = 2$, $z_1 = x$, $z_2 = y$, $f_1(\mathbf{z}) = y$, $f_2(\mathbf{z}) = 0$, $G_{11} = G_{12} = G_{21} = 0$, and $G_{22} = 1$. In this case (4.8) reduces to (4.4).

V. BEHAVIOR OF THE MEAN EXIT TIME

As noted earlier, we have not been able to obtain analytic solutions to the integral equations for the mean exit time or even to the differential equations Eqs. (3.12) and (3.13) appropriate for Markovian dichotomous fluctuations. Nevertheless, the derivation and the equations themselves provide a great deal of insight into the behavior of the mean exit time. We present this analysis for the case of Markovian dichotomous fluctuations in this section.

We begin by noting that three parameters characterize the mean exit time in this case: a, the absolute value of the acceleration; λ, the rate at which the acceleration switches from one value to the other; and L, the length of the interval. These three parameters combine in only one way into a single dimensionless parameter ϵ:

$$\epsilon = \frac{a}{\lambda^2 L}. \tag{5.1}$$

If times and distances are properly scaled into dimensionless form, then all results should depend only on the single parameter ϵ. Indeed, if times are expressed in units of λ^{-1}, i.e., $\hat{t} = \lambda t$, and distances in units of L, i.e., $\hat{x} = x/L$, then $\hat{T}$ is a function only of the initial conditions and ϵ:

$$\hat{T}(\hat{x}_0, \hat{y}_0) = f(\epsilon, \hat{x}_0, \hat{y}_0). \tag{5.2}$$

Another way of thinking about ϵ is to note that it is determined by the two time scales involved in the evolution of the random process. One is the average time the noise retains a given value,

$$\tau = 1/\lambda, \tag{5.3}$$

or the correlation time of the noise, $\tau_c = \tau/2$. The other is the time it takes the system to reach the boundaries in a deterministic way, τ^+ and τ^- (see Sec. I), denoted by τ_d. This time depends on the initial position and initial speed. For the sake of our specific argument we choose $x_0 = L/2$ and $y_0 = 0$ to fix the meaning of τ_d. In this case,

$$\tau^+ = \tau^- = \sqrt{L/a}. \tag{5.4}$$

Consequently,

$$\epsilon = \left(\frac{\tau}{\tau_d}\right)^2. \tag{5.5}$$

This relation allows one to identify two regimes in the behavior of the mean exit time $\hat{T}(\frac{1}{2}, 0)$. When $\epsilon \to 0$ ($\tau \ll \tau_d$), the system is driven by almost white noise. In the opposite limit, as $\epsilon \to \infty$ ($\tau_d \ll \tau$), $\hat{T}$ is dominated by the deterministic inertial behavior. The two regimes are characterized by the exponent that gives the scaling of $\hat{T}$ with ϵ,

$$\hat{T} \sim \epsilon^{\beta}. \tag{5.6}$$

When $\epsilon \to \infty$,

$$\hat{T}(1/2, 0) \sim \frac{1}{2}(\hat{\tau}^+ + \hat{\tau}^-) = \epsilon^{-1/2}, \tag{5.7}$$

and $\beta = -1/2$. In writing Eq. (5.7) we have assumed an equal probability for the two values of the initial acceleration. The exponent when $\epsilon \to 0$ is obtained with the following reasoning. For the free inertial system driven by white noise, the mean square displacement goes as t^3 [3],

$$\langle X^2 \rangle \sim t^3. \tag{5.8}$$

Therefore, unless there are unexpected unbounded moments in the problem we assume that the usual exponent reciprocity holds so that

$$\hat{T}(1/2, 0) \sim \epsilon^{-2/3}, \tag{5.9}$$

and therefore $\beta = -2/3$.

In order to test the crossover from $\beta = -1/2$ to $\beta = -2/3$ we carried out direct simulations of the random process $X(t)$ for different values of a, λ, and L to obtain $\hat{T}(\frac{1}{2}, 0)$ as a function of ϵ. The results are shown in Fig. 2, where the crossover from one value of β to the other is evident. The dashed line on this log-log plot corresponds to $\hat{T} = A\epsilon^{-2/3}$ with A fitted for the value of $\hat{T}$ obtained

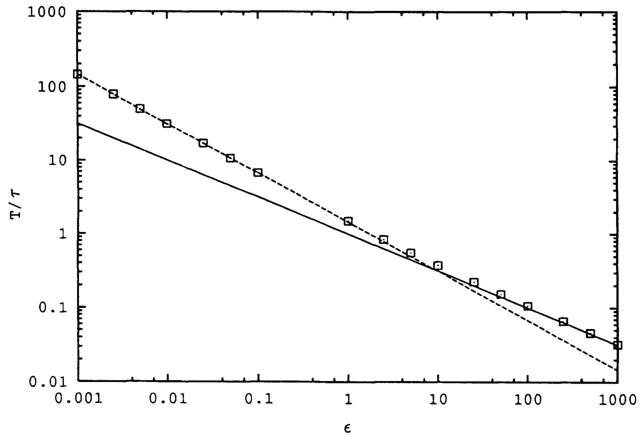

FIG. 2. Log-Log plot of $T(\frac{L}{2},0)/\tau$ vs $\epsilon = a/(\lambda^2 L)$. The square symbols are simulation results for different a, λ, and L. The dashed line gives the asymptotic behavior when $\epsilon \to 0$, that is, a line of slope $\beta = -2/3$. The solid line represents the asymptotic behavior when $\epsilon \to \infty$ and is given by a straight line of slope $\beta = -1/2$.

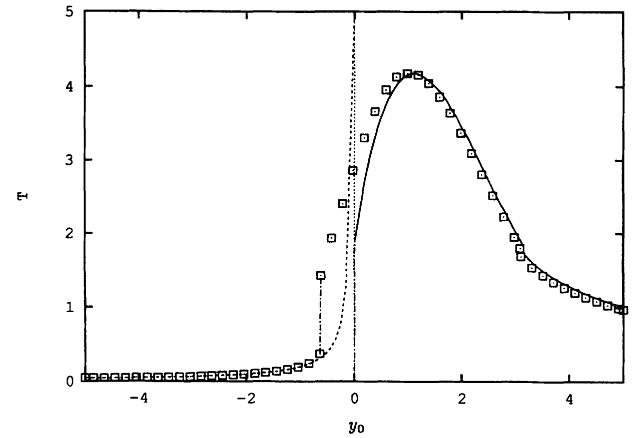

FIG. 3. Solid curve: $T(0, y_0)$ as a function of y_0 for $a = 1$, $\lambda = 1$, and $L = 5$. Square symbols: $T(0.2, y_0)$ as a function of y_0 for the same values of the parameters. Dashed line: approximation given by Eq. (5.11).

by simulation with $\epsilon = 10^{-5}$. The solid line corresponds to Eq. (5.7).

Much more interesting is the behavior of the mean exit time as a function of the initial position and velocity. It is easy to convince oneself that for a given value of ϵ the mean exit time has an absolute maximum for the case considered above, that is, when the process begins with zero velocity in the middle of the interval. For initial conditions other than $x_0 = L/2$ and $y_0 = 0$, the mean exit time decreases and is more and more determined by $\hat{\tau}^+$ and $\hat{\tau}^-$ as the magnitude of the initial velocity increases and/or the process starts nearer to one of the end points of the interval. Two asymptotic expressions are readily deduced from Eqs. (1.6)–(1.12). When $\hat{y}_0 \to +\infty$ or $\hat{x}_0 \to 1$ ($\hat{y}_0 > 0$),

$$\hat{T}(\hat{x}_0, \hat{y}_0) \sim \frac{1 - \hat{x}_0}{\hat{y}_0}; \tag{5.10}$$

when $\hat{y}_0 \to -\infty$ or $\hat{x}_0 \to 0$ ($\hat{y}_0 < 0$),

$$\hat{T}(\hat{x}_0, \hat{y}_0) \sim -\frac{\hat{x}_0}{\hat{y}_0}. \tag{5.11}$$

Results for other regimes are obtained from direct simulation of the process. These results are shown in Fig. 3, where we analyze the mean exit time as a function of the initial velocity $\hat{y}_0$ for two different values of the initial position $\hat{x}_0$. Several points are noteworthy in the figure. First, let us focus on the initial position $\hat{x}_0 = 0.2$. The simulation results for $\hat{T}(0.2, \hat{y}_0)$ are shown by the square symbols. The dashed line is the approximation (5.11) and the simulations clearly approach this asymptotic behavior with increasing negative initial velocities. The simulations show a maximum at around $\hat{y}_0 = 1$ and a decrease that would eventually approach the asymptote (5.10) with increasing positive initial velocities. The really interesting feature about the simulation is the discontinuity in the mean exit time observed at $\hat{y}_0 = -\sqrt{0.4}$. At this initial velocity the mean exit time jumps in value by about unity. The origin of this *real* discontinuity is discussed below.

Next we focus in Fig. 3 on the initial position $\hat{x}_0 = 0$. The simulation results for $\hat{T}(0, \hat{y}_0)$ are shown by the solid curve. Again a discontinuity, now even more pronounced, is observed at $\hat{y}_0 = 0$, where the mean exit time jumps by almost two units. Note that the mean exit time for negative velocities is identically zero, as it should be: if the process starts at the lower boundary with a negative velocity it indeed exits the interval immediately.

In order to understand the origin of these discontinuities we make use of the equations satisfied by the mean exit time. In fact, it follows from these equations that the mean exit time is discontinuous along two curves in the (x_0, y_0) plane. Although such a discontinuity is striking, it is not new: in noninertial systems driven by persistent-periodic dichotomous noise [10] the mean first passage time also exhibits this unsual behavior. The discontinuities arise from the presence of the barriers, which cause the phase space (x_0, y_0) to be divided into two regions for T^+ and T^- and into three for T. Let us consider the case of T^+ as example. The discontinuity of $\tau^+(x_0, y_0)$ [cf. (1.6) and (1.8)] along the curve

$$y_0 = -\sqrt{2ax_0}, \tag{5.12}$$

causes a discontinuity in T^+ along this curve. Indeed, if we fix x_0 and approach $y_c \equiv -\sqrt{2ax_0}$ from the right, $y_0 \to y_c^+$, and from the left, $y_0 \to y_c^-$, in Eq. (2.11), the difference $T^+(x_0, y_c^+) - T^+(x_0, y_c^-)$ reads

$$\Delta T^+(x_0, y_c) = \lim_{y_0 \to y_c^+} T^+(x_0, y_0) - \lim_{y_0 \to y_c^-} T^+(x_0, y_0)$$

$$= \Delta G^+(x_0, y_c) + \int_{\tau^+(x_0, y_c^-)}^{\tau^+(x_0, y_c^+)} dt\, \psi(t) \int_0^L$$

$$\times dx \int_{-\infty}^{\infty} dy\, T^-(x, y) h^+(x, y; t | x_0, y_0), \tag{5.13}$$

where

$$\Delta G^+(x_0,y_c) = \frac{1}{\lambda} e^{-\lambda\sqrt{2x_0/a}} \left(1 - e^{-\lambda\sqrt{2L/a}}\right). \quad (5.14)$$

In deducing Eq. (5.14), we have used the fact that an exponential $\psi(t)$ leads to [see Eq. (2.13)]

$$G^+(x_0,y_0) = \frac{1}{\lambda}\left(1 - e^{-\lambda\tau^+}\right). \quad (5.15)$$

Furthermore, from Eqs. (1.6) and (1.8) it follows that

$$\tau^+(x_0,y_c^+) = \sqrt{\frac{2x_0}{a}} + \sqrt{\frac{2L}{a}} > \tau^+(x_0,y_c^-) = \sqrt{\frac{2x_0}{a}}. \quad (5.16)$$

Therefore, since T^- is positive by definition, as is h^+, we conclude that $\Delta T(x_0,y_c) > 0$ and the mean exit time is discontinuous at y_c. Equation (5.14) gives a lower bound for the value of the discontinuity,

$$\Delta T^+(x_0,y_c) > \Delta G^+(x_0,y_c). \quad (5.17)$$

Similar reasoning leads to a discontinuity of $T^-(x_0,y_0)$ along the curve

$$y_0 = \sqrt{2a(L-x_0)}. \quad (5.18)$$

Since $T(x_0,y_0)$ involves both T^+ and T^- [see Eq. (3.16)], it will exhibit both discontinuities.

The entire phase space portrait of mean exit times shows these discontinuities most clearly (Fig. 3 shows two slices of this portrait). In Fig. 4 we show our simulation results for the surface $T^+(x_0,y_0)$ as a function of both x_0 and y_0 for a particular set of parameter values. The error bars are so small ($\sigma \sim 10^{-2}$) that simulation points were connected to give the impression of a smooth plot. The discontinuity along $y_0 = -\sqrt{2ax_0}$ has been outlined with a dashed line. Note also that the boundary conditions, Eqs. (3.18) and (3.19) are precisely satisfied. Simulation values for $T(x_0,y_0)$ for the same set of parameter values are shown in Fig. 5. Note that the symmetry relation

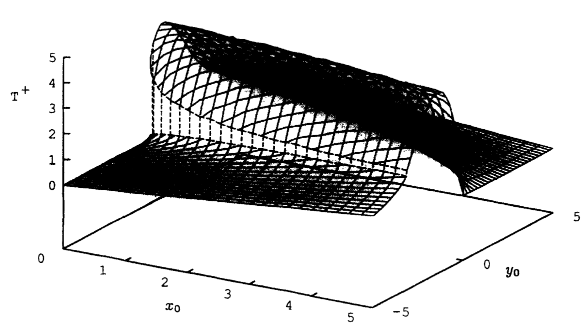

FIG. 4. Simulation results for $T^+(x_0,y_0)$ as a function of x_0 and y_0 for $a=1$, $\lambda=1$, and $L=5$. For our number of realizations made, 10^5, the error bars are so small that the simulation points can be connected with lines to obtain a smooth plot. The discontinuity is highlighted via a dashed curve for visual aid.

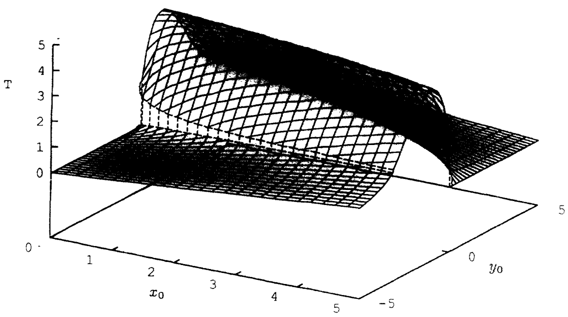

FIG. 5. Simulation results for $T(x_0,y_0)$ as a function of x_0 and y_0 for $a=1$, $\lambda=1$, and $L=5$. There are two discontinuities that are highlighted via dashed curves. The number of realizations is 2×10^5.

$$T(x_0,y_0) = T(L-x_0,-y_0) \quad (5.19)$$

is apparent in the figure. The discontinuities along the two curves discussed above are clearly visible.

The magnitude of the discontinuities depends on the values of the parameters that determine the mean exit time. This can be seen in Fig. 6, where we plot the intersection of the surface $T(x_0,y_0)$ and the plane $x_0 = L/2$ as a function of y_0 (Fig. 3 represents a similar figure but with intersections with places at $x_0 = L/25$ and $x_0 = 0$). In all cases we have taken $L=5$. The solid curve corresponds to the parameter values of Fig. 5, that is, $a=\lambda=1$. Simulation results for the parameter values $a=10$, $\lambda=100$ are represented by the dashed curve,

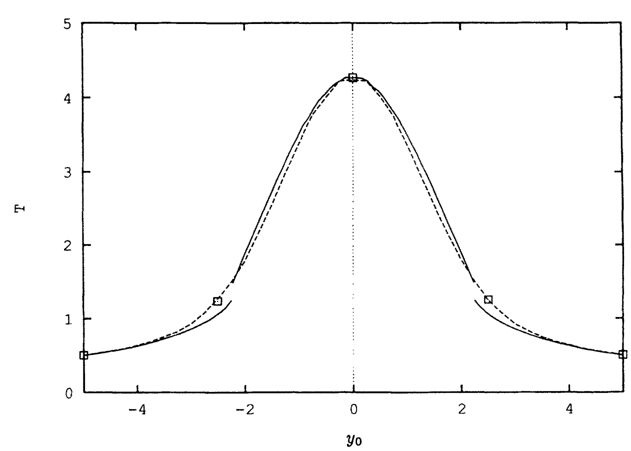

FIG. 6. $T(\frac{L}{2},y_0)$ as a function of the initial speed y_0. The solid curve was obtained for the parameters of Fig. 5, that is, this curve is the intersection of that figure with the plane $x_0=2.5$. The plot shows two discontinuities, one at $y_0=\sqrt{5}$ and the other at $y_0=-\sqrt{5}$. The dashed curve is obtained for $a=10$ and $\lambda=100$. Note that the discontinuities for these values of the parameters occur at $y_0=\sqrt{50}$ and $-\sqrt{50}$, that is, outside of the scale of this plot. The square symbols ($\square$) are obtained for $a=100$ and $\lambda=10^4$. As a and λ grow in such a way that $D=a^2/\lambda$ is finite, the results converge to those of the free inertial process driven by white noise of intensity D.

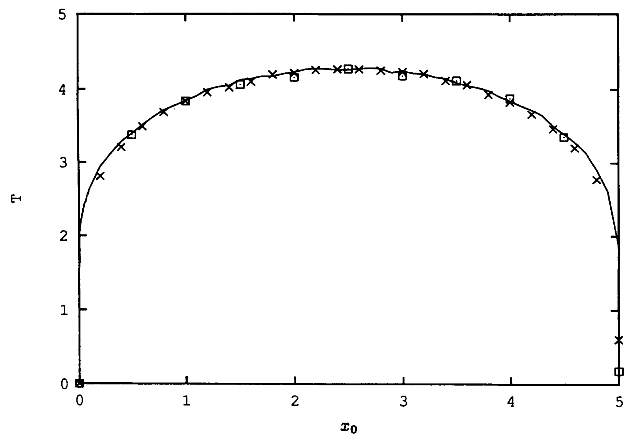

FIG. 7. $T(x_0, 0)$ as a function of the initial position x_0. The solid curve is for the same a, λ, and L as in Fig. 5. The curve is the intersection of that figure with the plane $y_0 = 0$. The crosses ($\times$) are simulation results for $a = 10$ and $\lambda = 100$, while the square symbols ($\Box$) are obtained with $a = 100$ and $\lambda = 10^4$. Aside from the differences in the values of the mean exit times at the boundaries ($x_0 = 0$ and 5), all the results lie on a single curve that corresponds to the mean exit time $T(x_0, 0)$ for the free inertial process driven by white noise of intensity $D = a^2/\lambda = 1$.

while the square symbols correspond to the values $a = 100$, $\lambda = 10^4$. Note that the discontinuities move off the graph because they occur at larger values of y_0 as a grows. Moreover, with increasing λ and a the curves converge to the curve of the mean exit time or mean first passage time for a free inertial system driven by white noise, as explained in Sec. IV.

Finally, Fig. 7 shows the intersection of the surface $T(x_0, y_0)$ and the plane $y_0 = 0$. More precisely, the intersection we plot is with the plane $y_0 \to 0^-$, as can be seen from the different values of the mean exit time at the boundaries. Thus, for instance, the mean exit time at $x_0 = 0$ vanishes while it does not at $x_0 = L$. The solid curve again shows our simulation results for the parameters of Fig. 5, that is, $a = \lambda = 1$. The squares and crosses are simulation results for higher values of a and λ while keeping a^2/λ constant at the value unity. As in Fig. 6, the values of the mean exit time as a and λ grow converge to those of the free inertial system driven by white noise.

VI. SUMMARY

The exit time out of an interval for a free inertial process driven by a dichotomous random acceleration has been analyzed both analytically and numerically. We have derived integral equations for the mean exit time. If the dichotomous process is Markovian, these integral equations reduce to partial differential equations that in turn reduce correctly to those that have been obtained by other methods in the limit of Gaussian white noise.

We have been unable to solve these equations analytically, but have found them useful in understanding the behavior of the mean exit time obtained from direct numerical simulations of the process. The most interesting aspect of this behavior is the appearance of discontinuities in the mean exit time as a function of the initial position and velocity of the process. We map out the phase space behavior of the mean exit time, identify the discontinuity curves, and provide a lower bound for the values of the discontinuities.

ACKNOWLEDGMENTS

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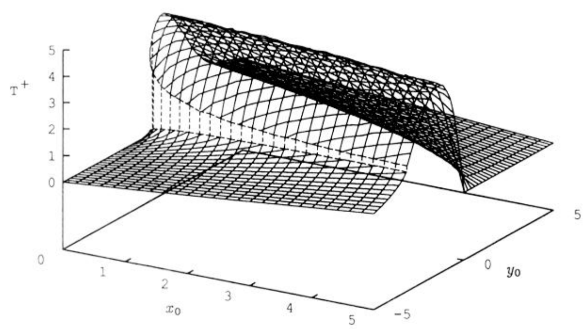

FIG. 4. Simulation results for $T^+(x_0, y_0)$ as a function of x_0 and y_0 for $a = 1$, $\lambda = 1$, and $L = 5$. For our number of realizations made, 10^5, the error bars are so small that the simulation points can be connected with lines to obtain a smooth plot. The discontinuity is highlighted via a dashed curve for visual aid.

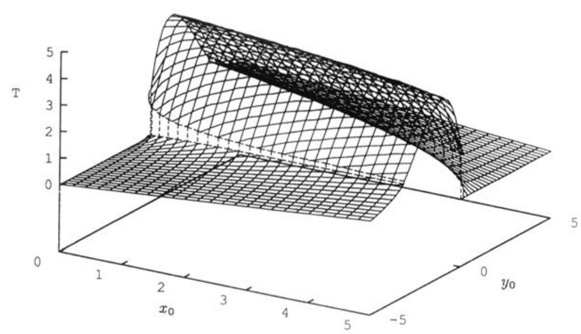

FIG. 5. Simulation results for $T(x_0, y_0)$ as a function of x_0 and y_0 for $a = 1$, $\lambda = 1$, and $L = 5$. There are two discontinuities that are highlighted via dashed curves. The number of realizations is 2×10^5.