Synchronization of spatiotemporal chaotic systems by feedback control

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We demonstrate that two identical spatiotemporal chaotic systems can be synchronized by (1) linking one or a few of their dynamical variables, and (2) applying a small feedback control to one of the systems. Numerical examples using the diffusively coupled logistic map lattice are given. The effect of noise and the limitation of the technique are discussed.

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I. INTRODUCTION

Recently, there has been a growing interest in the synchronization of chaotic dynamical systems [1–4] due to its potential application in secure communications [1,2]. Theoretically, the problem of synchronizing chaos itself is interesting because chaos, meaning sensitive dependence of a system's dynamical variables on initial conditions, seems to defeat synchronization of dynamical trajectories. The pioneering theoretical and experimental work by Pecora and Carroll [1] demonstrated that two chaotic systems can be synchronized if (i) some dynamical variables (the driving variables) are used to link the two systems and (ii) the subsystems excluding the driving variables possess only negative Lyapunov exponents (nonchaotic subsystems). Given a chaotic system, whether or not this type of synchronism can occur depends on the choice of the driving variables or, equivalently, the choice of the subsystem. For cases where nonchaotic subsystems cannot be found, an alternative approach [3] based on the idea of controlling chaos [5] has been suggested. In this method, small temporary parameter perturbations computed based on the difference of the two trajectories and the geometrical structure (stable and unstable directions) of the chaotic trajectory are applied to one of the two systems to synchronize them. This approach has also been implemented in experiments to synchronize two chaotic laser diodes [6]. This control approach [3], however, was applied only to systems described by two-dimensional maps or three-dimensional autonomous flow that can be reduced to two-dimensional maps on a Poincaré surface of section.

Spatiotemporal chaotic systems are high dimensional dynamical systems. Consider such a system that consists of a spatial network of chaotic elements. For the Pecora-Carroll type of synchronism to occur, it may be necessary to use a large number of driving variables spatially distributed among chaotic elements, and, as a matter of fact, that could be done [7]. Nonetheless, it is often the case that the subsystem obtained by excluding only a few driving variables will still be chaotic to a similar degree to the original system. That is, the subsystem has a number of positive Lyapunov exponents comparable to that of the original system. To illustrate this, consider the coupled logistic map lattice [8] (to be described in Sec. III) with 20 spatial sites (a 20-dimensional system). In certain parameter regimes, there are eight positive Lyapunov exponents. Linking an arbitrary dynamical variable yields a 19-dimensional subsystem which still has seven positive Lyapunov exponents. On the other hand, while synchronizable nonchaotic subsystems can be obtained by linking a sufficient number of dynamical variables, they are difficult to identify due to the high dimensionality of the system. The control strategy proposed in Ref. [3] is difficult to extend to high-dimensional systems because its success depends on the existence of unique stable and unstable directions at each trajectory point. Spatiotemporal chaotic systems usually have many unstable and stable directions at each trajectory point in the phase space.

In this paper, we demonstrate that by combining the Pecora-Carroll idea [1] and the control method in Ref. [3], it is possible to synchronize two nearly identical spatiotemporal systems. Specifically, by using a certain number of driving variables and by applying appropriately designed *feedback controls,* synchronization can be achieved for the two systems. The driving variables can be arbitrarily chosen and their number can be as few as *only 1.* The feedback control is applied to one of the two systems to be synchronized. The magnitude of the feedback control required can, in general, be very small. It should be mentioned that a similar method for synchronizing chaotic systems has been proposed by Pyragas [9]. In this method, the feedback control is directly proportional to the difference of a dynamical variable from two chaotic systems, and is applied to one of the systems.

The organization of the paper is as follows. In Sec. II, we describe the method of synchronization and design of the feedback control. In Sec. III, we apply the algorithm

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to the diffusively coupled logistic map lattice [8]. The effect of noise is also briefly discussed. In Sec. IV, we discuss the issue of long transient time preceding turn-on of the feedback control.

II. CONTROL METHOD

Our design of the feedback control is based on the principle of the Kalman filter [10], which tracks the system state by measuring a single scalar function of the system state. The Kalman filter is optimal for linear systems. For nonlinear or chaotic systems, a modified technique was developed by So, Ott, and Dayawansa [11] to deduce and track the state of the system from limited observation. Our design of the synchronization scheme is a direct application of this modified technique. Consider two identical spatiotemporal systems described by the following maps:

$$\mathbf{x}_{n+1}=\mathbf{F}(\mathbf{x}_n)\ ,\quad \hat{\mathbf{x}}_{n+1}=\mathbf{F}(\hat{\mathbf{x}}_n)\ , \tag{1}$$

where $\mathbf{x}$ and $\hat{\mathbf{x}}$ are N-dimensional state vectors. Following Pecora and Carroll [1], we decompose the system state into two parts: one is the N_d-dimensional driving system, which we denote $\mathbf{z}$ and $\hat{\mathbf{z}}$, and the other is the N_0-dimensional subsystems to be synchronized, denoted by $\mathbf{y}$ and $\hat{\mathbf{y}}$, where $N_d \ll N_0$. In general, we allow the subsystems $\mathbf{y}$ and $\hat{\mathbf{y}}$ to be chaotic. By definition of "driving," $\mathbf{z}$ and $\hat{\mathbf{z}}$ are equivalent and, hence, we denote $\mathbf{z}=\hat{\mathbf{z}}$. The equations for $\mathbf{y}$, $\hat{\mathbf{y}}$, and $\mathbf{z}$ are as follows:

$$\begin{aligned}\mathbf{y}_{n+1}&=\mathbf{F}_y(\mathbf{y}_n,\mathbf{z}_n)\ ,\\ \hat{\mathbf{y}}_{n+1}&=\mathbf{F}_y(\hat{\mathbf{y}}_n,\mathbf{z}_n)\ ,\\ \mathbf{z}_{n+1}&=\mathbf{F}_z(\mathbf{y}_n,\mathbf{z}_n)\ ,\end{aligned} \tag{2}$$

where $\mathbf{F}=[\mathbf{F}_y,\mathbf{F}_z]$. In the case where the full system $\mathbf{F}$ is chaotic, Pecora and Carroll showed that when the subsystem $\mathbf{F}_y$ has all negative Lyapunov exponents, $\mathbf{y}_n$ and $\hat{\mathbf{y}}_n$ can be synchronized [1]. Subsystems having only negative Lyapunov exponents are hard to identify when Eq. (1) is spatiotemporally chaotic and has many positive Lyapunov exponents and, hence, it is assumed that we do not know them. To achieve synchronization of $\mathbf{y}_n$ and $\hat{\mathbf{y}}_n$, we apply the following feedback control to one of the subsystems $\hat{\mathbf{y}}_n$:

$$\hat{\mathbf{y}}_{n+1}=\mathbf{F}_y(\hat{\mathbf{y}}_n,\mathbf{z}_n)-C_n\cdot[\mathbf{F}_z(\hat{\mathbf{y}}_n,\mathbf{z}_n)-\mathbf{F}_z(\mathbf{y}_n,\mathbf{z}_n)]\ , \tag{3}$$

where $\mathbf{C}_n$ is an $N_0\times N_d$ control matrix to be calculated at each time step. The synchronization scheme is schematically shown in Fig. 1. The feedback control $-\mathbf{C}_n\cdot[\mathbf{F}_z(\hat{\mathbf{y}}_n,\hat{\mathbf{z}}_n)-\mathbf{F}_z(\mathbf{y}_n,\mathbf{z}_n)]$ is applied only when $\mathbf{y}_n$ and $\hat{\mathbf{y}}_n$ are close [5,3]. The linearized dynamics in the neighborhood of $\mathbf{y}_n$ can therefore be written as

$$\begin{aligned}\delta\mathbf{y}_{n+1}=\hat{\mathbf{y}}_{n+1}-\mathbf{y}_{n+1}&=\mathbf{F}_y(\hat{\mathbf{y}}_n,\mathbf{z}_n)-\mathbf{F}_y(\mathbf{y}_n,\mathbf{z}_n)-\mathbf{C}_n\cdot[\mathbf{F}_z(\hat{\mathbf{y}}_n,\mathbf{z}_n)-\mathbf{F}_z(\mathbf{y}_n,\mathbf{z}_n)]\\ &=[\mathbf{DF}_y(\mathbf{y}_n,\mathbf{z}_n)-\mathbf{C}_n\cdot\mathbf{DF}_z(\mathbf{y}_n,\mathbf{z}_n)]\cdot\delta\mathbf{y}_n\equiv\mathbf{A}_n\cdot\delta\mathbf{y}_n\ ,\end{aligned} \tag{4}$$

where $\mathbf{DF}_y$ and $\mathbf{DF}_z$ are the $N_0\times N_0$ and $N_d\times N_0$ Jacobian matrices of $\mathbf{F}_y$ and $\mathbf{F}_z$, respectively, evaluated at $\mathbf{y}_n$ and $\mathbf{z}_n$. Since $\mathbf{F}_y$ is chaotic, $\hat{\mathbf{y}}_{n+1}$ will diverge from $\mathbf{y}_{n+1}$ exponentially without control. Our goal is to design the control matrix $\mathbf{C}_n$ so that $\delta\mathbf{y}_n\to 0$ as $n\to\infty$. To achieve this we assume that the subsystem $\mathbf{F}_y$ has N_u positive and N_s negative Lyapunov exponents, where $N_u+N_s=N_0$. Furthermore, we assume hyperbolicity for the subsystems $\hat{\mathbf{y}}$ and $\mathbf{y}$, i.e., every point on the asymptotic attractor of $\mathbf{F}_y$ has N_u unstable and N_s stable directions, the stable and unstable subspaces span the whole phase space, and the angles between stable and unstable subspaces are bounded away from zero [12,13]. However, the feedback control so designed applies to nonhyperbolic dynamical systems as well [11]. Let $\mathbf{e}_n^i$ $(i=1,\ldots,N_u)$ be the set of base column vectors in the unstable space at $\mathbf{y}_n$. If we restrict the control matrix $\mathbf{C}_n$ to the unstable space of $\mathbf{F}_y$ at $\mathbf{y}_{n+1}$, i.e.,

$$\mathbf{C}_n=\sum_{i=1}^{N_u}\mathbf{b}_n^i=\sum_{i=1}^{N_u}[C_1^i\mathbf{e}_{n+1}^i\cdot\mathbf{v}_1+\cdots+C_{N_d}^i\mathbf{e}_{n+1}^i\cdot\mathbf{v}_{N_d}]\ , \tag{5}$$

where $\{\mathbf{v}_j\}$ $(j=1,\ldots,N_d)$ are a complete set of row vectors that span the driving system $\mathbf{F}_z$ and $C_1^i,C_2^i,\ldots,C_{N_d}^i$ $(i=1,\ldots,N_u)$ are the set of $N_u\times N_d$ control coefficients, then it can be shown [11] that the matrix $\mathbf{A}_n$ $[=\mathbf{DF}_y(\mathbf{y}_n,\mathbf{z}_n)-\mathbf{C}_n\cdot\mathbf{DF}_z(\mathbf{y}_n,\mathbf{z}_n)]$ reduces to the following upper triangular form:

$$\mathbf{A}_n=\begin{bmatrix}\mathbf{U}_n & \mathbf{W}_n\\ 0 & \mathbf{S}_n\end{bmatrix}\ , \tag{6}$$

where $\mathbf{U}_n$ $(\mathbf{S}_n)$ is an $N_u\times N_u$ $(N_s\times N_s)$ matrix that causes a vector in the unstable (stable) space at $\mathbf{y}_n$ to evolve into a vector in the unstable (stable) space at $\mathbf{y}_{n+1}$, and $\mathbf{W}_n$ is an $N_u\times N_s$ matrix that takes a vector in the stable space at $\mathbf{y}_n$ into a vector in the unstable space at $\mathbf{y}_{n+1}$. In order to have $|\delta\mathbf{y}_n|\to 0$ as $n\to\infty$, it is required that all eigenvalues of the product matrix $\mathbf{A}_n\mathbf{A}_{n-1}\cdots\mathbf{A}_1$ vanish as $n\to\infty$. Since,

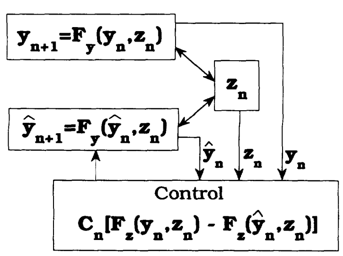

FIG. 1. The scheme of synchronizing two spatiotemporal chaotic systems by driving and feedback control.

$$\mathbf{A}_n\,\mathbf{A}_{n-1}\cdots\,\mathbf{A}_1=\begin{bmatrix}\mathbf{U}_n\mathbf{U}_{n-1}\cdots\mathbf{U}_1 & \sum_{i=1}^{n}\prod_{j=i+1}^{n}\mathbf{U}_j\mathbf{W}_i\prod_{k=1}^{i-1}\mathbf{S}_k\\ \mathbf{0} & \mathbf{S}_n\mathbf{S}_{n-1}\cdots\mathbf{S}_1\end{bmatrix}, \tag{7}$$

i.e., the product matrix $\mathbf{A}_n\,\mathbf{A}_{n-1}\cdots\,\mathbf{A}_1$ is upper triangular, and since the matrices $\mathbf{S}_n$ are already in the stable space along the trajectory (eigenvalues of the matrix product $\mathbf{S}_n\mathbf{S}_{n-1}\cdots\mathbf{S}_1\to 0$ as $n\to\infty$), the stability of the product $\mathbf{A}_n\,\mathbf{A}_{n-1}\cdots\,\mathbf{A}_1$ depends solely on the stability of $\mathbf{U}_n\mathbf{U}_{n-1}\cdots\mathbf{U}_1$. One way to make the product $\mathbf{U}_n\mathbf{U}_{n-1}\cdots\mathbf{U}_1$ stable is to let $\mathbf{U}_i$ be lower triangular and stable, i.e., all diagonal elements of $\mathbf{U}_i$ are eigenvalues of $\mathbf{U}_i$ and are less than 1. In this way, the product $\mathbf{U}_n\mathbf{U}_{n-1}\cdots\mathbf{U}_1$ is still lower triangular and has vanishing diagonal elements (eigenvalues) [11].

Now we define a set of contravariant row vectors $\mathbf{f}^i_{n+1}$ ($i=1,\ldots,N_u$) in the unstable space at $\mathbf{y}_{n+1}$ such that $\mathbf{f}^i_{n+1}\cdot\mathbf{e}^j_{n+1}=\delta_{ij}$, where $\delta_{ij}=0$ if $i\neq j$ and $\delta_{ii}=1$. The matrix elements of $\mathbf{U}_n$ are given by $U_{ij}=\mathbf{f}^i_{n+1}\cdot\mathbf{A}_n\cdot\mathbf{e}^j_n$ [11]. In order to make the matrices $\mathbf{U}_i$ lower triangular, we look at elements U_{ij} of the matrix $\mathbf{U}_n$, which can be expressed as follows:

$$\begin{aligned}U_{ij}&=\mathbf{f}^i_{n+1}\cdot\mathbf{A}_n\cdot\mathbf{e}^j_n=\mathbf{f}^i_{n+1}\cdot[\mathbf{DF}_y(\mathbf{y}_n,\mathbf{z}_n)-\mathbf{C}_n\cdot\mathbf{DF}_z(\mathbf{y}_n,\mathbf{z}_n)]\cdot\mathbf{e}^j_n\\ &=\mathbf{f}^i_{n+1}\cdot\mathbf{DF}_y(\mathbf{y}_n,\mathbf{z}_n)\cdot\mathbf{e}^j_n-\mathbf{f}^i_{n+1}\cdot\mathbf{b}^i_n\cdot\mathbf{DF}_z(\mathbf{y}_n,\mathbf{z}_n)\cdot\mathbf{e}^j_n\ ,\end{aligned} \tag{8}$$

where $\mathbf{f}^i_{n+1}\cdot\mathbf{b}^j_n=0$ for $i\neq j$ has been used [cf. Eq. (5)]. In order to have $U_{ij}=0$ for $j>i$, So, Ott, and Dayawansa [11] suggested the following procedure for choosing the unstable base vectors:

$$\begin{aligned}\lambda^1_n\mathbf{e}^1_{n+1}&=[\mathbf{DF}_y(\mathbf{y}_n,\mathbf{z}_n)]\cdot\mathbf{e}^1_n\ ,\\ \lambda^2_n\mathbf{e}^2_{n+1}&=[\mathbf{DF}_y(\mathbf{y}_n,\mathbf{z}_n)-\mathbf{b}^1_n\cdot\mathbf{DF}_z(\hat{\mathbf{y}}_n,\mathbf{z}_n)]\cdot\mathbf{e}^2_n\ ,\\ \cdots&=\cdots\\ \lambda^{N_u}_n\mathbf{e}^{N_u}_{n+1}&=\left[\mathbf{DF}_y(\mathbf{y}_n,\mathbf{z}_n)-\sum_{i=1}^{N_u-1}\mathbf{b}^i_n\cdot\mathbf{DF}_z(\hat{\mathbf{y}}_n,\mathbf{z}_n)\right]\cdot\mathbf{e}^{N_u}_n\ ,\end{aligned} \tag{9}$$

where λ^i_n ($i=1,\ldots,N_u$) are a set of numbers that can be related to the stretching rate of infinitesimal vectors along the unstable direction $\mathbf{e}^i_n$. It can then be shown that elements of the matrix $\mathbf{U}_n$ are given by

$$\begin{aligned}&U_{ij}=0,\quad j>i\\ &U_{ii}=\lambda^i_n-\mathbf{f}^i_{n+1}\cdot\mathbf{b}^i_n\cdot\mathbf{Dh}^i_n\\ &U_{ij}=-\mathbf{f}^i_{n+1}\cdot\mathbf{b}^i_n\cdot\mathbf{Dh}^j_n,\quad j<i\ ,\end{aligned} \tag{10}$$

where $\mathbf{Dh}^j_n\equiv\mathbf{DF}_z(\mathbf{y}_n,\mathbf{z}_n)\cdot\mathbf{e}^j_n$. To make the eigenvalues of the matrix $\mathbf{U}_n$ less than 1, we can adjust the $N_u\times N_d$ free control parameters C^i_j ($i=1,\ldots,N_u$, $j=1,\ldots,N_d$) such that all diagonal elements of $\mathbf{U}_n$ are less than 1. But this only provides N_u conditions, and there are still $N_u(N_d-1)$ free control parameters that we must set. The simplest choice is to set $C^i_j=0$ for $j>1$. Then setting the diagonal elements in Eq. (10) to zero gives $C^i_1=\lambda^i_n/[\mathbf{v}_1\cdot\mathbf{Dh}^i_n]$ ($i=1,\ldots,N_u$) and, consequently, the control matrix is given by

$$\mathbf{C}_n=\sum_{i=1}^{N_u}\frac{\lambda^i_n}{\mathbf{v}_1\cdot\mathbf{Dh}^i_n}\mathbf{e}^i_{n+1}\cdot\mathbf{v}_1\ . \tag{11}$$

In practice, the set of numbers λ^i_n and the set of unstable base vectors $\mathbf{e}^i_n$ can be computed by randomly initializing a set of base vectors $\mathbf{e}^i_n$ and evolving them in terms of Eq. (9). After a period of transience, the set of vectors so obtained converges to the real unstable directions. To assure that only small perturbations are applied, it is necessary to monitor the magnitude of the term in the denominator of Eq. (11). When $|\mathbf{v}_1\cdot\mathbf{Dh}^i_n|$ is below some small threshold, we set $\mathbf{C}_n=\mathbf{0}$. This will not result in a loss of control provided that it is done only occasionally [11]. We stress that the feedback control is derived under the applicability of linearized dynamics and, hence, the control is applied only when trajectories $\hat{\mathbf{y}}$ and $\mathbf{y}$ are sufficiently close. No control is applied when they are not close. This is the same idea used in controlling low-dimensional chaos [5]. Also note that the control law Eq. (11) has been derived under the condition of hyperbolicity, while there is no guarantee that spatiotemporal chaotic systems are hyperbolic. Nonetheless, as we will illustrate below, the control works for spatiotemporal systems modeled by coupled map lattices.

III. NUMERICAL EXAMPLES WITH THE DIFFUSIVELY COUPLED LOGISTIC MAP LATTICE

To illustrate the applicability of our control method, we consider the following system of diffusively coupled logistic maps that was first proposed by Kaneko [8] as a phenomenological model for spatiotemporal chaotic systems,

$$\begin{aligned}f[x_{n+1}(i)]&=(1-\epsilon)f[x_n(i)]\\ &\quad+\frac{\epsilon}{2}\{f[x_n(i+1)]+f[x_n(i-1)]\}\ ,\\ &\qquad i=1,\ldots,N\ ,\end{aligned} \tag{12}$$

where i and n denote discrete spatial sites and time, respectively, N is the total number of maps coupled in the lattice, ϵ denotes the coupling strength, and $f(x)$ is the one-dimensional logistic map $f(x)=ax(1-x)$. We assume periodic boundary condition: $x_n(N+1)=x_n(1)$.

Equation (12) exhibits extremely rich dynamical phenomena seen in real spatiotemporal systems and it is perhaps the most extensively studied model spatiotemporal system so far. In our examples we choose $a=4$, the parameter value for which the logistic map has a chaotic attractor.

Our first example is for $N=10$ and $\epsilon=0.7$. At this ϵ value, there are three positive Lyapunov exponents for Eq. (12). Figure 2(a) shows the corresponding Lyapunov spectrum for the full system Eq. (12), which plots λ_k versus the index k $(k=1,\ldots,N,\ \lambda_1\geq\lambda_2\geq\cdots\geq\lambda_N)$. To synchronize two such systems, we choose one of $x(i)$ $(i=1,\ldots,10)$ as the driving variable. Choosing a different $x(i)$ does not change the result due to symmetry of Eq. (12) with respect to site index i. The subsystems to be synchronized are therefore nine dimensional and still possess three positive Lyapunov exponents, as shown by the corresponding Lyapunov spectrum in Fig. 2(b). Thus the Pecora-Carroll-type synchronism will not occur for the subsystem. The control neighborhood is set to be $|\hat{\mathbf{y}}-\mathbf{y}|<r_0=0.015$. The control Eq. (11) is applied only when $|\mathbf{v}_1\cdot\mathbf{Dh}_n^j|\geq 10^{-3}$ $(j=1,2,3)$. With these control parameter settings, most randomly chosen initial conditions can be controlled. In general, the smaller the control neighborhood, the larger the probability that trajectories resulting from two randomly chosen initial conditions can be synchronized. In the case where one set of initial conditions fails to be synchronized, we disregard it and choose another set of initial conditions. Figures 3(a) and 3(b) show, when trajectories of the two subsystems resulting from a pair of randomly chosen initial conditions are within r_0, the error Δ_n, defined as

$$\Delta_n=|\hat{\mathbf{y}}_n-\mathbf{y}_n| , \tag{13}$$

and the control magnitude, defined as

$$|\mathbf{C}_n|=|\mathbf{C}_n\cdot[\mathbf{F}_z(\hat{\mathbf{y}}_n,\mathbf{z}_n)-\mathbf{F}_z(\mathbf{y}_n,\mathbf{z}_n)]| , \tag{14}$$

versus the time step n after the control is turned on. Clearly, two trajectories rapidly approach each other to within computer roundoff error ($\sim 10^{-14}$) after the control is applied, and the required feedback control decreases correspondingly to extremely small values.

Under the influence of small random noise, the degree to which two subsystems can be synchronized, or the value of $|\delta\mathbf{y}_n|$, is proportional to the amplitude of the noise. Figures 4(a) and 4(b) show Δ_n and $|C_n|$ versus time step n for the parameter setting of Fig. 3 when a noise term modeled by $h\sigma_n^i$ is added to each site of the lattice, where $h=10^{-7}$ is the noise amplitude and σ_n^i is a Gaussian random variable with zero mean and unit variance. In general, minimum values of Δ_n and $|C_n|$ have the same order of magnitude as h. Occasionally, both Δ_n

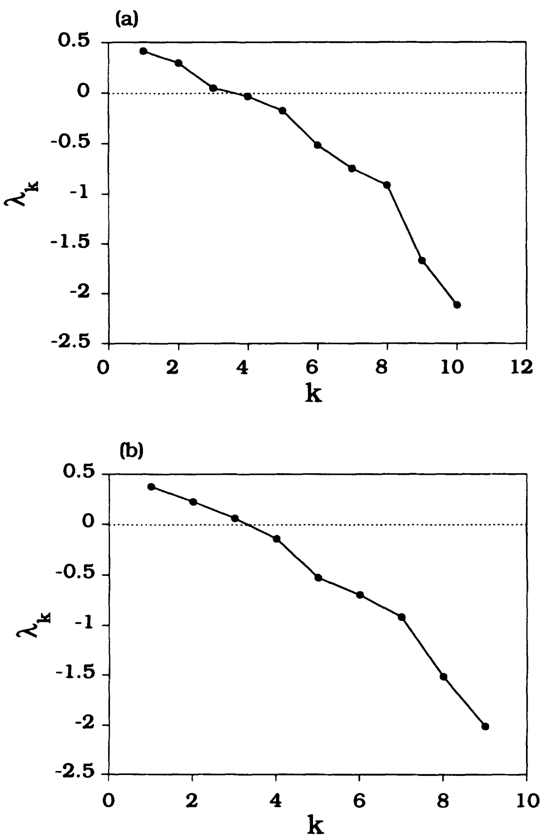

FIG. 2. The Lyapunov spectrum for $N=10$ and $\epsilon=0.7$ (a) of the full system and (b) of the reduced system by using one site as the driving signal.

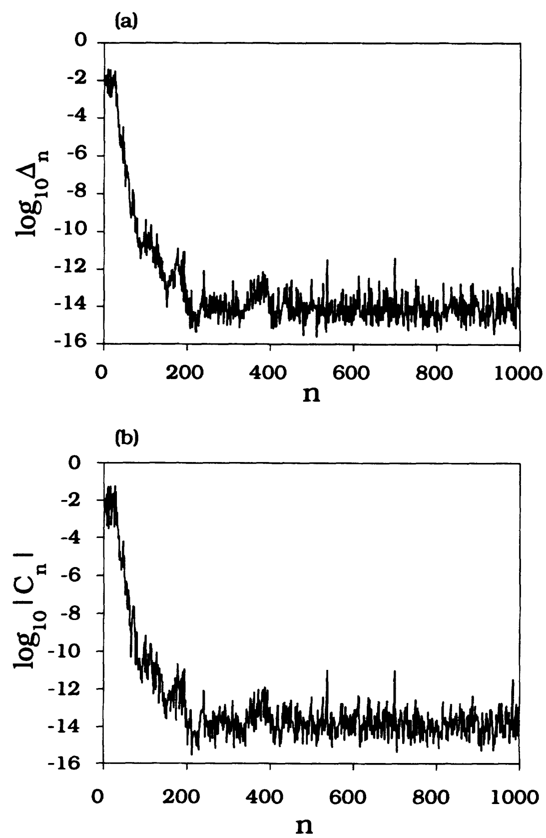

FIG. 3. Synchronization of two logistic map lattices ($N=10$, $\epsilon=0.7$). (a) The synchronization error $\log_{10}\Delta_n$ versus n and (b) the required feedback control magnitude $\log_{10}(|C_n|)$ versus n. The control neighborhood is set to be 0.015.

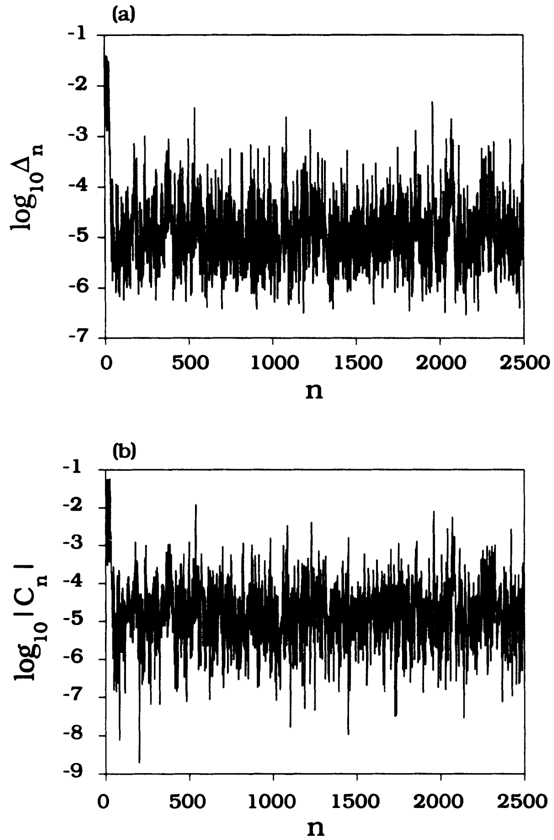

FIG. 4. (a) The synchronization error $\log_{10}\Delta_n$ and (b) the required feedback control magnitude $\log_{10}(|C_n|)$ versus n when a noise term $10^{-7}\sigma_n^i$ is added to each site of the lattice, where σ_n^i is a Gaussian random variable with zero mean and unit variance.

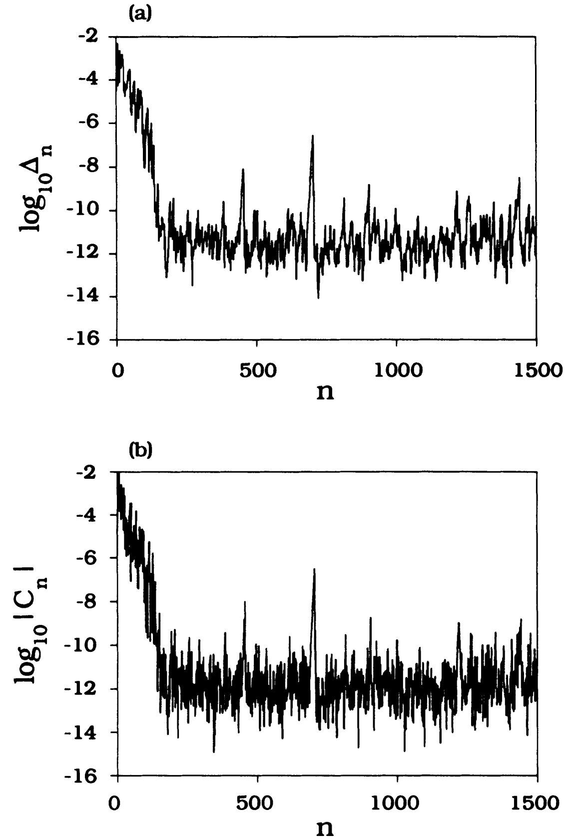

FIG. 6. (a) The synchronization error $\log_{10}\Delta_n$ and (b) the required feedback control magnitude $\log_{10}(|C_n|)$ versus n for $N=20$ and $\epsilon=0.5$. The control neighborhood needs to be reduced to 5×10^{-5} in order to achieve the control.

and $|C|_n$ can have values larger than 10^{-3}, indicating that the degree of synchronization decreases significantly at these time steps. In the worst situation, two subsystems can even be completely desynchronized. When this occurs, we turn off the control and let the systems evolve by themselves. Due to the ergodicity of the chaotic attractor, at some later time the two trajectories will come

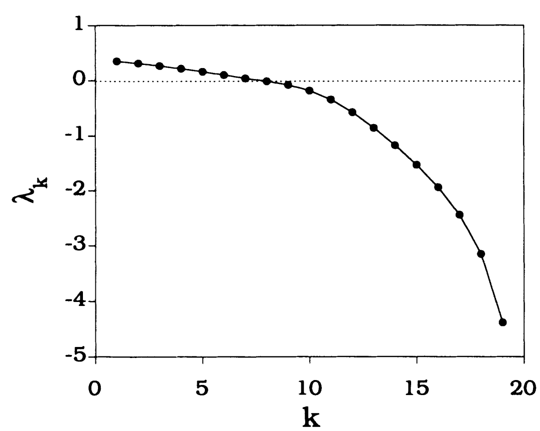

FIG. 5. The reduced Lyapunov spectrum for $N=20$ and $\epsilon=0.5$, where one site of the lattice is used as the driving signal.

close to each other and can be controlled again. This behavior of the control algorithm under the influence of noise is completely analogous to that of controlling low-dimensional chaos [5] and the synchronization of low-dimensional systems by control [3].

To demonstrate the applicability of our control algorithm in higher dimensions, we have performed control using $N=20$. In this case, we found that for $\epsilon=0.5$ there is a unique chaotic attractor with eight positive Lyapunov exponents. The subsystem obtained by using a driving signal $x_n(i)$, where i can be any number between 1 and 20, has seven positive Lyapunov exponents as shown in Fig. 5. In this case, the control neighborhood needs to be smaller for synchronization to occur. Besides, the quantity N_u used in the control algorithm needs to be slightly larger than the actual number of unstable directions [11]. We found that using $N_u=10$ suffices. Figures 6(a) and 6(b) show Δ_n and $|C_n|$ versus n, where the control is applied only when $|\delta\mathbf{y}_n|\le5\times10^{-5}$.

IV. DISCUSSIONS

In this paper, we have applied the principle of the extended Kalman filter to synchronize spatiotemporal chaotic systems by linking as few as one dynamical variable as the driving signal and using feedback control. The control algorithm works for the diffusively coupled logistic map lattice. The major advantage of this method is that only small amplitude control is required. While there is still one potential problem with our method, the work presented here represents an alternative approach in the general area of controlling and synchronizing spatiotemporal chaotic systems [7,14].

The difficulty concerns the transient time before control can be achieved. As we have demonstrated with Eq. (12), the control neighborhood needs to be reduced as the number of unstable directions increases. Going from three (Fig. 4, the $N=10$ case) to seven unstable directions (Fig. 6, the $N=20$ case) requires almost three orders of magnitude decrease in the size of the control neighborhood. As the size of the control neighborhood is decreased, the average transient time for two trajectories to get close increases algebraically with a scaling exponent determined by the Lyapunov spectrum of the chaotic attractor [5,3]. Thus, even for spatiotemporal systems with moderate sizes, the transient time required may be very long. The reason why an extremely small control neighborhood is needed is not clear but may be related to the noninvertibility and nonhyperbolicity of the coupled logistic map lattice. For instance, for noninvertible dynamical systems there may not be unique stable and unstable spaces at trajectory points [15], whereas our control algorithm is designed under the assumption that the dynamical systems possess unique and distinct stable and unstable spaces (invertibility and hyperbolicity).

ACKNOWLEDGMENT

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