Application of the slaving principle to nonlinear non-Markovian systems

Jing-hui Li
Department of Physics, Hebei Teachers College, Shijiazhuang 050091, Hebei, People's Republic of China

Li Cao and Da-jin Wu
Chinese Center of Advanced Science and Technology (World Laboratory), P.O. Box 8730, Beijing 100080, People's Republic of China and Department of Physics, Huazhong University of Science and Technology, Wuhan 430074, People's Republic of China

Shu-shan Xu
Department of Physics, Hebei Teachers College, Shijiazhuang, Hebei 050091, People's Republic of China
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We study a set of nonlinear stochastic equations that describe a large class of nonlinear multidimensional non-Markovian dynamical systems driven by Ornstein-Uhlenbeck (OU) noises, and a class of nonlinear multidimensional differential stochastic equations driven by dichotomous noises. By use of the stochastic generalization of the usual adiabatic approximation, the equations for the order parameters are obtained. The statistical properties of the new stochastic variables are studied. In the circumstances of OU noises or dichotomous noises, we derive the approximate Fokker-Planck equations (AFPE's) corresponding to the equations for the order parameters and calculate the stationary solutions of AFPE's.

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I. INTRODUCTION

The application of the slaving principle in stochastic systems is a method developed in recent years [1–10]. By using this method, we can deal with the stochastic systems not handled before, such as the semiclassical single-mode laser model [11], the system which contains quenching of fluctuations [12], and so on. This method was first advanced by Haken in the 1970s [1]. However, owing to some difficulties in practical calculation, it could not be used in stochastic systems. In 1981, Kaneko proposed a method of adiabatic elimination [2] by the eigenfunction expansion for two-dimensional systems driven by additive Gaussian white noises. Soon afterwards, Gardiner also developed a method [3]. In the process of using Gardiner's method, the Fokker-Planck equation (FPE) of the stochastic differential equations must be derived first. Then, by eliminating the slaved parameter, we can get approximate Fokker-Planck equations (AFPE's) of the order parameters. The defect of Gardiner's method is that its approximate degree is low. In 1986 and 1987, Schöner and Haken proposed a nice systematic approximation and systematic adiabatic approximation (Haken-Wunderlin-Schöner methods, or HWS methods) on the basis of Haken and Wunderlin's early work [4–6]. In their work, the slaving principle was generalized to the nonlinear systems driven by Gaussian white noises. But a set of new stochastic processes $Z_t^{(\nu)}$ $(\nu \geq 2)$ were introduced in the equations for the order parameters. It made the calculation and determination of the statistical properties of $Z_t^{(\nu)}$ $(\nu \geq 2)$ very complicated. Recently, Cao and Wu have developed a stochastic adiabatic approximation (CW method) in the stochastic systems [7–11]. By use of this method, some stochastic processes $Z_{t,1}$ and $Z_{t,2}^i$ $(i=1,2,3,\ldots,m)$ were introduced. It was easier to determine the statistical properties of $Z_{t,1}$ and $Z_{t,2}^i$ than to do those of $Z_t^{(\nu)}$.

Only the systems driven by Gaussian white noises can be treated with the methods mentioned above. The systems driven by Ornstein-Uhlenbeck (OU) noises, dichotomous noises, cannot be handled by them. The stochastic forces acting on the systems may be approximated to the white noises under some circumstances. But under the other circumstances, they may not be. They may probably be approximated to OU noises, dichotomous noises, etc. For example, owing to the impact of the quantum noise and the pump noise, a laser system must be a nonlinear non-Markovian one, the pump fluctuations in the system may be approximated to OU noises (colored noises) [12,13]. In a dye laser system, the quenching of fluctuations is the role of noise color [14]. How to use the slaving principle to solve the quenching of fluctuations in a laser system is a question which we must settle [15]. The task of this paper is to generalize the stochastic adiabatic approximation to stochastic systems driven by colored noises, such as OU noises or dichotomous noises. The paper is arranged as follows. In Sec. II we popularize the stochastic adiabatic approximation to the systems driven by OU noises. The equation for the order parameter and the statistical properties of the new stochastic variables are derived for a large class of nonlinear differential stochastic equations driven by OU noises. The AFPE of the systems and the AFPE stationary solution are calculated. Then the single-mode laser model driven by OU noises is studied by this method. In Sec. III the stochastic adiabatic approximation is generalized to the systems driven by dichotomous noises. We derive the equation for the order parameter, the statistical properties of the new stochastic variables for a set of nonlinear differential equations driven by dichotomous

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noises, the AFPE of the stochastic systems, and the AFPE stationary solution. Then a Brownian harmonic oscillator driven simultaneously by Gaussian white noise and dichotomous noise is solved by this method. Finally, the conclusions and discussions are presented in Sec. IV.

II. GENERALIZATION OF THE STOCHASTIC ADIABATIC APPROXIMATION IN A LARGE CLASS OF NONLINEAR NON-MARKOVIAN SYSTEMS DRIVEN BY OU NOISES

The nonlinear stochastic differential equations (NSDE's) of multidimensional non-Markovian processes driven by OU noises are

$$du_t=\lambda_u u_t dt+Q_0(u_t)S_t dt+Q_1(u_t)dt+[F_u(u_t)+F]\xi(t)dt\ , \tag{1}$$

$$dS_t=-\lambda_S S_t dt+P_0(u_t)S_t dt+P_1(u_t)dt+\sum_{i=1}^{m}[F_{s,i}(u_t)S_t+F_i(u_t)]\xi_i(t)dt\ , \tag{2}$$

where $\lambda_u\geq 0$, $\lambda_S>0$, and $\epsilon=\lambda_u/\lambda_S\ll 1$. The functions Q_0, Q_1, P_0, and P_1 contain nonlinear deterministic terms, while $[F_{S,i}(u_t)S_t+F_i(u_t)]$ and $[F_u(u_t)+F]$ are the coefficients of the independent stochastic forces $\xi_i(t)$ and $\xi(t)$ $(i=1,2,3,\ldots,m)$. $\xi_i(t)$ and $\xi(t)$ are OU noises. Their statistical properties are

$$\langle\xi(t)\rangle=0,\quad \langle\xi(t)\xi(t')\rangle=\frac{D}{\tau}e^{-|t-t'|/\tau}\ ,$$

$$\langle\xi_i(t)\rangle=0,\quad \langle\xi_i(t)\xi_i(t')\rangle=\frac{D_i}{\tau_i}\delta_{ij}e^{-|t-t'|/\tau_i}\ .$$

For the sake of convenience, Eqs. (1) and (2) are simply written as

$$du_t=\alpha(u_t)dt+Q_0(u_t)S_t dt+[F_u(u_t)+F]\xi(t)dt\ , \tag{3}$$

$$dS_t=-\beta(u_t)S_t dt+P_1(u_t)dt+\sum_{i=1}^{m}[F_{S,i}(u_t)S_t+F_i(u_t)]\xi_i(t)dt\ , \tag{4}$$

where $\alpha(u_t)=\lambda_u u_t+Q_1(u_t)$, $\beta(u_t)=\lambda_S-P_0(u_t)$.

A. The equation for the order parameter

The central idea of the stochastic adiabatic approximation is that for small ϵ, the slow variable u_t in Eq. (4) can be treated as a time- and chance-independent parameter. Hence Eq. (4) is just the linear stochastic differential equation for S. Its solution is

$$S_{\text{ad}}=S(t_0,u)\exp\left\{-\int_0^t\left[\beta(u)-\sum_{i=1}^{m}F_{S,i}(u)\xi_i(t')\right]dt'\right\}+\int_{t_0}^t\left[P_1(u)+\sum_{i=1}^{m}F_i(u)\xi_i(t')\right]\exp\left\{-\int_{t'}^t\left[\beta(u)-\sum_{i=1}^{m}F_{S,i}(u)\xi_i(s)\right]ds\right\}dt'\ , \tag{5}$$

where the subscript "ad" means that we have to take the stochastic adiabatic approximation, and regard u_t in Eq. (4) as an order parameter. Letting $t_0\to\infty$, $M_i(t)=\int_0^t\xi(t')dt'$ [make $dM_i(t)=\xi_i(t)dt$], Eq. (5) can be written as

$$S_{\text{ad}}=P_1(u)K_{t,1}+\sum_{i=1}^{m}F_i(u)K_{t,2}^i\ , \tag{6}$$

where

$$K_{t,1}=\int_{-\infty}^t\exp\left\{-\beta(u)(t-t')+\sum_{i=1}^{m}F_{S,i}(u)[M_i(t)-M_i(t')]\right\}dt'\ , \tag{7}$$

$$K_{t,2}^i=\int_{-\infty}^t\exp\left\{-\beta(u)(t-t')+\sum_{i=1}^{m}F_{S,i}(u)[M_i(t)-M_i(t')]\right\}dM_i(t')\quad (i=1,2,3,\ldots,m)\ . \tag{8}$$

Now, we derive the statistical properties of $K_{t,1}$ and $K_{t,2}^i$. It is easy to find that $K_{t,1}$ is the solution of SDE

$$dK_{t,1}=-\beta(u)K_{t,1}dt+\sum_{i=1}^{m}F_{S,i}(u)K_{t,1}\xi_i(t)dt+dt\ . \tag{9}$$

From Eq. (9), we have

$$\frac{d}{dt}\langle K_{t,1}\rangle=-\beta(u)\langle K_{t,1}\rangle+\sum_{i=1}^{m}F_{S,i}(u)\langle K_{t,1}\xi_i(t)\rangle+1\ . \tag{10}$$

Making the approximation of the small τ_i, and using Eqs. (2.18)–(2.23) of Ref. [16] and the Novikovian theorem [17], we can get

$$\langle K_{t,1}\xi_i(t)\rangle\simeq D_iF_{S,i}(u)\langle K_{t,1}\rangle-\tau_iD_iF_{S,i}(u)\ . \tag{11}$$

In Eq. (11), we only take account of the first order in τ_i. Substituting Eq. (11) into Eq. (10), we obtain

$$\frac{d}{dt}\langle K_{t,1}\rangle=-\beta(u)\langle K_{t,1}\rangle+\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2\langle K_{t,1}\rangle+1-\sum_{i=1}^{m}\tau_iD_i[F_{S,i}(u)]^2\ . \tag{12}$$

The solution of Eq. (12) is

$$\langle K_{t,1}\rangle=\int_{-\infty}^{t}\left[1-\sum_{i=1}^{m}\tau_i D_i[F_{S,i}(u)]^2\right]\exp\left\{-\left[\beta(u)-\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2\right](t-t')\right\}dt'$$

$$=\frac{1-\sum_{i=1}^{m}\tau_i D_i[F_{S,i}(u)]^2}{\beta(u)-\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2}\ . \tag{13}$$

To derive the correlation function, we write from (9)

$$\frac{d}{dt}\langle K_{t,1}K_{t',1}\rangle=-\beta(u)\langle K_{t,1}K_{t',1}\rangle+\sum_{i=1}^{m}F_{S,i}(u)\langle K_{t,1}K_{t',1}\xi_i(t)\rangle+\langle K_{t',1}\rangle\ . \tag{14}$$

By using Eqs. (2.26)–(2.30) of Ref. [16], we can find (we only take account of the first order in τ_i)

$$\langle K_{t,1}K_{t',1}\xi_i(t)\rangle=D_iF_{S,i}(u)\langle K_{t,1}K_{t',1}\rangle-\tau_iD_iF_{S,i}(u)\langle K_{t',1}\rangle$$
$$+D_iF_{S,i}(u)\langle K_{t,1}K_{t',1}\rangle\exp[-(t-t')/\tau_i]-\tau_iD_iF_{S,i}(u)\langle K_{t',1}\rangle\exp[-(t-t')/\tau_i]\ . \tag{15}$$

Substituting Eq. (15) into Eq. (14), we obtain

$$\frac{d}{dt}\langle K_{t,1}K_{t',1}\rangle=-\beta(u)\langle K_{t,1}K_{t',1}\rangle+\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2\{1+\exp[-(t-t')/\tau_i]\}\langle K_{t,1}K_{t',1}\rangle$$
$$-\sum_{i=1}^{m}D_i\tau_i[F_{S,i}(u)]^2\{1+\exp[-(t-t')/\tau_i]\}\langle K_{t,1}\rangle+\langle K_{t,1}\rangle\ . \tag{16}$$

Neglecting the terms proportional to $e^{-(t-t')/\tau_i}$, Eq. (16) becomes

$$\frac{d}{dt}\langle K_{t,1}K_{t',1}\rangle=-\beta(u)\langle K_{t,1}K_{t',1}\rangle+\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2\langle K_{t,1}K_{t',1}\rangle+\left[1-\sum_{i=1}^{m}\tau_iD_i[F_{S,i}(u)]^2\right]\langle K_{t,1}\rangle\ . \tag{17}$$

The solution of Eq. (17) is

$$\langle K_{t,1}K_{t',1}\rangle=\frac{1-\sum_{i=1}^{m}\tau_iD_i[F_{S,i}(u)]^2}{\beta(u)-\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2}\langle K_{t,1}\rangle+\left[\frac{1-2\sum_{i=1}^{m}\tau_iD_i[F_{S,i}(u)]^2}{\beta(u)-2\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2}\langle K_{t,1}\rangle-\frac{1-\sum_{i=1}^{m}\tau_iD_i[F_{s,i}(u)]^2}{\beta(u)-\sum_{i=1}^{m}D_i[F_{s,i}(u)]^2}\right]$$
$$\times\exp\left\{-\left[\beta(u)-\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2\right]|t-t'|\right\}\ . \tag{18}$$

Similarly, by means of the Novikovian theorem [17,18] and Ref. [16], the mean value and correlation function of $K_{t,2}^{i}$ are

$$\langle K_{t,2}^{i}\rangle=\frac{D_iF_{S,i}(u)[1-\tau_i\beta(u)]}{\beta(u)-\sum_{j=1}^{m}D_j[F_{S,j}(u)]^2}, \tag{19}$$

$$\langle K_{t,2}^{i}K_{t',2}^{i}\rangle=\frac{D_iF_{s,i}(u)[1-\tau_i\beta(u)]\langle K_{t,2}^{i}\rangle}{\beta(u)-\sum_{j=1}^{m}D_j[F_{s,j}(u)]^2}$$
$$+\left[\frac{D_i\{1-\tau_i\beta(u)+D_iF_{S,i}(u)[3-2\tau_i\beta(u)]\}\langle K_{t,2}^{i}\rangle}{\beta(u)-2\sum_{j=1}^{m}D_j[F_{S,j}(u)]^2}-\frac{D_iF_{S,i}(u)[1-\tau_i\beta(u)]\langle K_{t,2}^{i}\rangle}{\beta(u)-\sum_{j=1}^{m}D_j[F_{S,j}(u)]^2}\right]$$
$$\times\exp\left\{-\left[\beta(u)-\sum_{j=1}^{m}D_j[F_{S,j}(u)]^2\right]|t-t'|\right\}\ . \tag{20}$$

The formulas (13) and (18)–(20) will return to corresponding formulas of Ref. [7] in the limit of $\tau_i \to 0$.

Substituting Eq. (6) into Eq. (3), we can obtain the equation for the order parameter

$$\frac{d}{dt}u_t=\alpha(u)+Q_0(u)\left[P_1(u)K_{t,1}+\sum_{i=1}^{m}F_i(u)K_{t,2}^i\right]+[F_u(u)+F]\xi(t) . \qquad (21)$$

By introducing the stochastic processes $\eta_1(t)$ and $\eta_2^i(t)$ defined by

$$\eta_1(t)=K_{t,1}-\langle K_{t,1}\rangle,\quad \eta_2^i(t)=K_{t,2}^i-\langle K_{t,2}^i\rangle ,$$

we have from Eq. (21)

$$\frac{d}{dt}u_t=f(u)+\bar{P}_1(u)\eta_1(t)+\sum_{i=1}^{m}\bar{F}_i(u)\eta_2^i(t)+[F_u(u)+F]\xi(t) , \qquad (22)$$

where

$$f(u)=\alpha(u)+\bar{P}_1(u)\langle K_{t,1}\rangle+\sum_{i=1}^{m}\bar{F}_i(u)\langle K_{t,2}^i\rangle ,$$

$$\bar{P}_1(u)=Q_0(u)P_1(u),\quad \bar{F}_i(u)=Q_0(u)F_i(u) .$$

From $\eta_1(t)=K_{t,1}-\langle K_{t,1}\rangle$ and $\eta_2^i(t)=K_{t,2}^i-\langle K_{t,2}^i\rangle$, we can get

$$\langle\eta_1(t)\rangle=0 ,$$

$$\langle\eta_1(t)\eta_1(t')\rangle=N_1(u)\exp[-G(u)|t-t'|] ,$$

$$\langle\eta_2^i(t)\rangle=0 ,$$

$$\langle\eta_2^i(t)\eta_2^l(t')\rangle=N_2^i(u)\delta_{il}\exp[-G(u)|t-t'|] ,$$

$$\langle\eta_1(t)\eta_2^i(t')\rangle=0 ,$$

where $i=1,2,3,\ldots,m$, $G(u)=\beta(u)-\sum_{j=1}^{m}D_j[F_{S,j}(u)]^2$,

$$N_1(u)=\frac{1-2\sum_{i=1}^{m}\tau_iD_i[F_{S,i}(u)]^2}{\beta(u)-2\sum_{i=1}^{m}D_i[F_{S,i}(u)]^2}\langle K_{t,1}\rangle-\frac{1-\sum_{j=1}^{m}D_j\tau_j[F_{S,j}(u)]^2}{G(u)} ,$$

and

$$N_2^i(u)=\frac{D_i[1-\tau_i\beta(u)]+D_iF_{S,i}(u)[3-2\tau_i\beta(u)]\langle K_{t,2}^i\rangle}{\beta(u)-2\sum_{j=1}^{m}D_j[F_{S,j}(u)]^2}-\frac{D_iF_{S,i}(u)[1-\tau_i\beta(u)]\langle K_{t,2}^i\rangle}{G(u)} .$$

Equation (22) is the equation of order parameter for Eqs. (1) and (2).

The formulas (13) and (18)–(20) are suitable for the multiplicative noise case. When $F_{S,i}(u)=0$, i.e., the additive OU noise case, we need not make small τ_i. Right now, Eq. (4) reduces to

$$\frac{dS_t}{dt}=-\beta(u)S_t+P_1(u)+\sum_{i=1}^{m}F_i(u)\xi_i(t) . \qquad (23)$$

The stochastic adiabatic solution of S can be obtained from Eq. (23) (u is regarded as a time-independent parameter):

$$S_{\rm ad}=\frac{1}{\beta(u)}P_1(u)+\sum_{i=1}^{m}F_i(u)K_{t,2}^i ,$$

with

$$K_{t,2}^i=\int_{-\infty}^{t}\exp[-\beta(u)(t-\tau)]\xi_i(\tau)d\tau . \qquad (24)$$

The statistical properties of $K_{t,2}^i$ are

$$\langle K_{t,2}^i\rangle=0 ,$$

$$\langle K_{t,2}^iK_{t,2}^j\rangle=\delta_{ij}\left\{-F_1^i\exp\left[-\frac{|t-t'|}{\tau_i}\right]+F_2^i\exp[-\beta(u)|t-t'|]\right\} , \qquad (24')$$

where

$$F_1^i=\frac{D_i\tau_i}{1-\tau_i^2\beta^2(u)} ,\quad F_2^i=\frac{D_i}{[1-\tau_i^2\beta^2(u)]\beta(u)} .$$

[We give the derivation of Eq. (24′) in Appendix A.] The formulas (24′) can return to the corresponding ones of Ref. [7] in the limit of $\tau_i \to 0$. In this moment, the equation of order parameter is

$$du=g(u)dt+\sum_{i=1}^{m}\bar{F}_i(u)K_{t,2}^idt+[F_u(u)+F]\xi(t)dt , \qquad (24'')$$

where

$$g(u)=\alpha(u)+Q_0(u)\frac{P_1(u)}{\beta(u)},\quad \bar{F}_i(u)=Q_0(u)F_i(u) .$$

B. The approximate Fokker-Planck equation and its stationary solution

In Ref. [19] the authors introduce the interaction representation, and then make

$$V(t)=-\exp[\partial_q A(q)t]\times\partial_q[F(\bar{\alpha}+\xi(t))g(q)-\langle F(\bar{\alpha}+\xi(t))g(q)\rangle]\times\exp[-\partial_q A(q)t]$$

[Eq. (6) of Ref. [19]]. We think that this formula is wrong. It should be

$$V(t)=-\exp[-\partial_q A(q)t] \times\partial_q[F(\bar{\alpha}+\xi(t))g(q)-\langle F(\bar{\alpha}+\xi(t))g(q)\rangle] \times\exp[\partial_q A(q)t] .$$

Imitating the method proposed by the authors [19] we get the equation for the approximate probability density of Eq. (22) (N_1, N_2^i, G^{-1}, D, and τ are small values) as follows:

$$\partial_t P(u,t)=-\partial_u f(u)P(u,t)+LP(u,t) , \tag{25}$$

where

$$\begin{aligned} L=&\partial_u\bar{P}_1(u)\int_0^\infty dt'\langle\eta_1(t)\eta_1(t-t')\rangle\{\exp[t'\partial_u f(u)]\}\partial_u\bar{P}_1(u)\\ &+\sum_{i=1}^m\partial_u\bar{F}_i(u)\int_0^\infty dt'\langle\eta_2^i(t)\eta_2^i(t-t')\rangle\exp[t'\partial_u f(u)]\partial_u\bar{F}_i(u)\\ &+\partial_u F_u(u)\int_0^\infty dt'\langle\xi(t)\xi(t-t')\rangle\{\exp[t'\partial_u f(u)]\}\partial_u F_u(u)+\partial_u F\int_0^\infty dt'\langle\xi(t)\xi(t-t')\rangle\{\exp[t'\partial_u f(u)]\}\partial_u F\\ =&\partial_u\bar{P}_1(u)\frac{N_1(u)}{G(u)}\frac{1}{1-[G(u)]^{-1}\partial_u f(u)}\partial_u\bar{P}_1(u)+\sum_{i=1}^m\partial_u\bar{F}_i(u)\frac{N_2^i(u)}{G(u)}\frac{1}{1-[G(u)]^{-1}\partial_u f(u)}\partial_u\bar{F}_i(u)\\ &+\partial_u F_u(u)D\frac{1}{1-\tau\partial_u f(u)}\partial_u F_u(u)+F^2\partial_u^2 D\frac{1}{1-\tau\partial_u f(u)} . \end{aligned}\tag{26}$$

If the intensities of the noises $\eta_1(t)$, $\eta_2^i(t)$, and $\xi(t)$ are small enough, in the case of small G^{-1} and τ, we can use u_S, which satisfies $f(u_S)=0$ to replace variable u in $\{1-[G(u)]^{-1}\partial_u f(u)\}^{-1}$ and $[1-\tau\partial_u f(u)]^{-1}$ of Eq. (26) (and in the following we shall calculate the stationary solution of the approximate probability density, which is also a reason for using u_S to replace u). We refer to this ansatz as the "Hanggi-like ansatz." The "Hanggi-like ansatz" is not the "Hanggi ansatz." We imply that this "ansatz" and the "Hanggi ansatz" are somewhat alike. See P. Hanggi *et al.*, Physica **22A**, 695 (1985). In this moment, we can get the approximate Fokker-Planck equation

$$\begin{aligned} \partial_t P(u,t)=&-\partial_u f(u)P(u,t)+\partial_u\bar{P}_1(u)\bar{N}_1(u)\left[1+\frac{\partial_u f(u)}{G(u)}\right]_{u=u_S}\partial_u\bar{P}_1(u)P(u,t)\\ &+\sum_{i=1}^m\partial_u\bar{F}_i(u)\bar{N}_2^i(u)\left[1+\frac{\partial_u f(u)}{G(u)}\right]_{u=u_S}\partial_u\bar{F}_i(u)P(u,t)\\ &+D\partial_u F_u(u)[1+\tau\partial_u f(u)]_{u=u_S}\partial_u F_u(u)P(u,t)+DF^2[1+\tau\partial_u f(u)]_{u=u_S}\partial_u^2 P(u,t) , \end{aligned}\tag{27}$$

where $\bar{N}_1(u)=N_1(u)/G(u)$ and $\bar{N}_2^i(u)=N_2^i(u)/G(u)$. In Eq. (27), we have fetched $1/\{1-[G(u)]^{-1}\partial_u f(u)\}\approx\{1+[G(u)]^{-1}\partial_u f(u)\}$ and $1/[1-\tau\partial_u f(u)]\approx[1+\tau\partial_u f(u)]$ so that we can easily compare Eq. (27) with the result obtained by the method of Refs. [2,22].

The stationary solution of Eq. (27) under the natural boundary condition is [20,21]

$$P_S(u)=N\left[\int du\frac{\Delta}{AA'+BD\{[F_u(u)]^2+F^2\}}\right] , \tag{28}$$

where

$$P_S(u)=\lim_{t\to\infty}P(u,t) ,$$

$$\Delta=f(u)-\bar{P}_1(u)\bar{N}_1(u)A\partial_u\bar{P}_1(u)-\sum_{i=1}^m\bar{F}_i(u)\bar{N}_2^i(u)A\partial_u\bar{F}_i(u)-DF_u(u)B\partial_u F_u(u) ,$$

$$A=\left[1+\frac{\partial_u f(u)}{G(u)}\right]_{u=u_S},\quad B=[1+\tau\partial_u f(u)]_{u=u_S} ,$$

$$A'=[\bar{P}_1(u)]^2\bar{N}_1(u)+\sum_{i=1}^m[\bar{F}_1(u)]^2\bar{N}_2^i(u) .$$

When the stochastic system is driven by one OU noise, under the circumstances of "Hanggi-like ansatz," after using u_S satisfying $f(u_S)=0$ to replace in $\{1-[G(u)]^{-1}\partial_u f(u)\}^{-1}$ and $[1-\tau\partial_u f(u)]^{-1}$ of Eq. (26), the result obtained by this method accords with that obtained by the method of Refs. [2,22].

Similarly, we can, respectively, calculate the AFPE of Eq. (24″) and the AFPE stationary solution (in the circumstances of small τ, τ_i, $1/\beta(u)$, D, F_1^i, F_2^i):

$$\partial_t P(u,t)=-\partial_u g(u)P(u,t)+\sum_{i=1}^{m}\partial_u \bar{F}_i(u)(-F_1^i)\tau_i[1+\tau_i\partial_u g(u)]_{u=u_S}\partial_u \bar{F}_i(u)P(u,t)$$
$$+\sum_{i=1}^{m}\partial_u \bar{F}_i(u)F_2^i\frac{1}{\beta(u)}[1+\partial_u g(u)/\beta(u)]_{u=u_S}\partial_u \bar{F}_i(u)P(u,t)$$
$$+D\partial_u F_u(u)[1+\tau\partial_u g(u)]_{u=u_S}\partial_u F_u(u)P(u,t)+DF^2[1+\tau\partial_u g(u)]_{u=u_S}\partial_u^2 P(u,t)\ ,$$

and

$$P_S(u)=N\exp\left[\int du\frac{\Delta'}{C+EE'+B'D\{[F_u(u)]^2+F^2\}}\right]\ ,$$

where

$$C=-\sum_{i=1}^{m}[\bar{F}_i(u)]^2F_1^i\tau_i[1+\tau_i\partial_u g(u)]_{u=u_S}\ ,$$

$$E=\sum_{i=1}^{m}[\bar{F}_i(u)]^2F_2^i\frac{1}{\beta(u)},\quad E'=[1+\partial_u g(u)/\beta(u)]_{u=u_S}\ ,$$

$$B'=[1+\tau\partial_u g(u)]_{u=u_S}\ ,$$

$$\Delta'=g(u)-\sum_{i=1}^{m}\bar{F}_i(u)\{-F_1^i\tau_i[1+\tau_i\partial_u g(u)]_{u=u_S}+\frac{F_2^i}{\beta(u)}E'\}\partial_u\bar{F}_i(u)$$
$$-DF_u(u)B\partial_u F_u(u)\ .$$

C. The single-mode laser model with multiplicative OU noise

The stochastic differential equations for the single-mode laser model are

$$du_t=(\alpha u_t+au_tS_t)dt+F_u\xi_1(t)dt\ , \qquad (29)$$

$$dS_t=-\beta S_t dt-(bu_t^2S_t+cu_t^2)dt+[F_{S,1}S_t+F_S]\xi_2(t)dt\ , \qquad (30)$$

where α, β, a, b, c, F_u, $F_{S,1}$, and F_S are constants. $\xi_1(t)$ and $\xi_2(t)$ are two independent OU noises satisfying

$$\langle\xi_1(t)\rangle=0,\quad \langle\xi_1(t)\xi_1(t')\rangle=\frac{D_1}{\tau_1}\exp[-|t-t'|/\tau_1]\ .$$

$$\langle\xi_2(t)\rangle=0\ ,\langle\xi_2(t)\xi_2(t')\rangle=\frac{D_2}{\tau_2}\exp[-|t-t'|/\tau_2]\ .$$

In Eqs. (29) and (30), u_t corresponds to field amplitude and S_t to inversion. The purpose of the example is to illustrate the application of the stochastic adiabatic approximation in laser systems. In fact, the laser fields have the complex parameters. But we only emphasize the basic structure of Eqs. (29) and (30), and keep away the complicated details of physics. Thus u_t will be considered as the parameter of real number. Comparing Eqs. (29) and (30) with Eqs. (3) and (4), we have $\alpha(u)=\alpha u$, $Q_0(u)=au$, $F_u(u)=0$, $F=F_u$, $\beta(u)=\beta+bu^2$, $P_1(u)=-cu^2$, $F_{S,1}(u)=F_{S,1}$, $F_{S,i}(u)=0$ $(i\geq 2)$, $F_1(u)=F_S$, $F_i(u)=0$ $(i\geq 2)$. Substituting these equations into Eq. (22), we can obtain

$$\frac{d}{dt}u_t=f(u)-acu^3\eta_1(t)+F_Sau\eta_2(t)+F_u\xi_1(t)\ , \qquad (31)$$

where

$$f(u)=\alpha u-acu^3\frac{1-\tau_2D_2F_{S,1}^2}{\beta+bu^2-D_2F_{S,1}^2}+aF_Su\frac{D_2F_{S,1}[1-\tau_2(\beta+bu^2)]}{\beta+bu^2-D_2F_{S,1}^2}\ .$$

It is easy to find

$$\langle\eta_1(t)\rangle=0\ ,$$

$$\langle\eta_1(t)\eta_1(t')\rangle=N_1(u)\exp\{-[(\beta+bu^2)-D_2F_{S,1}^2]|t-t'|\},$$

$$\langle\eta_2(t)\rangle=0\ ,$$

$$\langle\eta_2(t)\eta_2(t')\rangle=N_2(u)\exp\{-[(\beta+bu^2)-D_2F_{S,1}^2]|t-t'|\},$$

in which

$$N_1(u)=\frac{(1-2\tau_2D_2F_{S,1}^2)(1-\tau_2D_2F_{S,1}^2)}{(\beta+bu^2-2D_2F_{S,1}^2)(\beta+bu^2-D_2F_{S,1}^2)}-\frac{1-\tau_2D_2F_{S,1}^2}{\beta+bu^2-D_2F_{S,1}^2}\ ,$$

$$N_2(u)=\frac{D_2[1-\tau_2(\beta+bu^2)]+D_2F_{S,1}[3-2\tau_2(\beta+bu^2)]\dfrac{D_2F_{S,1}[1-\tau_2(\beta+bu^2)]}{\beta+bu^2-D_2F_{S,1}^2}}{\beta+bu^2-2D_2F_{S,1}^2}-\left[\frac{D_2F_{S,1}[1-\tau_2(\beta+bu^2)]}{\beta+bu^2-D_2F_{S,1}^2}\right]^2\ .$$

Equation (31) is the equation of order parameter for Eqs. (29) and (30). The AFPE of Eq. (31) can be obtained from (27),

$$\partial_t P(u,t) = -\partial_u f(u)P(u,t) + \partial_u (acu^3)\bar{N}_1(u)A\partial_u(acu^3)P(u,t) + \partial_u(aF_S u)\bar{N}_2(u)A\partial_u(aF_S u)P(u,t) + D\partial_u F_u B\partial_u F_u P(u,t) , \quad (32)$$

where

$$\bar{N}_1(u) = \frac{N_1(u)}{\beta + bu^2 - D_2F_{S,1}^2}, \quad \bar{N}_2(u) = \frac{N_2(u)}{\beta + bu^2 - D_2F_{S,1}^2} ,$$

$$A = \left[1 + \frac{\partial_u f(u)}{\beta + bu^2 - D_2F_{S,1}^2}\right]_{u=u_S}, \quad B = [1+\tau_1\partial_u f(u)]_{u=u_S} .$$

The stationary solution of Eq. (32) is

$$P_S(u) = N\exp\left\{\int du \frac{\Delta}{A[(acu^3)^2\bar{N}_1(u) + (aF_S u)^2\bar{N}_2(u)] + BDF_u^2}\right\} ,$$

with

$$\Delta = f(u) - 3(ac)^2u^5\bar{N}_1(u)A - (aF_S)^2u\bar{N}_2(u)A .$$

III. GENERALIZATION OF THE STOCHASTIC ADIABATIC APPROXIMATION IN A CLASS OF NONLINEAR NON-MARKOVIAN SYSTEMS DRIVEN BY DICHOTOMOUS NOISES

NSDE's of non-Markovian processes discussed by us are

$$du_t = \alpha(u_t)dt + Q_0(u_t)S_t dt + [F_u(u_t) + F]\xi(t)dt , \quad (33)$$

$$dS_t = -\beta(u_t)S_t dt + P_1(u_t)dt + [F_{S,1}(u_t)S_t + F_1(u_t)]\xi'(t)dt , \quad (34)$$

where $\alpha(u_t) = \lambda_u u_t + Q_1(u_t)$, $\beta(u_t) = \lambda_S - P_0(u_t)$, $\lambda_u \geq 0$, $\lambda_S > 0$, and $\lambda_u/\lambda_S \ll 1$. $\xi(t)$ and $\xi'(t)$ are independent dichotomous noises. The statistical properties of $\xi(t)$ and $\xi'(t)$ are

$$\langle \xi(t)\rangle = 0, \quad \langle \xi(t)\xi(t')\rangle = E_1^2\exp[-\lambda_1|t-t'|] ,$$

$$\langle \xi'(t)\rangle = 0, \quad \langle \xi'(t)\xi'(t')\rangle = E_2^2\exp[-\lambda_2|-t-t'|] .$$

Here the values of $\xi(t)$ and $\xi'(t)$ are, respectively, $\pm E_1$ and $\pm E_2$. The transition rate of $\xi(t)$ from E_1 to $-E_1$ and these from $-E_1$ to E_1 are denoted by $\lambda_1/2$. The homologue of $\xi'(t)$ is $\lambda_2/2$.

A. The equation for the order parameter

Let $M(t) = \int_0^t \xi'(t')dt'$. Using the similar method which has been utilized in Sec. II A, we can get

$$S_{\rm ad} = P_1(u)K_{t,1} + F_1(u)K_{t,2} , \quad (35)$$

where

$$K_{t,1} = \int_{-\infty}^t dt'\exp\{-\beta(u)(t-t') + F_{S,1}(u)[M(t) - M(t')]\} , \quad (36)$$

$$K_{t,2} = \int_{-\infty}^t dM(t')\exp\{-\beta(u)(t-t') + F_{S,1}(u)[M(t) - M(t')]\} . \quad (37)$$

We now calculate the statistical properties of $K_{t,1}$ and $K_{t,2}$. From Eq. (36), we find $K_{t,1}$ is the solution of SDE

$$\frac{d}{dt}K_{t,1} = -\beta(u)K_{t,1} + F_{S,1}(u)\xi'(t)K_{t,1} + 1 . \quad (38)$$

From Eq. (38) we have

$$d\langle K_{t,1}\rangle = -\beta(u)\langle K_{t,1}\rangle dt + F_{S,1}(u)\langle \xi'(t)K_{t,1}\rangle dt + dt . \quad (39)$$

By using the differential formula in Ref. [23] [i.e., Eq. (2) in it], we can get

$$\partial_t\langle \xi'(t)K_{t,1}\rangle = \langle \xi'(t)\partial_t K_{t,1}\rangle - \lambda_2\langle \xi'(t)K_{t,1}\rangle . \quad (40)$$

Substituting Eq. (38) into Eq. (40) and noting $[\xi'(t)]^2 = E_2^2$, we obtain

$$\partial_t\langle \xi'(t)K_{t,1}\rangle = -[\beta(u)+\lambda_2]\langle \xi'(t)K_{t,1}\rangle + F_{S,1}(u)E_2^2\langle k_{t,1}\rangle . \quad (41)$$

The solution of Eq. (41) is

$$\langle \xi'(t)K_{t,1}\rangle = \frac{E_2^2F_{S,1}(u)}{\beta(u)+\lambda_2}\langle k_{t,1}\rangle . \quad (42)$$

Substituting (42) into Eq. (39), we get the solution

$$\langle K_{t,1}\rangle = \frac{\beta(u)+\lambda_2}{[\beta(u)]^2 + \lambda_2\beta(u) - E_2^2[F_{S,1}(u)]^2} . \quad (43)$$

Similarly, we can derive

$$\langle (K_{t,1})^2\rangle = \frac{2\beta(u) + \lambda_2 + \dfrac{2E_2^2[E_{S,1}(u)]^2}{\beta(u)+\lambda_2}}{2[\beta(u)]^2 + \lambda_2\beta(u) - 2[F_{S,1}(u)]^2E_2^2}\langle K_{t,1}\rangle . \quad (44)$$

From Eq. (38), the equation of the correlation function of $K_{t,1}$ is

$$\frac{d}{dt}\langle K_{t,1}K_{t',1}\rangle=-\beta(u)\langle K_{t,1}K_{t',1}\rangle +F_{S,1}(u)\langle \xi'(t)K_{t,1}K_{t',1}\rangle+\langle K_{t',1}\rangle \ . \quad (45)$$

By using the differential formula in Ref. [23] [i.e., Eq. (2) in it], we obtain

$$\partial_t\langle \xi'(t)K_{t,1}K_{t',1}\rangle=\langle \xi'(t)\partial_t(k_{t,1}K_{t',1})\rangle -\lambda_2\langle \xi'(t)K_{t,1}K_{t',1}\rangle \ . \quad (46)$$

Substituting

$$\frac{d}{dt}(K_{t,1}K_{t',1})=-\beta(u)K_{t,1}K_{t',1} +F_{S,1}(u)K_{t,1}K_{t',1}\xi'(t)+K_{t',1}$$

into Eq. (46) and noting $[\xi'(t)]^2=E_2^2$, we find

$$\partial_t\langle \xi'(t)k_{t,1}K_{t',1}\rangle=-[\beta(u)+\lambda_2]\langle \xi'(t)K_{t,1}K_{t',1}\rangle +E_2^2F_{S,1}(u)\langle k_{t,1}K_{t',1}\rangle+\langle \xi'(t)K_{t',1}\rangle \ . \quad (47)$$

From Eqs. (45) and (47), we can get

$$\frac{d^2}{dt^2}\langle K_{t,1}K_{t',1}\rangle+[2\beta(u)+\lambda_2]\frac{d}{dt}\langle k_{t,1}K_{t',1}\rangle+\{[\beta(u)]^2+\lambda_2\beta(u)-E_2^2[F_{S,1}(u)]^2\}\langle K_{t,1}K_{t',1}\rangle =[\beta(u)+\lambda_2]\langle K_{t,1}\rangle+\frac{E_2^2[F_{S,1}(u)]^2}{\beta(u)+\lambda_2}\langle K_{t,1}\rangle e^{-\lambda_2|t-t'|} \ . \quad (48)$$

In the process of deriving Eq. (48), we utilized

$$\langle \xi'(t)K_{t',1}\rangle=\langle \xi'(t)K_{t,1}\rangle e^{-\lambda_2|t-t'|} =\frac{E_2^2F_{S,1}(u)}{\beta(u)+\lambda_2}\langle K_{t,1}\rangle e^{-\lambda_2|t-t'|} \ . \quad (49)$$

[The proof of Eq. (49) is given in Appendix B.] The solution of Eq. (48) is

$$\langle K_{t,1}K_{t',1}\rangle=C_1e^{-r_1|t-t'|}+C_2e^{-r_2|t-t'|}+(\langle K_{t,1}\rangle)^2+Ae^{-\lambda_2|t-t'|} \ , \quad (50)$$

where

$$r_1=\tfrac{1}{2}[2\beta(u)+\lambda_2+\omega],\quad r_2=\tfrac{1}{2}[2\beta(u)+\lambda_2-\omega] \ ,$$

$$C_1=\frac{2\beta(u)+\lambda_2-\omega}{2\omega}A+\frac{2\beta(u)+\lambda_2-3\omega}{2\omega}(\langle K_{t,1}\rangle^2-\langle K_{t,1}^2\rangle) \ ,$$

$$C_2=\frac{2\beta(u)+\lambda_2}{\omega}A-\frac{2\beta(u)+\lambda_2+\omega}{2\omega}(\langle K_{t,1}\rangle^2-\langle K_{t,1}^2\rangle) \ ,$$

$$\omega=\{\lambda_2^2+4E_2^2[F_{S,1}(u)]^2\}^{1/2} \ ,$$

$$A=\frac{E_2^2[F_{S,1}(u)]^2\langle K_{t,1}\rangle}{\{[\beta(u)]^2-\lambda_2\beta(u)-E_2^2[F_{S,1}(u)]^2\}[\beta(u)+\lambda_2]} \ .$$

Similarly, we can derive the mean value, square mean value, and correlation function of $K_{t,2}$ from (37). They are

$$\langle K_{t,2}\rangle=\frac{F_{S,1}(u)E_2^2}{[\beta(u)]^2+\lambda_2\beta(u)-E_2^2[F_{S,1}(u)]^2} \ , \quad (51)$$

$$\langle K_{t,2}^2\rangle=\frac{E_2^2F_{S,1}(u)\left[2+\dfrac{\lambda_2+2\beta(u)}{\lambda_2+\beta(u)}\right]\langle K_{t,2}\rangle+E_2^2\dfrac{\lambda_2+2\beta(u)}{\lambda_2+\beta(u)}}{2[\beta(u)]^2+\lambda_2\beta(u)-2E_2^2[F_{S,1}(u)]^2} \ , \quad (52)$$

and

$$\langle K_{t,2}K_{t',2}\rangle = C_1' e^{-r_1'|t-t'|} + C_2' e^{-r_2'|t-t'|} + \langle K_{t,2}\rangle^2 + A' e^{-\lambda_2|t-t'|} , \quad (53)$$

where

$$r_1' = \tfrac{1}{2}[2\beta(u)+\lambda_2+\omega'], \quad r_2' = \tfrac{1}{2}[2\beta(u)+\lambda-\omega'] ,$$

$$C_1' = \frac{2\beta(u)+\lambda_2}{\omega'} A' + \frac{2\beta(u)+\lambda_2+\omega'}{2\omega'}(\langle K_{t,2}\rangle^2 - \langle K_{t,2}^2\rangle) ,$$

$$C_2' = \frac{2\beta(u)+\lambda_2-\omega'}{2\omega'} A' + \frac{2\beta(u)+\lambda_2-3\omega'}{2\omega'}(\langle K_{t,2}\rangle^2 - \langle K_{t,2}^2\rangle) ,$$

$$\omega' = \{\lambda_2^2 + E_2^2[F_{S,1}(u)]^2\}^{1/2} ,$$

$$A' = \frac{E_2^2\beta(u)[F_{S,1}(u)\langle K_{t,2}\rangle + 1]}{\{[\beta(u)]^2 - \lambda_2\beta(u) - E_2^2[F_{S,1}(u)]^2\}[\lambda_2+\beta(u)]} .$$

The formulas (43), (50) (51), and (53) will return to the corresponding results in Ref. [7] at the limit of white noises [24]:

$$E_2 \to +\infty, \quad \lambda_2 \to +\infty, \quad \frac{E_2^2}{\lambda_2} = D_2 = \text{const } (=2) .$$

Substituting Eq. (35) into Eq. (33), we can get the equation for the order parameter of Eqs. (33) and (34) as follows:

$$du_t = \alpha(u)dt + \bar{P}_1(u)K_{t,1}dt + \bar{F}_1(u)K_{t,2}dt + [F_u(u)+F]\xi(t)dt , \quad (54)$$

where

$$\bar{P}_1(u) = Q_0(u)P_1(u), \quad \bar{F}_1(u) = Q_0(u)F_1(u) .$$

We find that the methods of Ref. [19] can be applied to the stochastic systems driven by dichotomous noises whose correlation times and intensity are small. Let $\eta_1(t) = K_{t,1} - \langle K_{t,1}\rangle$, $\eta_2(t) = K_{t,2} - \langle K_{t,2}\rangle$. Then $\eta_1(t)$ and $\eta_2(t)$ become dichotomouslike noises. Using the methods used in Ref. [19] which are similar to those in Sec. II B, we can finally get the AFPE of Eq. (54) as follows (in the circumstance of small $1/\lambda_1$, $1/\lambda_2$, $1/r_1$, $1/r_2$, C_1, C_2, A, C_1', C_2', and A'):

$$\partial_t P(u,t) = -\partial_u g'(u)P(u,t) + \partial_u \bar{P}_1(u)A_0\partial_u \bar{P}_1(u)P(u,t) + \partial_u \bar{F}_1(u)B_0\partial_u \bar{F}_1(u)P(u,t) + E_1^2\partial_u[F_u(u)+F]C\partial_u[F_u(u)+F]P(u,t) , \quad (55)$$

where

$$g'(u) = \alpha(u) + \bar{P}_1(u)\langle K_{t,1}\rangle + \bar{F}_1(u)\langle K_{t,2}\rangle ,$$

$$A_0 = \frac{C_1}{r_1(u)}\left[\frac{1}{1-[r_1(u)]^{-1}\partial_u g'(u)}\right]_{u=u_S} + \frac{C_2}{r_2(u)}\left[\frac{1}{1-[r_2(u)]^{-1}\partial_u g'(u)}\right]_{u=u_S} + \frac{A}{\lambda_2}\left[\frac{1}{1-\lambda_2^{-1}\partial_u g'(u)}\right]_{u=u_S} ,$$

$$B_0 = \frac{C_1'}{r_1'(u)}\left[\frac{1}{1-[r_1'(u)]^{-1}\partial_u g'(u)}\right]_{u=u_S} + \frac{C_2'}{r_2'(u)}\left[\frac{1}{1-[r_2'(u)]^{-1}\partial_u g'(u)}\right]_{u=u_S} + \frac{A'}{\lambda_2}\left[\frac{1}{1-\lambda_2^{-1}\partial_u g'(u)}\right]_{u=u_S} ,$$

and

$$C = \left[\frac{1}{\lambda_1 - \partial_u g'(u)}\right]_{u=u_S} .$$

And $u = u_S$ means that we must take the value of u by u_S, the steady value of equation

$$\frac{d}{dt}u = g'(u) + \bar{P}_1(u)\eta_1(t) + \bar{F}_1(u)\eta_2(t) + [F_u(u)+F]\xi(t) . \quad (55')$$

The stationary solution of Eq. (55) under the natural boundary condition is

$$P_S(u) = N \exp\left\{\int du \frac{\Delta}{A_0A_0' + B_0B_0' + CE_1^2\{[F_u(u)]^2 + F^2\}}\right\} , \quad (56)$$

where

$$\Delta = g(u) - \bar{P}_1(u)A_0\partial_u\bar{P}_1(u) - \bar{F}_1(u)B_0\partial_u\bar{F}_1(u) - E_1^2 CF_u(u)\partial_u F_u(u) ,$$

$$A_0' = [\bar{P}_1(u)]^2, \quad B_0' = [\bar{F}_1(u)]^2 .$$

B. The Brownian harmonic oscillator with dichotomous noise

The stochastic dynamical equations of the Brownian harmonic oscillator driven by dichotomous noise and Gaussian white noise are

$$\frac{d}{dt}u(t)=S(t) , \quad (57)$$

$$\frac{d}{dt}S(t)=-2\alpha S(t)-[\Omega_0^2+\xi(t)]u(t)+\eta(t) , \quad (58)$$

where α, Ω^2 are real, and $\alpha>0$. $u(t)$ and $S(t)$ are, respectively, the generalized displacement and the generalized momentum. Ω_0 is the mean value of the frequency, and $\xi(t)$ the fluctuating part of Ω_0^2. $\eta(t)$ is a Gaussian white noise satisfying

$$\langle\eta(t)\rangle=0, \quad \langle\eta(t)\eta(t')\rangle=4\alpha k_B T\delta(t-t') ,$$

$$\langle\eta(t)\xi(t')\rangle=0 ,$$

where k_B is Boltzmann's constant and T is temperature. Equations (57) and (58) are a typical stochastic model. Some physical phenomena conform to it. It has been studied by West *et al.* when $\xi(t)$ is Gaussian white noise [25]. Hemández-Machado and San Maguel have studied it when $\xi(t)$ is OU noise [16]. In this paper, we fetch $\xi(t)$ for dichotomous noise whose statistical properties are

$$\langle\xi(t)\rangle=0, \quad \langle\xi(t)\xi(t')\rangle=E^2e^{-\lambda|t-t'|} ,$$

where the values of $\xi(t)$ are $\pm E$. The transition rates of $\xi(t)$ from $+E$ to $-E$ and those from $-E$ to E are denoted by $\lambda/2$.

The purpose of this example is to illustrate how to use the stochastic adiabatic approximation to deal with the systems driven by dichotomous noises. Treating u as a time-independent parameter, the solution of SDE (58) can be written as

$$S_{\rm ad}=\int_{-\infty}^{t}\{-[\Omega_0^2+\xi(t')]u+\eta(t')\}\times\exp\left[-\int_{t'}^{t}2\alpha ds\right]dt' ,$$

namely,

$$S_{\rm ad}=-\frac{\Omega_0^2}{2\alpha}u-uK_{t,2}+Z_{t,2} , \quad (59)$$

where

$$K_{t,2}=\int_{-\infty}^{t}\xi(t')dt'\exp[-2\alpha(t-t')] ,$$
$$Z_{t,2}=\int_{-\infty}^{t}\eta(t')dt'\exp[-2\alpha(t-t')] . \quad (60)$$

For the statistical properties of $Z_{t,2}$, according to Ref. [7], we have

$$\langle Z_{t,2}\rangle=0, \quad \langle Z_{t,2}Z_{t',2}\rangle=k_BT\exp[-2\alpha|t-t'|] .$$

The statistical properties of $K_{t,2}$ can be calculated by formulas (51)–(53). Comparing (60) with (37), we find

$$F_{S,1}(u)=0, \quad \beta(u)=2\alpha, \quad \lambda_2=\lambda, \quad E_2=E .$$

Substituting these equations into formulas (51)–(53), we get

$$\langle K_{t,2}\rangle=0 ,$$
$$\langle K_{t,2}K_{t',2}\rangle=C_1'\exp[-(2\alpha+\lambda)|t-t'|]+C_2'\exp[-2\alpha|t-t'|]+A'\exp[-\lambda|t-t'|] ,$$

where

$$A'=\frac{E^2}{4\alpha^2-\lambda^2}, \quad C_2'=\frac{2\alpha E^2}{\lambda(4\alpha^2-\lambda^2)}-\frac{2\alpha-\lambda}{\lambda}\langle K_{t,2}^2\rangle ,$$
$$C_1'=\frac{(4\alpha+\lambda)E^2}{\lambda(4\alpha^2-\lambda^2)}-\frac{2\alpha+\lambda}{\lambda}\langle K_{t,2}^2\rangle ,$$
$$\langle K_{t,2}^2\rangle=\frac{E^2}{2\alpha(\lambda+2\alpha)} .$$

Substituting Eq. (59) into SDE (57), we can finally obtain the equation for the order parameter

$$\frac{d}{dt}u(t)=-\frac{\Omega_0^2}{2\alpha}u(t)-u(t)K_{t,2}+Z_{t,2} . \quad (61)$$

Under the conditions that $1/(2\alpha+\lambda)$, $1/2\alpha$, $1/\lambda$, C_1', C_2', E, and A' are small values, using the methods used in Ref. [19] which are similar to those in Secs. II B and III, the AFPE of SDE (61) is

$$\partial_t P(u,t)=-\partial_u\left[-\frac{\Omega_0^2}{2\alpha}u\right]P(u,t)+\partial_u(-u)\int_0^\infty dt'\langle K_{t,2}K_{t',2}\rangle\exp\left[t'\partial_u\left[-\frac{\Omega_0^2}{2\alpha}u\right]\right]\partial_u(-u)P(u,t)$$
$$+\partial_u\int_0^\infty dt'\langle Z_{t,2}Z_{t',2}\rangle\exp\left[t'\partial_u\left[-\frac{\Omega_0^2}{2\alpha}u\right]\right]\partial_u P(u,t)$$
$$=\frac{\Omega_0^2}{2\alpha}\partial_u uP(u,t)+\partial_u u\left[c_1'\frac{1}{2\alpha+\lambda+\frac{\Omega_0^2}{2\alpha}}+c_2'\frac{1}{2\alpha+\frac{\Omega_0^2}{2\alpha}}+A'\frac{1}{\lambda+\frac{\Omega_0^2}{2\alpha}}\right]\partial_u uP(u,t)+k_BT\partial_u^2\frac{1}{2\alpha+\frac{\Omega_0^2}{2\alpha}}P(u,t)$$
$$=L\partial_u uP(u,t)+\partial_u uB\partial_u uP(u,t)+H\partial_u^2 P(u,t) . \quad (62)$$

Under the natural boundary condition, its solution is

$$P_S(u)=N\left[\frac{H}{B}+u^2\right]^{-(1/2)(L/B+1)},$$

where

$$H=\frac{2\alpha k_B T}{4\alpha^2+\Omega_0^2},$$

$$B=\frac{2\alpha c_1'}{(2\alpha+\lambda)2\alpha+\Omega_0^2}+\frac{2\alpha c_2'}{4\alpha^2+\Omega_0^2}+\frac{2\alpha A'}{2\alpha\lambda+\Omega_0^2},$$

$$L=\frac{\Omega_0^2}{2\alpha}.$$

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we have generalized the CW stochastic adiabatic approximation to the stochastic systems driven by OU noises or dichotomous noises. We find that the procedure of the stochastic adiabatic approximation of the stochastic systems driven by OU noises or dichotomous noises, in Stratonovich's case [20], is basically similar to eliminating one of the stochastic adiabatic approximations of stochastic systems driven by Gaussian white noises. The differences between $Z_{t,1}$, $Z_{t,2}^i$ in Ref. [7] and $K_{t,1}$, $K_{t,2}^i$ (or $K_{t,2}$) are that $Z_{t,1}$ and $Z_{t,2}^i$ are the solutions of equations driven by Gaussian white noises, while $K_{t,1}$ and $K_{t,2}^i$ (or $K_{t,2}$) are the solutions of equations driven by OU noises (or dichotomous noises). Thus the statistical properties of $K_{t,1}$ and $K_{t,2}^i$ (or $K_{t,2}$) are different from those of $Z_{t,1}$ and $Z_{t,2}^i$. After we made $\eta_1(t)=K_{t,1}-\langle K_{t,1}\rangle$ and $\eta_2^i(t)=K_{t,2}^i-\langle K_{t,2}^i\rangle$ [or $\eta_2(t)=K_{t,2}-\langle K_{t,2}\rangle$], $\eta_1(t)$ and $\eta_2^i(t)$ [or $\eta_2(t)$] are OU-like noises (or dichotomous-like noises). The correlation functions of these noises have probably two correlation times.

The central idea of the paper is that for small ϵ, the slow variable u_t in SDE's (4) and (34) can be treated as a time- and chance-independent parameter. The above approximation is the stochastic generalization of the usual adiabatic approximation [1]. To see this, we write S_t formally as a function of u_t, t, and K_t^i, that is,

$$S_t=S_t(u_t,t,K_t^i)\,. \tag{63}$$

Thus, by use of the usual chain rule, we can write from Eq. (63)

$$dS_{\rm ad}=\frac{\partial S}{\partial t}dt+\sum_i\frac{\partial S}{\partial K_t^i}dK_t^i\,. \tag{64}$$

Obviously, Eq. (64) is a generalization of the usual (deterministic and autonomous) adiabatic approximation

$$\frac{dS_{\rm ad}}{dt}=0\,.$$

In this paper, we studied a class of two-dimensional non-Markovian systems driven by OU noises or dichotomous noises. When the stochastic systems are three- or multi-three-dimensional ones, the method (the stochastic adiabatic approximation) generalized in this paper is wholly applicable. When the stochastic systems are driven by OU noises, in the multiplicative noise case we must make the approximation of the small τ_i for $\xi_i(t)$. Formulas (11), (15), etc. can be expanded to different order in τ_i according to our demands. But in this paper, we only expand them to first order in τ_i. When the stochastic systems are driven by dichotomous noises or additive OU noises, in the process of calculating the statistical properties for the new stochastic variables, we need not make approximation of small τ_i. In the limits of white noises, this method will return to that of CW.

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APPENDIX A: PROOF OF (24′)

From (24), we can get

$$\langle K_{t,2}^i K_{t',2}^j\rangle=\int_{-\infty}^{t}d\tau\int_{-\infty}^{t'}d\tau'\langle\xi_i(\tau)\xi_j(\tau')\rangle\times\exp\{-\beta(u)(t'-\tau')-\beta(u)(t-\tau)\} \tag{A1}$$

(obviously, $t\ge\tau$, $t'\ge\tau'$). Equation (A1) can be written

$$\langle K_{t,2}^i K_{t',2}^j\rangle=\delta_{ij}\int_{-\infty}^{t}d\tau\int_{-\infty}^{t'}d\tau'\frac{D_i}{\tau_i}\exp\left\{-\beta(u)[(t-\tau)+(t'-\tau')]-\frac{|\tau-\tau'|}{\tau_i}\right\}. \tag{A2}$$

First, we let $t'>t$. For the sake of removing the signs of absolute values, we divide the integral range $-\infty<\tau'\le t'$ into $-\infty<\tau'\le\tau$ and $\tau\le\tau'\le t'$. Thus

$$\begin{aligned}\langle K_{t,2}^i K_{t',2}^j\rangle&=\delta_{ij}\int_{-\infty}^{t}d\tau\left[\int_{-\infty}^{\tau}+\int_{\tau}^{t'}\right]d\tau'\frac{D_i}{\tau_i}\exp\left[-\beta(u)[(t-\tau)+(t'-\tau')]-\frac{|\tau-\tau'|}{\tau_i}\right]\\&=\delta_{ij}\frac{D_i}{\tau_i}\exp[-\beta(u)(t+t')]\left\{\int_{-\infty}^{t}\exp[\beta(u)\tau]d\tau\int_{-\infty}^{\tau}\left[\exp\left[\beta(u)\tau'-\frac{\tau-\tau'}{\tau_i}\right]\right]d\tau'\right.\\&\qquad\left.+\int_{-\infty}^{t}\exp[\beta(u)\tau]d\tau\int_{\tau}^{t'}\left[\exp\left[\beta(u)\tau'-\frac{\tau'-\tau}{\tau_i}\right]\right]d\tau'\right\}\\&=\delta_{ij}\left\{-\frac{D_i\tau_j}{1-\beta^2(u)\tau_i^2}\exp(-|t-t'|/\tau_i)+\frac{D_i}{\beta(u)[1-\beta^2(u)\tau_i^2]}\exp[-\beta(u)|t-t'|]\right\}.\end{aligned} \tag{A3}$$

As $t' < t$, using a similar method we can find that the results of (A2) are also (A3).

Obviously, from Eq. (24) we can obtain $\langle K_{t,2}^{i} \rangle = 0$.

APPENDIX B: PROOF OF (49)

Using the differential formula in Ref. [23] [i.e., Eq. (2) in it], we can get

$$\partial_t \langle \xi'(t) K_{t',1} \rangle = -\lambda_2 \langle \xi'(t) K_{t',1} \rangle \ . \tag{B1}$$

The solution of (B1) is

$$\begin{aligned}\langle \xi'(t) K_{t',1} \rangle &= \langle \xi'(t) K_{t,1} \rangle \exp[-\lambda_2 |t-t'|] \\ &= \frac{E_2^2 F_{S,1}(u)}{\beta(u)+\lambda_2} \langle K_{t,1} \rangle \exp[-\lambda_2 |t-t'|] \ .\end{aligned} \tag{B2}$$

[1] H. Haken, *Advanced Synergetics* (Springer-Verlag, Berlin, 1983).
[2] Kunihiko Kaneko, Prog. Theor. Phys. **66**, 129 (1981).
[3] C. W. Gardiner, Phys. Rev. A **29**, 2814 (1984); **29**, 2823 (1984); **29**, 2834 (1984).
[4] G. Schöner and H. Haken, Z. Phys. B **63**, 493 (1986); **68**, 89 (1987).
[5] A. Wunderlin and H. Haken, Z. Phys. B **44**, 135 (1981).
[6] H. Haken and A. Wunderlin, Z. Phys. B **47**, 179 (1982).
[7] Da-jin Wu and Li Cao, Z. Phys. B **81**, 131 (1990); **81**, 451 (1990).
[8] Li Cao and Da-jin Wu, Commun. Theor. Phys. **14**, 21 (1990).
[9] Li Cao and Da-jin Wu, Phys. Lett. A **145**, 159 (1990).
[10] Jing-Hui Li and Li Cao, Phys. Lett. A(to be published).
[11] Li Cao and Da-jin Wu, Phys. Lett. A **155**, 257 (1991).
[12] S. Zhu, A. W. Yu, and R. Roy, Phys. Rev. A **34**, 4333 (1986).
[13] E. Peacock-Lopez, F. J. de la Rubia, B. J. West, and K. Linderberg, Phys. Rev. A **39**, 4026 (1989).
[14] R. G. K. Habiger, H. Risken, M. James, Frank Moss, and W. Schleich, Phys. Rev. A **41**, 3950 (1990).
[15] Hu Gang and H. Haken, Z. Phys. B **76**, 537 (1989).
[16] A. Hemández-Machado and M. San Maguel, J. Math. Phys. **25**, 1066 (1984).
[17] E. A. Novikov, Sov. Phys. JETP **20**, 1290 (1965).
[18] J. M. Sancho *et al.*, Phys. Rev. A **26**, 1589 (1982).
[19] L. Cao, D. J. Wu, and H. X. Wang, Phys. Lett. A **133**, 476 (1988).
[20] C. W. Gardiner, *Handbook of Stochastic Method for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, Berlin, 1983).
[21] N. G. Van Kampen, *Stochastic Processes in Physics and Chemistry* (North-Holland, Amsterdam, 1981).
[22] J. M. Sancho and M. San Miguel, Z. Phys. B **36**, 357 (1980).
[23] V. E. Shapiro and W. M. Loginov, Physica A **91**, 563 (1978).
[24] C. Van Den Broeck, J. Stat. Phys. **31**, 467 (1983).
[25] Bruce J. West *et al.*, Physica A **102**, 470 (1980).