Projection dynamics of highly dissipative systems

I. A. Lubashevskii[1] and V. V. Gafiychuk[2]
[1]*Academy of Technology Science of Russia Federation, 117333 Vavilova 46/92 Moscow, Russia*
[2]*Institute for Applied Problems of Mechanics and Mathematics, Ukrainian Academy of Science, 3b, Naukova strasse L'viv, 290601, Ukraine*
(Received 28 April 1993; revised manuscript received 10 March 1994)

We propose a method of investigation of highly dissipative systems, which is based on an approximation of the attractor by some manifold. The projection dynamic equations for the general form of such a manifold of the dissipative dynamic system are obtained. The dynamics of the dissipative structures in a concrete reaction diffusion system is considered.

PACS number(s): 05.70.Ln, 05.40.+j, 02.90.+p

I. INTRODUCTION

For the last few years there has been great interest in self-organization phenomena arising in nonequilibrium distributed dissipative systems. A wide variety of such phenomena can be theorized as the origination, development, and evolution of complex spatial structures. The characteristic properties of such stationary stable structures have been partly investigated (for a review see, e.g., [1–3]). However, at the present time dynamics of nonstationary spatial structures especially complex in geometry is actually an unresolved challenging problem. The latter is caused by the lack of well developed general analytical mathematical methods for studying nonlinear dynamics of substantially nonuniform fields in dissipative systems. The available methods are specially designed for certain particular models only and, thus, nonstationary spatial structures arising in dissipative systems are usually investigated by numerical simulation.

In the present paper we develop a general approach to investigating dynamics of nonlinear distributed dissipative systems. In order to state the problem under consideration we represent the evolution of dissipative distributed systems in terms of the following nonlinear equations:

$$\frac{\partial \psi_i}{\partial t} = F_i\{\psi_1, \ldots, \psi_N; A_1, \ldots, A_M\}. \tag{1.1}$$

Here $i = 1, 2, 3, \ldots, N$ where N is a given integer number, $\{\psi_i(\vec{r}, t)\}$ are certain fields specifying a state of this system and are regarded as real functions of the time t and the spatial coordinates $\vec{r}$, $\{F_i\}$ are the components of a nonlinear evolution operator F which depends on both the fields $\{\psi_i\}$ and the external parameters $A_1, \ldots, A_M$ (M is also an integer number). Due to dissipation, the system tends to a certain state in the space Ψ of the functions $\{\psi_i\}$ as $t \to \infty$. This state is conventionally treated as a certain set Ω^* called the attractor of the dissipative system. Therefore one of the main problems in the description of dissipative systems is analysis of the attractor geometry and the system motion in the vicinity of the corresponding attractor. In the following we shall confine ourselves to this problem.

For certain systems there are a large number of experimental data and results obtained by numerical modeling that allow one to approximately imagine the general form of the attractors beforehand. In more exact terms, these results show the general form of the fields $\{\psi_i(\vec{r}, t)\}$ being the asymptotic solution of Eqs. (1.1) as $t \to \infty$ and, thereby, enable one to construct some manifold Ω in the space Ψ that characterizes such solutions (Fig. 1). Therefore, for this system we can specify its attractor by paths in the space Ψ that go in a small neighborhood of the manifold Ω. The given manifold may be of finite dimension p and in this case it is possible to describe it in terms of

$$\Omega = \{\psi_i(\vec{r}) = \Phi_i(\vec{r}, u_1, \ldots, u_p)\}, \tag{1.2a}$$

where $\Phi_i(\vec{r}, u_1, \ldots, u_p)$ are certain functions of the spatial coordinates $\vec{r}$ and the collection of real variables $(u_1, \ldots, u_p)$. The system $\{\Phi_i\}$, as a vector function of $\vec{r}$, gives the position of the physical system in the space Ψ and, as a vector function of $(u_1, \ldots, u_p)$, determines the geometry of the manifold Ω. The time dependence of the variables $(u_1, \ldots, u_p)$ approximately describes the physical system motion along the attractor Ω^*. Besides, there are cases where the manifold Ω is of infinite dimen-

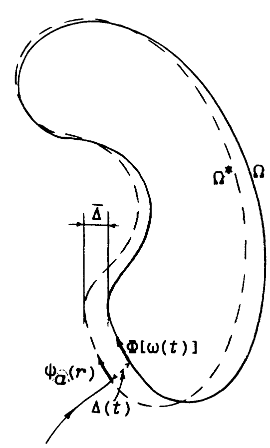

FIG. 1. The schematic view of the attractor Ω^* and the considered manifold Ω.

sion and can be determined by the collection of nonlinear operators

$$\Omega = \{\psi_i(r) = \Phi_i(\vec{r}, u_1, \ldots, u_p, v_1(\vec{r}), \ldots, v_q(\vec{r}))\}. \tag{1.2b}$$

Here the operator Φ_i depends on the real functions $v_1(\vec{r}), \ldots, v_q(\vec{r})$ as well as on the spatial coordinates $\vec{r}$ and the real variables $(u_1, \ldots, u_p)$.

The basic idea of the approach to be developed in the present paper is to reduce the system of equations (1.1) to some evolution equations that contain solely the variables $(u_1, \ldots, u_p)$ and may be $v_i(\vec{r}), \ldots, v_q(\vec{r})$, whose time dependence characterizes the motion of the system along the attractor. For some systems solving such evolution equations can be found to be more simple than solving the system of equations (1.1).

We note that a similar idea has been much used in describing the evolution of quasiconservative systems (see, e.g., [4]). Nevertheless the general procedure has been developed for the first order perturbation technique only. In order to obtain an approximation at the next order individual analysis is required for each particular system. The aim of the present paper is to develop such a procedure for dissipative systems that enables one to find approximations to any order in a small parameter in a regular way. For dynamical systems described by ordinary differential equations a similar technique has already been designed [5].

II. PERTURBATION TECHNIQUE

Let us analyze the motion of a dissipative system where the evolution operator $F = (F_1, \ldots, F_N)^T$ involves two parts. The first one (F_0) determines the fast motion towards a manifold Ω and the second one (ϵF_p) gives rise to the slow motion along the manifold Ω (Fig. 1). In mathematical terms the motion of this system is described by the equation

$$\frac{\partial \psi}{\partial t} = F_0\{\psi\} + \epsilon F_p\{\psi\}, \tag{2.1}$$

where $\psi = (\psi_1, \ldots, \psi_N)^T$, ϵ is a small parameter, and the operator $F_0\{\psi\}$ becomes zero at the manifold Ω, i.e.,

$$F_0\{\psi\}\,|_{\psi=\phi} = 0 \tag{2.2}$$

at any point ϕ of the manifold Ω. The manifold Ω is supposed to be specified in the space Ψ by the expression

$$\Omega = \{\psi = \phi(\omega)\}, \tag{2.3}$$

where $\phi = (\Phi_1, \ldots, \Phi_N)^T$ and the generalized coordinates ω of the manifold Ω may involve both the real variables $u_1, u_2, \ldots, u_p$ and the real functions $v_1(\vec{r}), \ldots, v_q(\vec{r})$ [see expressions (1.2a) and (1.2b)].

We represent the solution of Eq. (2.1) as the sum

$$\psi(t) = \phi(\omega(t)) + \Delta(t), \tag{2.4}$$

where the first term on the right-hand side describes the motion along the manifold Ω. The second one is the small deviation of the system from the manifold Ω. Expression (2.4) enables us to expand the operators $F_0\{\psi\}$ and $F_p\{\psi\}$ into the Taylor series of Δ:

$$F_0\{\psi\} = F_0^{(1)}\{\phi \mid \Delta\} + \frac{1}{2}F_0^{(2)}\{\phi \mid \Delta\} + \cdots$$
$$\equiv F_0^{(1)}(\phi)\Delta + \frac{1}{2}F_0^{(2)}(\phi)(\Delta, \Delta) + \cdots \tag{2.5}$$

and

$$F_p\{\psi\} = F_p(\phi) + F_p^{(1)}(\phi)\Delta + \frac{1}{2}F_p^{(2)}(\phi)(\Delta, \Delta) + \cdots . \tag{2.6}$$

Here the term $F^{(n)}(\phi)(\Delta, \Delta, \ldots, \Delta)$ denoted also as $F^{(n)}\{\phi \mid \Delta\}$ is the nth order differential operator which is a homogeneous operator of degree n with respect to Δ, i.e.,

$$F^{(n)}(\phi)(\beta\Delta, \ldots, \beta\Delta) = \beta^n F^{(n)}(\phi)(\Delta, \ldots, \Delta).$$

It should be pointed out that the term $F^{(1)}(\phi)$ is a linear operator with respect to Δ and conventionally represented as

$$F^{(1)}(\phi)\Delta = L(\phi)\Delta, \tag{2.7}$$

where $L(\phi)$ is the Frisher derivative of the operator $F\{\psi\}$ at the point $\psi = \phi$. The term $F^{(2)}(\phi)(\Delta, \Delta)$ is a symmetrical bilinear operator with respect to Δ.

The Frisher derivative $L_0\{\phi\} = dF_0/d\psi\,|_{\psi=\phi}$ of the operator $F_0\{\psi\}$, given at the manifold Ω, plays an important role in the perturbation technique. Thus we, first, consider its properties in detail. We assume that for any point $\phi \in \Omega$ the eigenvectors $\{\psi_\lambda(\phi)\}$ of the Frisher derivative $L_0\{\phi\}$ form a complete system of linearly independent vectors in the space Ψ. In other words, any vector of the space Ψ, in particular, Δ, can be expanded relative to the basis $\{\psi_\lambda(\phi)\}$;

$$\Delta(t) = \sum_\lambda \mathcal{A}_\lambda(\phi)\psi_\lambda(\phi) \tag{2.8}$$

and, thereby,

$$\psi = \phi + \sum_\lambda \mathcal{A}_\lambda(\phi)\psi_\lambda(\phi), \tag{2.9}$$

where $\mathcal{A}_\lambda$ are certain constants and the sum runs over all the vectors $\{\psi_\lambda(\phi)\}$. The symbol λ stands for the eigenvalue corresponding to a given eigenvector ψ_λ. In the general case the eigenvalues, as well as the eigenvectors, depend on the point ϕ of the manifold Ω. Considering an infinitely small displacement $\delta\psi = (d\phi/d\omega)\cdot\delta\omega$ along the manifold Ω from condition (2.2) we find

$$\left.\frac{dF_0}{d\psi}\right|_\phi \frac{d\phi}{dw}\delta w = L_0(\psi)\frac{d\phi}{d\omega}\delta\omega = 0. \tag{2.10}$$

For possible different values of $\delta\omega$ the system $\{(d\phi/d\omega)\delta\omega\}$ can form the basis of the plane T_ϕ tangent to the manifold Ω at the point ϕ in the space Ψ. Therefore, as follows from (2.10), any vector belonging to the tangent plane T_ϕ is an eigenvector of the Frisher derivative L_0 which matches the zero eigenvalue.

Since condition (2.2) practically defines the manifold Ω any eigenvector of the Frisher derivative $L_0(\phi)$ corresponding to the zero eigenvalue belongs to the plane T_ϕ. Let us single out a region Ω_r on the manifold Ω that is characterized by the condition $\lambda \leq 0$ for all the eigenvalues of the operator $L_0\{\phi\}$ at any point ϕ belonging to Ω_r. It is near this part of the manifold Ω that the system will go practically along it because at these points the evolution operator F_0 causes the system motion towards the manifold Ω. When the system leaves the region Ω_r the evolution operator F_0 must give rise to the fast motion of the system away from the manifold Ω. So such motion can be treated as a fast transition of the system between different points of the region Ω_r and be described by the equation

$$\frac{\partial\psi}{\partial t} = F_0\{\psi\}.$$

At this stage of motion the effect of the small operator $\epsilon F_p\{\psi\}$ can be ignored. Analysis of the fast transitions on the basis of this equation is a problem in its own right and requires that the system be concretized. Therefore in the following we shall consider the system motion near the region Ω_r only.

As follows from expansion (2.8) in the case under consideration the motion of the system can be represented as the motion of the point ϕ (the shadow) on the manifold Ω and the time variation of the coefficients $\mathcal{A}_\lambda$. At the present stage the motion of the shadow is not uniquely determined because the motion along the manifold Ω is independently described by the motion of the point ϕ and the time variations of the coefficients $\mathcal{A}_\lambda(t)|_{\lambda=0}$. Since the evolution operator $F_0\{\psi\}$ has no effect on the system motion along the manifold Ω in the general case the coefficients $\mathcal{A}_\lambda(t)|_{\lambda=0}$ may increase beyond all bounds. In order to analyze the system dynamics in terms of the shadow motion, the distance between the point ψ, showing the real system position in the space Ψ, and the point ϕ must be small. In other words, the last term in expression (2.9) must be small too. Therefore it is reasonable to specify the shadow motion in such way that at every instant of time all the coefficients $\mathcal{A}_\lambda(t)|_{\lambda=0}$ be equal to zero. This procedure is equivalent to eliminating singular terms in the evolution equation for the coefficients $\mathcal{A}_\lambda$, obtained by perturbation technique.

Keeping the latter in mind we note that the procedure developed in the present work, the Bogolubov-Metropolskii method of averages [6], the perturbation technique designed for ordinary differential equations [5], as well as the perturbation technique for nonlinear waves proposed in [7] are similar in eliminating singular terms.

We now proceed to formal construction of the perturbation technique. Substituting (2.4)–(2.6) and (2.9) into (2.1) we get

$$\frac{d\phi}{dw}\dot{w} + \sum_\lambda \left[\dot{\mathcal{A}}_\lambda \psi_\lambda(\phi) + \mathcal{A}_\lambda \frac{d\psi_\lambda}{d\omega}\dot{\omega}\right] = \sum_\lambda \lambda \mathcal{A}_\lambda \psi_\lambda + P. \tag{2.11}$$

Here

$$P = \sum_{m=2}^{\infty} \frac{1}{m!} F_0^{(m)} \left\{ \phi \middle| \sum_\lambda \mathcal{A}\psi_\lambda \right\}_\lambda + \epsilon F_p\{\phi\} + \epsilon \sum_{m=1}^{\infty} \frac{1}{m!} F_p^{(m)} \left\{ \phi \middle| \sum_\lambda \mathcal{A}_\lambda \psi_\lambda \right\}, \tag{2.12}$$

$d\phi/d\omega$ and $d\psi_\lambda/d\omega$ are the corresponding Frisher derivatives, and we have also taken into account that

$$F_0^{(1)} \left\{ \phi \middle| \sum_\lambda \mathcal{A}_\lambda \psi_\lambda \right\} = L_0(\phi) \sum_\lambda \mathcal{A}_\lambda \psi_\lambda = \sum_\lambda \lambda \mathcal{A}_\lambda \psi_\lambda.$$

Equation (2.11) is completed by the conditions

$$\mathcal{A}_\lambda|_{\lambda=0} = 0. \tag{2.13}$$

In order to find the explicit expansion of the vector P relative to the basis $\{\psi_\lambda(\phi)\}$ we introduce the linear operator $G_\delta\{\phi\}$ defined by the formula

$$G_\delta\{\phi\}\,(L_0\{\phi\} - E\delta) = -E, \tag{2.14}$$

where the regularization parameter $\delta \to +0$ and E is the unit operator. The operator G_δ possesses the same set of the eigenvectors $\{\psi_\lambda\}$ and its eigenvalues are $\{-1/(\lambda - \delta)\}$, respectively. The second type of operator that we need are the projection operator

$$\mathcal{P} = \delta \lim_{\delta\to+0} G_\delta \tag{2.15}$$

and the operator, called the Green operator,

$$\mathcal{G} = \lim_{\delta\to+0} \left[G_\delta - \frac{1}{\delta}\mathcal{P} \right]. \tag{2.16}$$

As follows from definition (2.14)–(2.16) the actions of these operators on an arbitrary vector

$$\psi = \sum b_\lambda \psi_\lambda$$

of the space Ψ are specified by the expressions

$$\mathcal{P}\psi = \sum_{\lambda=0} b_\lambda \psi_\lambda \tag{2.17}$$

and

$$\mathcal{G}\psi = -\sum{}' \frac{1}{\lambda} b_\lambda \psi_\lambda. \tag{2.18}$$

The operators $\mathcal{P}$ and $\mathcal{G}$ enable us to divide Eq. (2.11) into two parts governing the motion of the system towards and along the manifold Ω.

$$\mathcal{G}\sum_{\lambda}\left[\dot{\mathcal{A}}_\lambda\psi_\lambda(\phi)+\mathcal{A}_\lambda\frac{d\psi_\lambda}{dw}\dot{w}\right]=-\sum{}'\mathcal{A}_\lambda\psi_\lambda+\mathcal{G}P, \tag{2.19}$$

$$\frac{d\phi}{dw}\dot{w}+\sum{}'\mathcal{A}_\lambda\mathcal{P}\frac{d\psi_\lambda}{dw}\dot{w}=\mathcal{P}P. \tag{2.20}$$

Here the prime on the sums indicates that the terms matching the zero eigenvalue are omitted and the following identities resulting from (2.13), (2.17), and (2.18) have been taken into account:

$$\mathcal{G}\frac{d\phi}{dw}\dot{w}\equiv 0,\quad \mathcal{G}\sum_{\lambda}\lambda\mathcal{A}_\lambda\psi_\lambda\equiv-\sum{}'\mathcal{A}_\lambda\psi_\lambda,$$

$$\mathcal{P}\frac{d\phi}{dw}\dot{w}\equiv\frac{d\phi}{dw}\dot{w},\quad \mathcal{P}\sum_{\lambda}\lambda\dot{\mathcal{A}}_\lambda\psi_\lambda(\phi)\equiv 0,$$

$$\mathcal{P}\sum_{\lambda}\lambda\mathcal{A}_\lambda\psi_\lambda\equiv 0.$$

Under conditions (2.13) the vector Δ takes the form

$$\Delta=\sum{}'\mathcal{A}_\lambda\psi_\lambda(\phi). \tag{2.21}$$

Expression (2.21) allows us to rewrite Eq. (2.19) as

$$\mathcal{G}\frac{d}{dt}\Delta=-\Delta+\mathcal{G}P. \tag{2.22}$$

We consider such systems, that will be called the highly dissipative systems, for which the transient term in Eq. (2.22) can be treated as a small perturbation. Physically, this means that from viewpoint of the fast motion towards the manifold Ω_r the system motion along it may be regarded as quasistationary. In this case Eq. (2.22) describing evolution of the vector Δ can be reduced to the explicit relationship determining the vector Δ in terms of projection dynamics, viz.,

$$\begin{aligned}\Delta&=\left[1+\mathcal{G}\frac{d}{dt}\right]^{-1}\mathcal{G}P\\ &\equiv\mathcal{G}P-\mathcal{G}\dot{w}\frac{d}{dw}[\mathcal{G}P]+\mathcal{G}\dot{w}\frac{d}{dw}\left[\mathcal{G}\dot{w}\frac{d}{dw}(\mathcal{G}P)\right]\\ &\quad+\mathcal{G}^2\left[\ddot{w}\frac{d}{dw}(\mathcal{G}P)\right]+\cdots.\end{aligned} \tag{2.23}$$

In mathematical terms the highly dissipative systems are characterized by convergence of the latter series. In order to verify whether a given system belongs to this class one should analyze in detail the spectrum of the Frisher operator. In particular, if there is a finite gap separating the zero eigenvalue ($\lambda=0$) from other ones ($\mathrm{Re}\lambda<0$ for Ω_r) then, as follows from (2.18), for an arbitrary ψ the vector $\mathcal{G}\psi$ will be finite. In this case due to the system motion along the manifold Ω being caused by the perturbation operator ϵF_p the time derivative $d\Delta/dt$ according to (2.23) may be estimated as $(d\Delta/dt)\sim\epsilon\Delta$ and thus the transient term in (2.22) is small in comparison with the first one on the right-hand side. Therefore such a system can be classified as a highly dissipative system.

As follows from (2.17) for any eigenvector ψ_λ corresponding to nonzero eigenvalue $(\lambda\neq 0)\mathcal{P}\psi_\lambda=0$ at each point $\phi=\phi(w)$ of the manifold Ω. Differentiating the latter identity with respect to the time we get

$$\mathcal{P}\frac{d\psi_\lambda}{dw}\dot{w}=-\dot{w}\frac{d\mathcal{P}}{dw}\psi_\lambda.$$

This expression and (2.21) enable us to rewrite Eq. (2.20) in the form

$$\frac{d\phi}{dw}\dot{w}-\dot{w}\frac{d\mathcal{P}}{dw}\Delta=\mathcal{P}P. \tag{2.24}$$

It should be pointed out that the vector Δ and the point ϕ of the manifold Ω directly determine the value of $\mathcal{P}$. Indeed, according to (2.12)

$$\begin{aligned}P&=\sum_{m=2}^{\infty}\frac{1}{m!}F_0^{(m)}\{\phi\mid\Delta\}+\epsilon F_p\{\phi\}\\ &\quad+\epsilon\sum_{m=1}^{\infty}\frac{1}{m!}F_p^{(m)}\{\phi\mid\Delta\}.\end{aligned} \tag{2.25}$$

Therefore Eq. (2.24) along with expressions (2.23) and (2.25) completely describe the projection dynamics of the system under consideration. In the general case Eq. (2.23) is of complex form and can contain all the time derivatives of w.

We note that expression (2.23) and Eq. (2.24) contain apart from the operators $F_0\{\psi\}$, $F_p\{\psi\}$ and these derivatives which are determined at all points of the space Ψ, the operators $\mathcal{G}$ and $\mathcal{P}$ as well as their derivatives determined at the manifold Ω only. Therefore it would be desirable to find the derivatives of $\mathcal{G}$ and $\mathcal{P}$ along the manifold as functions of this operator and certain derivatives of the evolution operator $F_0\{\psi\}$. As shown in Appendix A such an expression for the first derivatives of the operators $\mathcal{G}$ and $\mathcal{P}$ along the manifold is of the form

$$\dot{w}\frac{d\mathcal{P}}{dw}=\mathcal{P}\left(\dot{w}\frac{dL_0}{dw}\right)\mathcal{G}+\mathcal{G}\left(\dot{w}\frac{dL_0}{dw}\right)\mathcal{P}, \tag{2.26}$$

$$\begin{aligned}\dot{w}\frac{d\mathcal{G}}{dw}&=\mathcal{G}\left(\dot{w}\frac{dL_0}{dw}\right)\mathcal{G}-\mathcal{P}\left(\dot{w}\frac{dL_0}{dw}\right)\mathcal{G}^2\\ &\quad-\mathcal{G}^2\left(\dot{w}\frac{dL_0}{dw}\right)\mathcal{P}.\end{aligned} \tag{2.27}$$

Successively differentiating expressions (2.26) and (2.27) we can obtain the desired formulas for the higher order derivatives of the operators $\mathcal{G}$ and $\mathcal{P}$.

In particular, substituting (2.26) into (2.24), taking into account expression (2.23) and the identity $\mathcal{P}\mathcal{G}=0$, we get

$$\frac{d\phi}{dw}\dot{w}=\mathcal{P}\mathcal{F}. \tag{2.28}$$

where

$$\mathcal{F} = \dot{w}\frac{dL_0}{dw}\mathcal{G}\Delta + P$$

is the generalized evolution operator for the system motion along the manifold Ω. The explicit expressions for P and Δ as functionals of ϕ can be obtained by successive iteration of Eqs. (2.23), (2.25), (2.28), and at lower order in ϵ from (2.25) we get

$$\mathcal{F}^{(1)} = P^{(1)} = \epsilon F_p\{\phi\}, \quad (2.29)$$

$$\Delta^{(1)} = \mathcal{G}P^{(1)} \equiv \epsilon \mathcal{G} F_p\{\phi\}, \quad (2.30)$$

and thus the shadow motion equation at first order in ϵ takes the form

$$\frac{d\phi}{dw}\dot{w} = \mathcal{P}\mathcal{F}^{(1)} \equiv \epsilon \mathcal{P} F_p\{\phi\}. \quad (2.31)$$

To the next order in ϵ from (2.25) and (2.28) we find

$$P^{(2)} = \epsilon F_p\{\phi\} + \epsilon^2 F_p^{(1)}\left\{\phi \mid \Delta^{(1)}\right\} + \frac{1}{2}F_0^{(2)}\left\{\phi \mid \Delta^{(1)}\right\} \quad (2.32)$$

and

$$\frac{d\phi}{dw}\dot{w} - \mathcal{P}\left(\dot{w}^{(1)}\frac{dL_0}{dw}\right)\mathcal{G}\Delta^{(1)} = \mathcal{P}P^{(2)}. \quad (2.33)$$

Substituting (2.30) and (2.31) into (2.32) and (2.33) we obtain the shadow motion equation to second order in ϵ

$$\frac{d\phi}{dw}\dot{w} = \mathcal{P}\mathcal{F}^{(2)}, \quad (2.34)$$

where

$$\begin{aligned}\mathcal{F}^{(2)} = {} & \epsilon F_p\{\phi\} + \epsilon^2 F_p^{(1)}(\phi)(\mathcal{G}F_p\{\phi\}) \\ & + \epsilon^2 \frac{1}{2} F_0^{(2)}(\phi)(\mathcal{G}F_p\{\phi\};\, \mathcal{G}F_p\{\phi\}) \\ & + \epsilon^2 F_0^{(2)}(\phi)(\mathcal{P}F_p\{\phi\};\, \mathcal{G}^2 F_p\{\phi\}). \end{aligned} \quad (2.35)$$

In obtaining Eq. (2.34) we have taken into account the relation

$$\dot{w}\frac{dL_0}{dw}\mathcal{G}\Delta = F_0^{(2)}(\phi)\left(\dot{w}\frac{d\phi}{dw};\, \mathcal{G}\Delta\right)$$

and substituted (2.31) into the latter equality. It should be noted that the term of second order in ϵ can play an essential part when the operator F_p is degenerate at the manifold Ω and the first order approximation is inadequate to give the right results.

Equation (2.28) is actually of the vector form whose components are specified by the coordinate system, given initially in the space Ψ. Therefore the formal dimension of Eq. (2.28) coincides with the dimension of the space Ψ. However, in actual truth, the amount of the independent equations as well as the independent variables is determined by the dimension of the manifold Ω and can be substantially less than the dimension of the space Ψ. So concluding the present section we also obtain a possible form of such independent equations of the shadow motion.

The plane T_ϕ tangent to the manifold Ω at the point $\psi = \phi(\omega)$ can be specified by the set of vectors $\{(d\phi/d\omega)\cdot\delta\omega\}$ where the vector $\delta\omega$ runs all the possible values. Let the collection of vectors $\{e_\alpha\}$ form a basis in the space $\{\delta\omega\}$, and thus the vector system $\{(d\phi/d\omega)e_\alpha\}$ be a basis of the plane T_ϕ in the space Ψ.

Since Eq. (2.28) contains solely the vectors lying in the plane T_ϕ it can be equivalently represented as the system of equations

$$\left\langle \frac{d\phi}{dw}\dot{w} - \mathcal{P}\mathcal{F} \middle| \frac{d\phi}{dw}e_\alpha \right\rangle = 0. \quad (2.36)$$

The convenience of the given equation system is that it contains the complete collection of independent equations explicitly describing the system motion in terms of time variations of the parameters ω only.

III. EXAMPLE OF THE PERTURBATION TECHNIQUE APPLICATION: THEORY OF DISSIPATIVE STRUCTURE OSCILLATION

In this section, as an example of the developed method application, we analyze nonlinear dynamics of spatial structures in a highly dissipative system described by the following one-dimensional reaction-diffusion equations [8]:

$$\tau_\theta\dot{\theta} = l^2\nabla^2\theta + \theta - \theta^3 + \eta, \quad (3.1)$$

$$\tau_\eta\dot{\eta} = L^2\nabla^2\eta - g\eta - (\theta - A). \quad (3.2)$$

Here θ and η are dimensionless order parameters, τ_θ, τ_η, and l, L are the characteristic scales of temporal and spatial variations of these parameters, and g and A are given constants. We assume that $\tau_\theta \ll \tau_\eta$ and $l \ll L$. When the uniform distribution of the fields θ, η becomes unstable, contrast spatial structures typically occur (Fig. 2) in such systems.

From the viewpoint of the order parameter θ these structures involve two types of domains where the value θ is about ± 1, which are separated by "walls," i.e., by thin regions inside which the field θ varies abruptly [3]. As a rule spatial periods of these structures are equal to $(lL)^{1/2}$ in order. Under some conditions such a spatial structure in turn can become unstable and in this case spatially nonuniform oscillations develop in the system. Qualitatively, these spatially nonuniform oscillations may be treated as motion of the walls. Keeping the aforementioned in mind we consider the motion of a single wall in the region shown in Fig. 3 whose size $2\mathcal{L}$ satisfies the inequalities

$$l \ll \mathcal{L} \ll L \quad (3.3)$$

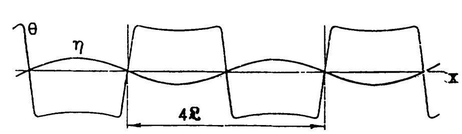

FIG. 2. The solution of the reaction-diffusive system in the form of periodical dissipative structures.

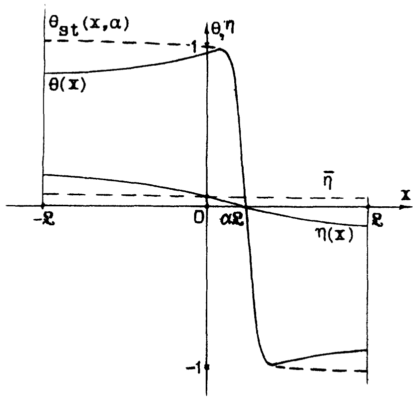

FIG. 3. The fragment of the dissipative structure placed in the interval $[-\mathcal{L}, \mathcal{L}]$.

and at whose boundaries $x = \pm\mathcal{L}$ the fields θ and η are subject to the conditions

$$\nabla\theta\,|_{x=-\mathcal{L}} = \nabla\theta\,|_{x=\mathcal{L}} = 0, \tag{3.4}$$

$$\nabla\eta\,|_{x=-\mathcal{L}} = \nabla\eta\,|_{x=\mathcal{L}} = 0. \tag{3.5}$$

This model practically describes characteristics of spatially nonuniform oscillations of periodic structures.

As follows from Eq. (3.1) and (3.2) time variations of the field θ, η are characterized at least by three temporal scales, viz., τ_θ, τ_η, and $\tau_\eta(\mathcal{L}/L)^2$. The scale τ_θ is associated with the formation of the wall, the second one (τ_η) characterizes time variations in the field η caused by generation or dissipative processes in the system and on the third time scale $\tau_\eta(\mathcal{L}/L)^2 \ll \tau_\eta$ the diffusion processes control formation of nonuniform distribution of the variable η.

The couple of equations (3.1),(3.2) may be regarded as the motion equation of form (2.1) for the vector

$$\psi = \left\|\begin{matrix}\theta\\ \eta\end{matrix}\right\|. \tag{3.6}$$

Taking into account the time hierarchy mentioned above $[\tau_\theta, \tau_\eta(\mathcal{L}/L)^2 \ll \tau_\eta]$ we divide the total evolution operator $F\{\psi\}$ of this motion equation into two parts: $F = F_0 + \epsilon F_p$ where

$$F_0 = \left\|\begin{matrix}(l^2\nabla^2\theta + \theta - \theta^3)\frac{1}{\tau_\theta}\\ \frac{1}{\tau_\eta}L^2\nabla^2\eta\end{matrix}\right\| \tag{3.7}$$

and

$$\epsilon F_p = \left\|\begin{matrix}\frac{1}{\tau_\theta}\eta\\ \frac{1}{\tau_\eta}[-g\eta - (\theta - A)]\end{matrix}\right\| \tag{3.8}$$

and treat ϵF_p as a small perturbation operator. The solution of the equation $F_0\{\psi\} = 0$ is of the form

$$\theta(x) = \theta_{st}(x,\alpha) = \tanh\left[\frac{\alpha\mathcal{L} - x}{\sqrt{2}l}\right], \tag{3.9}$$

$$\eta(x) = \bar{\eta}, \tag{3.10}$$

where $\bar{\eta}$ and $-1 < \alpha < 1$ are certain arbitrary constants. Here by virtue of (3.3) we have ignored boundary condition (3.4), because at $x = \pm\mathcal{L}$ the derivative $\frac{d}{dx}\theta_{st}$ differs from zero by a value of order $\exp\{-\mathcal{L}/l\}$. Therefore the manifold $\Omega = \{\psi\colon\ \psi = \phi(\vec{r}, w)\}$ at which $F_0 = 0$ can be specified in the form

$$\psi(x) = \phi(x,\alpha,\bar{\eta}) = \left\|\begin{matrix}\theta_{st}(x,\alpha)\\ \bar{\eta}\end{matrix}\right\| \tag{3.11}$$

and we may regard the variables α, $\bar{\eta}$ as the generalized coordinates of the manifold Ω.

Following the procedure developed in Sec. II we calculate the Frisher derivative $dF_0/d\psi$. From (3.7) we find

$$L_0\{\phi\} = \left.\frac{dF_0}{d\phi}\right|_\Omega = \left\|\begin{matrix}\frac{1}{\tau_\theta}\left[l^2\nabla^2 + [1 - 3\theta_{st}^2]\right] & 0\\ 0 & \frac{1}{\tau_\eta}L^2\nabla^2\end{matrix}\right\|, \tag{3.12}$$

The given operator $L_0\{\phi\}$ is Hermitian, thus we may omit the projection operator $\mathcal{P}$ in Eqs. (2.36) because $\mathcal{P} = \mathcal{P}^+ = E$ for the vectors of T_ϕ. Besides, as seen from the final results obtained below, in this case the evolution operator $\epsilon F\{\phi\}$ is degenerate. Therefore we have to use the second order approximation (2.34) of the evolution equation. Expression (3.11) specifying the manifold Ω parametrization enables us to regard the vectors

$$e_\alpha = \left\|\begin{matrix}\mathcal{L}\left(-\frac{\partial\theta_{st}}{\partial x}\right)\\ 0\end{matrix}\right\|, \qquad e_\eta = \left\|\begin{matrix}0\\ 1\end{matrix}\right\|, \tag{3.13}$$

as an orthogonal basis $\{e_\alpha, e_\eta\}$ of the plane T_ϕ tangent to the manifold Ω at the point ϕ. In these terms

$$\dot{\phi} = \frac{d\phi}{dw}\dot{w} = e_\alpha\dot{\alpha} + e_\eta\dot{\eta} \tag{3.14}$$

and starting, for example, from the motion equation of form (2.36) and omitting the projection operator $\mathcal{P}$ we obtain the following equations governing the motion of the parameters α and η:

$$\langle e_\alpha \mid e_\alpha\rangle\dot{\alpha} = \langle e_\alpha \mid \mathcal{F}^{(2)}\rangle, \tag{3.15}$$

$$\langle e_\eta \mid e_\eta\rangle\dot{\eta} = \langle e_\eta \mid \mathcal{F}^{(2)}\rangle. \tag{3.16}$$

From (3.9) and (3.13) we find

$$\langle e_\alpha \mid e_\alpha\rangle = \frac{2\sqrt{2}}{3l}\mathcal{L}^2, \quad \langle e_\eta \mid e_\eta\rangle = 2\mathcal{L}. \tag{3.17}$$

By virtue of (3.7), (3.8), and (3.11) for the given system expression (2.35) can be rewritten in terms of

$$\mathcal{F}^{(2)} = \begin{Vmatrix} \frac{1}{\tau_\theta}(\bar\eta + \delta\eta_1(x) - 3\theta_{st}(x)\{[\delta\theta_1(x)]^2 + 2\delta\theta_2(x)\delta\theta_3(x)\}) \\ \frac{1}{\tau_\eta}\{-g\bar\eta - [\theta_{st}(x,\alpha) - A] - g\delta\eta_1(x) - \delta\theta_1(x)\} \end{Vmatrix}. \quad (3.18)$$

Here

$$\begin{Vmatrix} \delta\theta(x) \\ \delta\eta(x) \end{Vmatrix}_1 = \mathcal{G}\epsilon F_p|_\Omega , \quad \begin{Vmatrix} \delta\theta(x) \\ \delta\eta(x) \end{Vmatrix}_2 = \mathcal{G}^2\epsilon F_p|_\Omega ; \quad \begin{Vmatrix} \delta\theta(x) \\ \delta\eta(x) \end{Vmatrix}_3 = \mathcal{P}\epsilon F_p|_\Omega, \quad (3.19)$$

where by virtue of (3.8) and (3.11)

$$\epsilon F_p|_\Omega = \begin{Vmatrix} \frac{1}{\tau_\theta}\bar\eta \\ \frac{1}{\tau_\eta}\{-g\bar\eta - [\theta_{st}(x,\alpha) - A]\} \end{Vmatrix}$$

and $\mathcal{G}, \mathcal{P}$ are the regular Green and projection operators. According to definition (2.14), the total Green operator G_δ satisfies the relation $(L_0[\phi] - \delta E)G_\delta = -E$ for $\delta = +0$, thus, due to (3.12) it is of the form

$$G_\delta = \begin{Vmatrix} G_\delta^\theta & 0 \\ 0 & G_\delta^\eta \end{Vmatrix}, \quad (3.20)$$

where the operators G_δ^θ and G_δ^η in the coordinate representation are the functions $G_\delta^\theta(x,x_0), G_\delta^\eta(x,x_0)$ (calculation of which is presented in Appendix B) obeying the equations

$$\frac{1}{\tau_\theta}\{l^2\nabla_x^2 + [1 - 3\theta_{st}^2(x)]\}G_\delta^\theta(x,x_0) - \delta G_\delta^\theta(x,x_0) = -\delta(x - x_0), \quad (3.21)$$

$$\frac{1}{\tau_\eta}L^2\nabla_x^2 G_\delta^\eta(x,x_0) - \delta G_\delta^\eta(x,x_0) = -\delta(x - x_0), \quad (3.22)$$

subject to the boundary conditions

$$\nabla_x G^\theta|_{x=\pm\mathcal{L}} = 0, \quad (3.23a)$$

$$\nabla_x G^\eta|_{x=\pm\mathcal{L}} = 0. \quad (3.23b)$$

Solving Eq. (3.22) under boundary conditions (3.23b) and separating the regular part from the obtained result we get

$$G^\eta(x,x_0) = \frac{\tau_\eta}{4\mathcal{L}L^2}\begin{cases} (\mathcal{L}+x)^2 + (\mathcal{L}-x_0)^2 - (4/3)\mathcal{L}^2 & \text{if } x < x_0 \\ (\mathcal{L}-x)^2 + (\mathcal{L}+x_0) - (4/3)\mathcal{L}^2 & \text{if } x > x_0. \end{cases} \quad (3.24)$$

As follows from (3.21) the Green function $G_\delta^\theta(x,x_0)$ practically does not depend on the form of boundary conditions (3.23a) because the characteristic scale on which it varies significantly is about l. The solution of Eq. (3.21), i.e., the total Green function $G_\delta^\theta(x,x_0)$ as well as its regular part $G^\theta(x,x_0), (G^\theta)^2(x,x_0)$, and the projection operator $\mathcal{P}(x,x_0)$ with full details are presented in Appendix B. There, in particular, it is shown that the functions $\delta\theta_1(x)$, $\delta\theta_2(x)$, and $\delta\theta_3(x)$ [see (3.19)] must be even functions about the point $x = \alpha\mathcal{L}$. Besides, we may set

$$G^\theta(x,x_0) \cong \frac{\tau_\theta}{2\sqrt{2}l}\exp\left\{-\frac{\sqrt{2}}{l}\mid x - x_0\mid\right\} \quad (3.25)$$

except for the points belonging to a small neighborhood of the point $x = \alpha\mathcal{L}$ whose radius is about l.

The analysis of the Green operator G_δ and the Frisher derivative $L_0(\phi)$ also shows that there is a gap between the zero eigenvalue and other negative eigenvalues. The latter proves that the system under consideration can be regarded as highly dissipative.

Substituting the obtained results for the regular Green operator

$$G = \begin{Vmatrix} G^\theta & 0 \\ 0 & G^\eta \end{Vmatrix} \quad (3.26)$$

into (3.19) we find the following expressions for $\delta\eta_1(x)$:

$$\delta\eta_1(x) = -\int_{-\mathcal{L}}^{\mathcal{L}} G^\eta(x,x_0)\theta_{st}(x_0)dx_0 \quad (3.27)$$

and, in particular,

$$\delta\eta_1(\alpha) = \frac{2}{3}\frac{\mathcal{L}^2}{L^2}\alpha(1-\alpha^2) \quad (3.28)$$

and for $\delta\theta_1(x)$

$$\delta\theta_1(x) \cong \frac{1}{2}\bar\eta \quad (3.29)$$

when the point x does not belong to a small neighborhood of $x = \alpha$.

Then from (3.13) and (3.18) we obtain

$$\langle e_\alpha \mid \mathcal{F}^{(2)}\rangle = \int_{-\mathcal{L}}^{\mathcal{L}} dx\mathcal{L}\left(-\frac{\partial\theta_{st}}{\partial x}\right)(\bar\eta + \delta\bar\eta(x) - 3\theta(x)_{st}\{[\delta\theta_1(x)]^2 + 2\delta\theta_2(x)\delta\theta_3(x)\}). \quad (3.30)$$

By virtue of (3.9) the function $\frac{\partial\theta_{st}}{\partial x}$ differs from zero at a small neighborhood of the point $x = \alpha$. Keeping in mind that the functions $\partial\theta_{st}/\partial x$, $\delta\theta_1(x)$, $\delta\theta_2(x)$, $\delta\theta_3(x)$ are even functions about $x = \alpha$ whereas $\theta_{st}(x)$ is an odd one, and, in addition, that the function $\delta\eta(x)$ cannot vary substantially on the scale l, from (3.30) we get

$$\langle e_\alpha \mid \mathcal{F}^{(2)}\rangle \cong \mathcal{L}[\bar{\eta} + \delta\eta(\alpha)]\frac{1}{\tau_\theta} \tag{3.31}$$

and according to (3.28)

$$\langle e_\alpha \mid \mathcal{F}^{(2)}\rangle \cong \mathcal{L}\left[\bar{\eta} + \frac{2}{3}\frac{\mathcal{L}^2}{L^2}\alpha(1-\alpha^2)\right]\frac{1}{\tau_\theta}. \tag{3.32}$$

The behavior of the function $\delta\theta_1(x)$ in a small neighborhood of the point $x = \alpha$ has practically no effect on the value of $\langle e_\eta \mid \mathcal{F}^{(2)}\rangle$. So in order to obtain $\langle e_\eta \mid \mathcal{F}^{(2)}\rangle$ expression (3.29) may be used. Then substituting this expression together with (3.28) into (3.18) we find

$$\langle e_\eta \mid \mathcal{F}^{(2)}\rangle \cong -\frac{2\mathcal{L}}{\tau_\eta}\left[\left(g + \frac{1}{2}\right)\bar{\eta} + \left(2\alpha - \frac{A}{2}\right)\right]. \tag{3.33}$$

Expressions (3.17), (3.32), and (3.33) enable us to rewrite Eqs. (3.15), (3.16) governing the motion of the parameters α and η in the desired form:

$$\tau_\theta\dot{\alpha} = \frac{1}{\mathcal{L}}\frac{3l}{2\sqrt{2}}\left[\bar{\eta} + \frac{2}{3}\frac{\mathcal{L}^2}{L^2}\alpha(1-\alpha^2)\right], \tag{3.34}$$

$$\tau_\eta\dot{\bar{\eta}} = -\left[\left(g + \frac{1}{2}\right)\bar{\eta} + \left(2\alpha - \frac{A}{2}\right)\right]. \tag{3.35}$$

As follows from the stability linear analyses the stationary solution (α_s, $\bar{\eta}_s$) of the equation system (3.34), (3.35) is unstable when

$$[1 - 3\alpha_s^2] > \sqrt{2}\left(g + \frac{1}{2}\right)\frac{\tau_\theta}{\tau_\eta}\frac{L^2}{l\mathcal{L}} \tag{3.36}$$

and the unstability in the wall attitude occurs through oscillations with the frequency

$$w \approx \left[\frac{3}{\sqrt{2}}\frac{l}{\mathcal{L}}\frac{1}{\tau_\eta\tau_\theta}\right]^{1/2}. \tag{3.37}$$

The conventional bifurcation analysis of Eqs. (3.34) and (3.35) shows that in the given system the supercritical bifurcation takes place. For example, when $A = 0$ and, thus, $\alpha_s = 0$, $\bar{\eta}_s = 0$, i.e., the stationary position of the wall is the middle point, the amplitude α_A of the appearing oscillations is

$$\alpha_A = \frac{2}{\sqrt{3}}\left[1 - \sqrt{2}\left(g + \frac{1}{2}\right)\frac{\tau_\theta}{\tau_\eta}\frac{L^2}{l\mathcal{L}}\right]^{1/2}. \tag{3.38}$$

It should be pointed out that according to (3.38) the oscillation amplitude attains the value $\alpha_A = 1$ when

$$\sqrt{2}\left(g + \frac{1}{2}\right)\frac{\tau_\theta}{\tau_\eta}\frac{L^2}{l\mathcal{L}} = \frac{1}{4}. \tag{3.39}$$

In this case the wall will be broken down.

Concluding this section we note that, as has been demonstrated, the developed projection dynamic method enables one not only to obtain some qualitative results but also to study nonlinear dynamics of dissipative structures in sufficient detail.

IV. CONCLUSION

We have considered a highly dissipative system whose attractor Ω^* is located inside a small neighborhood of the known manifold Ω. The motion of the system along the attractor Ω^* has been described as motion of its projection (shadow) onto the manifold Ω. We have developed the perturbation technique which enables us to find equations governing the shadow motion to a given accuracy. These equations contain evolution operators determined at the manifold Ω only. By way of example, dissipative structure oscillations have been analyzed on the basis of the developed method.

ACKNOWLEDGMENT

This work was supported in part by the Ukrainian National Committee of Science and Technology.

APPENDIX A: PROOF OF EXPRESSIONS (2.26) AND (2.27)

First, we shall show that the operator G_δ to first order in δ can be represented as

$$G_\delta = \frac{1}{\delta}\mathcal{P} + \mathcal{G} - \delta\mathcal{G}^2. \tag{A1}$$

Indeed, according to definition (2.14) the operator G_δ and the Frisher derivative $L_0\{\phi\}$ possess the same set of eigenvectors $\{\psi_\lambda\}$ which match the eigenvalues $\{1/(\delta - \lambda)\}$ and $\{\lambda\}$, respectively. Therefore the action of the operator G_δ on an arbitrary vector $\psi = \sum \alpha_\lambda\psi_\lambda$ is specified by the formula

$$G_\delta\psi = \frac{1}{\delta}\sum_{\lambda=0}\alpha_\lambda\psi_\lambda + \sum{}'\frac{1}{\delta - \lambda}\alpha_\lambda\psi_\lambda. \tag{A2}$$

For $\lambda \neq 0$ to first order in δ

$$\frac{1}{\delta - \lambda} = \frac{1}{-\lambda} - \frac{\delta}{\lambda^2}.$$

Thus, to the same order in δ

$$G_\delta\psi = \frac{1}{\delta}\sum_{\lambda=0}\alpha_\lambda\psi_\lambda + \sum{}'\left(-\frac{1}{\lambda}\right)\alpha_\lambda\psi_\lambda - \delta\sum{}'\frac{1}{\lambda^2}\alpha_\lambda\psi_\lambda. \tag{A3}$$

Comparing (2.17), (2.18), and (A3) we get expression (A1). Differentiating (2.14) with respect to ω we find

$$\frac{dG_\delta}{dw}(L_0 - \delta E) + G_\delta \frac{dL_0}{dw} = 0 \tag{A4}$$

and, thereby,

$$\frac{dG_\delta}{dw} = G_\delta \frac{dL_0}{dw} G_\delta \tag{A5}$$

because, by definition, $(L_0 - \delta E)G_\delta = -E$. Substitution of (A1) into (A5) to zeroth order in δ yields

$$\frac{1}{\delta}\frac{d\mathcal{P}}{dw} + \frac{d\mathcal{G}}{dw} = \frac{1}{\delta^2}\mathcal{P}\frac{dL_0}{dw}\mathcal{P} + \frac{1}{\delta}\left(\mathcal{P}\frac{dL_0}{dw}\mathcal{G} + \mathcal{G}\frac{dL_0}{dw}\mathcal{P}\right)$$
$$+\mathcal{G}\frac{dL_0}{dw}\mathcal{G} - \mathcal{P}\frac{dL_0}{dw}\mathcal{G}^2 - \mathcal{G}^2\frac{dL_0}{dw}\mathcal{P}. \tag{A6}$$

The identity $L_0\mathcal{P} \equiv \mathcal{P}L_0 \equiv 0$ leads to the expression

$$\frac{dL_0}{dw}\mathcal{P} = -L_0\frac{d\mathcal{P}}{dw}$$

and, thus, the first term on the right-hand side of expression (A6) is equal to zero. Equating the terms of the same order in δ from (A6) we obtain

$$\frac{d\mathcal{P}}{dw} = \mathcal{P}\frac{dL_0}{dw}\mathcal{G} + \mathcal{G}\frac{dL_0}{dw}\mathcal{P} \tag{A7}$$

and

$$\frac{d\mathcal{G}}{dw} = \mathcal{G}\frac{dL_0}{dw}\mathcal{G} - \mathcal{P}\frac{dL_0}{dw}\mathcal{G}^2 - \mathcal{G}^2\frac{dL_0}{dw}\mathcal{P}. \tag{A8}$$

Formulas (2.26) and (2.27) immediately result from (A7) and (A8).

APPENDIX B: THE GREEN FUNCTION AND THE PROJECTION OPERATOR

In order to solve Eq. (3.21) we, first, consider the eigenvalue problem for the operator [9]

$$\hat{H} = \frac{1}{\tau_\theta}\left\{l^2\nabla_x^2 + \left[1 - 3\theta_{st}^2\right]\right\} - \delta \tag{B1}$$

for $\alpha = 0$. Substituting (3.9) into (B1) and converting to the new variable $Z = x/(\sqrt{2}l)$ we represent the operator $\hat{H}$ as

$$\hat{H} = \frac{1}{2\tau_\theta}\left[\hat{h} - 2(2 + \delta\tau_\theta)\right], \tag{B2}$$

where

$$\hat{h} = \frac{d^2}{dZ^2} + \frac{6}{\cosh^2 Z}. \tag{B3}$$

In this way the eigenvalue problem for the operator $\hat{H}$ is reduced to solving the equation $\hat{h}\theta_k = k\theta_k$ where k is such a constant that the solution of this equation is bounded as $Z \to \pm\infty$. Using the conventional transformations [10] we convert from Z to $\xi = \sinh Z$ and represent θ as

$$\theta = (\cosh Z)^3\left(\frac{d}{d\xi}\right)^3 \chi = \hat{f}\chi, \tag{B4}$$

where

$$\hat{f} = \frac{d^3}{dZ^3} - 3\frac{\sinh Z}{\cosh Z}\frac{d^2}{dZ^2} + \left[2 - \frac{3}{(\cosh Z)^2}\right]\frac{d}{dZ}. \tag{B5}$$

Then, from the latter equation we obtain

$$\frac{d^3}{d\xi^3}\left[\frac{d^2}{dZ^2}\chi - k\chi\right] = 0. \tag{B6}$$

In the present work we assume that $l \ll L$ and consider the wall being far (in unit l) from the boundaries $x = \pm\mathcal{L}$. In this case boundary condition (3.4) or (3.23a) is not the factor and we may choose any boundary conditions for convenience. In particular, we shall assume that the function χ meets the Born–von Kármán conditions at $Z = \pm Z_\infty$ where $Z_\infty \to +\infty$.

For every $k < 0$ there are two bounded solutions of Eq. (B6) of the form

$$\chi_+(Z) = \cos(\sqrt{|k|}Z),$$
$$\chi_-(Z) = \sin(\sqrt{|k|}Z), \tag{B7}$$

which meet the Born–von Kármán boundary conditions for

$$k_n = -\left(\frac{\pi}{Z_\infty}\right)^2 n^2, \tag{B8}$$

where $n = 1, 2, \ldots$. Thus the eigenfunctions of the operator $\hat{h}$ corresponding to the eigenvalue k_n (B8) and normalized to unity can be written as

$$\theta_{n\pm} = \frac{1}{\sqrt{2Z_\infty}}U_n\hat{f}\chi_\pm, \tag{B9}$$

where

$$U_n = \left[\frac{2}{9k_n^2 - k_n(k_n+2)^2}\right]^{1/2}.$$

As results from analysis of Eq. (B6) the operator h also possesses the following three eigenfunctions:

$$\theta_1(Z) = \frac{\sqrt{3}}{2}\frac{1}{(\cosh Z)^2},$$

$$\theta_2(Z) = \left(\frac{3}{2}\right)^{1/2}\frac{\sinh Z}{(\cosh Z)^2}, \tag{B10}$$

$$\theta_3(Z) = \frac{1}{\sqrt{2Z_\infty}}\left[1 - \frac{3}{2(\cosh Z)^2}\right],$$

which match the eigenvalues $k = 4, 2, 0$, respectively. The operator $\hat{H}$ possesses the same collection of eigenfunctions $\{\theta_k \equiv \theta_\lambda\}$ and the corresponding eigenvalues are

$$\left\{\lambda = \frac{1}{2\tau_\theta}[k - 2(2 + \delta\tau_\theta)]\right\}.$$

Due to the collection of eigenfunctions $\{\theta_k\}$ being orthonormal we may write

$$\delta(Z - Z_0) = \sum_k \theta_k(Z)\theta_k(Z_0) \tag{B11}$$

and, thereby, the solution of Eq. (3.21) can be represented as

$$G_\delta^\theta(x, x_0) = \frac{1}{\sqrt{2}l}\sum_k \frac{2\tau_\theta}{[2(2+\delta\tau_\theta) - k]}\theta_k\left(\frac{x - \alpha\mathcal{L}}{\sqrt{2}l}\right) \times\theta_k\left(\frac{x_0 - \alpha\mathcal{L}}{\sqrt{2}l}\right). \tag{B12}$$

For $\delta = 0$ all the eigenvalues $\lambda = \frac{1}{2\tau_\theta}[k - 4]$ differ from zero except for the eigenvalue corresponding to the eigenfunction $\theta_1(Z)$. Thus, from (B12) and the definitions of the operators $\mathcal{G}$ and $\mathcal{P}$ it follows that

$$\mathcal{P}^\theta(x, x_0) = \frac{1}{\sqrt{2}l}\theta_1\left(\frac{x - \alpha\mathcal{L}}{\sqrt{2}l}\right)\theta_1\left(\frac{x_0 - \alpha\mathcal{L}}{\sqrt{2}l}\right) \tag{B13}$$

and

$$\mathcal{G}^\theta(x, x_0) = \frac{1}{\sqrt{2}l}\sum_{k, k\neq 4} \frac{2\tau_\theta}{[4 - k]}\theta_k\left(\frac{x - \alpha\mathcal{L}}{\sqrt{2}l}\right) \times\theta_k\left(\frac{x_0 - \alpha\mathcal{L}}{\sqrt{2}l}\right). \tag{B14}$$

In particular, as it results from (B14), the operator $(\mathcal{G}^\theta)^2$ can be represented in terms of

$$(\mathcal{G}^\theta)^2(x, x_0) = \frac{1}{\sqrt{2}l}\sum_{k, k\neq 4}\left(\frac{2\tau_\theta}{4 - k}\right)^2\theta_k\left(\frac{x - \alpha\mathcal{L}}{\sqrt{2}l}\right)\theta_k \times\left(\frac{x_0 - \alpha\mathcal{L}}{\sqrt{2}l}\right). \tag{B15}$$

Besides, when $x - \alpha\mathcal{L}$, $x_0 - \alpha\mathcal{L} \gg l$ or $\alpha\mathcal{L} - x$, $\alpha\mathcal{L} - x_0 \gg l$ in formula (B14) we may take into account solely the terms corresponding to $k = k_n$ and summing over all $n = 1, 2, \ldots$ we obtain

$$G^\theta(x, x_0) = \frac{\tau_\theta}{2\sqrt{2}l}\exp\left\{-\frac{\sqrt{2}}{l}\mid x - x_0\mid\right\}. \tag{B16}$$

It should be pointed out that expressions of the type

$$\int_{-\mathcal{L}}^{\mathcal{L}} dx_0 G^\theta(x, x_0),$$

$$\int_{-\mathcal{L}}^{\mathcal{L}} dx_0 (G^\theta)^2(x, x_0),$$

$$\int_{-\mathcal{L}}^{\mathcal{L}} dx_0 \mathcal{P}^\theta(x, x_0) \tag{B17}$$

that we meet in finding functions (3.19) are practically reduced to the series of form

$$\sum_k A_k\theta_k\left(\frac{x - \alpha\mathcal{L}}{\sqrt{2}l}\right)\int_{-Z_\infty}^{Z_\infty} dZ\theta_k(Z), \tag{B18}$$

where A_k is a certain function of k. Since the given collection of the eigenfunctions $\{\theta_k(Z)\}$ contains odd and even functions of Z, series (B18) actually contains the even eigenfunctions only. Whence it follows that the functions $\delta\theta_1(x - \alpha\mathcal{L}), \delta\theta_2(x - \alpha\mathcal{L})$, and $\delta\theta_3(x - \alpha\mathcal{L})$ are even functions with regard to the transformation $(x - \alpha\mathcal{L}) \to -(x - \alpha\mathcal{L})$.

[1] G. Nicolis and I. Prigogine, *Exploring Complexity* (Freeman, New York, 1989).
[2] H. Haken, *Information and Self-organization: A Macroscopic Approach to Complex Systems* (Springer-Verlag, Berlin, 1988).
[3] B. S. Kerner and V. V. Osipov, *Autosolitons* (Nauka, Moscow, 1991) (in Russian).
[4] D. McLaughlin and A. C. Scott, in *Solitons in Action*, edited by K. Lonngren and A. Scott (Academic, New York, 1978).
[5] V. P. Bogaevsky and A. Ya. Povzner, *Algebraic Methods in Nonlinear Perturbation Theory* (Nauka, Moscow, 1987) (in Russian).
[6] N. N. Bogolubov and Iu. A. Mitropolsky, *Asymptotic Methods in Theory of Nonlinear Oscillation* (Nauka, Moscow, 1974) (in Russian).
[7] V. N. Biktashev, Physica D **40**, 83 (1989).
[8] Iu. M. Romanovsky *et al.*, *Mathematical Modeling in Biophysics* (Moscow, Nauka, 1984) (in Russian).
[9] S. Flugge, *Practical Quantum Mechanics* (Springer-Verlag, Berlin, 1971).
[10] E. Kamke, *Differentialgleichungen: Lösungsmethoden und Lösungen, Vol. 1: Gewöhnliche Differentialgleichungen*, 3rd ed. (Chelsea, New York, 1948).