Stationary convection in a cylindrical plasma

L. Gomberoff
Departamento de Física, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile
(Received 21 January 1994)

It is shown that in a current-carrying cylindrical plasma with a free boundary, viscosity and thermal conductivity can lead to large scale steady convection. The physical situation is identical to the case when the plasma column is limited by perfectly conducting walls, in spite of the fact that the boundary conditions are different.

PACS number(s): 52.20.−j, 52.25.−b, 52.35.Py, 52.30.−q

Large scale stationary convection in a current-carrying cylindrical plasma has been the subject of several studies during the past years. Thus, it has been shown that for magnetic fields satisfying $(B_\theta/B_z)\gg 1$, nonideal effects, such as viscosity and thermal conductivity lead to a physical mode characterized by $k_\parallel=0$ which, for a given value of thermal conductivity and perpendicular viscosity, triggers large scale steady convection in the plasma [1]. It has also been shown that the combined effect of resistivity and thermal conductivity can also lead to steady convection. In this case, convection occurs when either $(B_\theta/B_z)\gg 1$ or when $(B_\theta/B_z)\ll 1$. Otherwise, convection takes place for large azimuthal wave numbers [2]. When all three nonideal effects are considered, namely, thermal conductivity, resistivity, and viscosity, there are four states which can lead to large scale steady convection [3]. It has also been shown that when Hall currents are taken into account, the stability properties are drastically changed and convection is still possible under some conditions [4,5]. In these studies, the system was assumed to consist of a cylindrical plasma column limited by perfectly conducting walls.

Recently, with the purpose of studying more realistic systems, like a θ or a z pinch, or a screw pinch with uniform longitudinal magnetic field, where the obtained magnetic and density profiles are more similar to those of the present model, a cylindrical plasma column with a free boundary was considered. It was shown that thermal conductivity and resistivity also lead to convection [6]. If convection can be achieved, it can lead to stable configurations which, in the context of nonlinear theory, are usually thought to give rise to turbulence. Moreover, it is clear that stationary convection can lead to anomalous transport effects.

Here we shall show that the combined effect of thermal conductivity and viscosity can also lead to convection when the plasma column is surrounded by vacuum. The conditions for convection are identical to the case when the plasma is bounded by conducting walls, even though the boundary conditions are different.

The basic equations are

$$\rho\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\mathbf{v}=\frac{1}{c}\mathbf{j}\times\mathbf{B}-\nabla p-u_\perp\nabla\times(\nabla\times\mathbf{v}) , \quad (1)$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{v})=0 , \quad (2)$$

$$\frac{\partial p}{\partial t}+\mathbf{v}\cdot\nabla p-\tfrac{2}{3}\kappa\nabla^2 p-S_0=-\gamma p\nabla\cdot\mathbf{v} , \quad (3)$$

$$\mathbf{E}+\frac{1}{c}\mathbf{v}\times\mathbf{B}=\mathbf{0} , \quad (4)$$

$$\nabla\cdot\mathbf{B}=0 , \quad (5)$$

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{j} , \quad (6)$$

$$\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E} , \quad (7)$$

where $\mu_\perp$ is the perpendicular viscosity [7], κ is the thermal conductivity, and S_0 is a constant heat source which maintains the equilibrium pressure profile.

The system consists of a cylindrical current-carrying plasma of radius a surrounded by vacuum. The equilibrium is characterized by a magnetic field given by

$$\mathbf{B}^p=\left(0,B_I\frac{r}{a}\ ,\ B_0\right) ,\quad \mathbf{B}^v=\left(0,B_I\frac{a}{r}\ ,\ B_0\right) , \quad (8)$$

where p and v stand for plasma and vacuum, respectively, and B_I and B_0 are constants.

The equilibrium velocity is zero, and the equilibrium pressure is given by

$$p^{(0)}=p_0-(B_I^2/4\pi)(r/a)^2 , \quad (9)$$

where p_0 is a constant.

The rotational transform q is constant and, therefore, the magnetic field is shearless:

$$q=\frac{2\pi r B_z^{(0)}}{LB_\theta^{(0)}}=\frac{2\pi a B_0}{LB_I} , \quad (10)$$

where L is the length of the cylinder.

Assuming the density to be nearly constant, $\rho\approx\rho_0$, the motion incompressible, and that all perturbed quantities behave like

$$f^{(1)}(r,\theta,z)=f^{(1)}(r)\exp(im\theta+ikz+\Omega t) , \quad (11)$$

the following equation for the perturbed velocity is obtained:

$$\hat{\Omega}^2+\hat{\Omega}\hat{\mu}_\perp\beta^2a^2+(m-nq)^2]\xi = -a^2\nabla\hat{p}^{(1)}-2i|m-nq|(\xi_\theta\hat{e}_r-\xi_r\hat{e}_\theta) , \quad (12)$$

where

$$\hat{\Omega}=(4\pi a^2\rho_0/B_I^2)^{1/2}\Omega ,$$

$$\xi=(1/\Omega)\mathbf{v}^{(1)} ,$$

$$\hat{\mu}_\perp=(4\pi/\rho_0 B_I^2a^2)^{1/2}\mu_\perp ,$$

$$\nabla\times\mathbf{B}^{(1)}=\beta\mathbf{B}^{(1)} ,$$

and

$$\hat{p}^{(1)}=(4\pi/B_I^2)[p^{(1)}+(\mathbf{B}\cdot\mathbf{B})^{(1)}/8\pi]$$

is the total perturbed pressure.

Taking the divergence of ξ by using its components, and setting it equal to zero yields:

$$\nabla^2\hat{p}^{(1)}+k^2\sigma^2\hat{p}^{(1)}=0 , \quad (13)$$

where α is a constant.

This is Bessel's equation whose regular solution at the axis of the cylinder, $r=0$, is

$$\hat{p}^{(1)}=\alpha J_m[k(\sigma^2-1)^{1/2}r] , \quad (14)$$

where

$$\sigma=2(m-nq)/[\hat{\Omega}^2+\hat{\Omega}\hat{\mu}_\perp\beta^2a^2+(m-nq)^2] , \quad (15)$$

$n=-kL/2\pi$, and $\beta=k\sigma$.

The perturbed magnetic field in the plasma is given by

$$\mathbf{B}^{(1)}=\frac{iB_I}{a}(m-nq)\xi . \quad (16)$$

The value of β can be obtained by taking the curl of Eq. (12) and using the last equation. The result is $\beta=k\sigma$.

On the other hand, the perturbed magnetic field in the vacuum region is given by [6]

$$B_r^{(1)}=AkK_m'(kr) , \quad (17)$$

$$B_\theta^{(1)}=\frac{imK_m}{krK_m'}B_r^{(1)} , \quad (18)$$

$$B_z^{(1)}=\frac{iK_m}{K_m'}B_r^{(1)} , \quad (19)$$

where A is a constant, K_m are Bessel's functions of the second kind, and K_m' is the first derivative of K_m.

At the plasma-vacuum boundary, the following condition must be satisfied [6]:

$$\tilde{p}^{(1)}(a)+\tilde{p}^{(0)}(a+\xi_r) = \frac{1}{8\pi}[\mathbf{B}^{(0)}(a+\xi_r)+\mathbf{B}^{(1)}(a)]\cdot[\mathbf{B}^{(0)}(a+\xi_r) +\mathbf{B}^{(1)}(a)]|^v , \quad (20)$$

where

$$\tilde{p}=p+\frac{\mathbf{B}\cdot\mathbf{B}}{8\pi} \quad (21)$$

is the total pressure in the plasma.

From Eq. (21) it follows that

$$\tilde{p}(a)=\frac{1}{4\pi}\left[\mathbf{B}^{(0)}\cdot\frac{\partial\mathbf{B}^{(0)}}{\partial r}\bigg|_{r=a}^{v}\xi_r(a)-4\pi\frac{\partial\tilde{p}^{(0)}}{\partial r}\xi_r(a) +\mathbf{B}^{(0)}\cdot\mathbf{B}^{(1)}|^v\right] . \quad (22)$$

Using Eqs. (8) and (9), the last equation reduces to

$$\tilde{p}^{(1)}(a)=\frac{1}{4\pi}(\mathbf{B}^{(0)}\cdot\mathbf{B}^{(1)})|^v . \quad (23)$$

Since the tangential components of the electric field at the plasma boundary must be equal, i.e.,

$$E_{tg}^p=E_{tg}^v|_{r=a} , \quad (24)$$

from Eqs. (4) and (17), it follows that

$$B_r^{(1)}|^v=\frac{iB_I}{a}(m-nq)\xi_r . \quad (25)$$

Therefore, Eq. (23) reduces to

$$\tilde{p}^{(1)}(a)=-\frac{B_I^2}{4\pi a^2}\frac{K_m}{kK_m'}(m-nq)^2\xi_r(a) . \quad (26)$$

Noting that

$$\hat{p}^{(1)}(a)=\frac{4\pi}{B_I^2}\tilde{p}^{(1)}(a) , \quad (27)$$

and using Eqs. (12) and (14), Eq. (27) reduces to

$$\frac{\sigma}{\sigma^2-1}\left[\frac{ka(\sigma^2-1)^{1/2}J_m'}{J_m}+m\sigma\right]=-\frac{2kaK_m'}{(m-nq)K_m} , \quad (28)$$

where J_m' is the first derivatives of J_m.

Equation (28) is the boundary condition to be satisfied at the plasma-vacuum boundary. In general, the last equation is satisfied for some $\sigma=\sigma_0$. For such value, $ka(\sigma_0^2-1)^{1/2}=x_0$. Before solving for σ_0, we shall discuss the dispersion relation.

By setting $\sigma=\sigma_0$ in Eq. (15), it follows that the dispersion relation is given by

$$\hat{\Omega}=-\tfrac{1}{2}\hat{\mu}_\perp k^2\sigma_0^2a^2+\left[\frac{1}{4}\hat{\mu}_\perp^4k^4\sigma_0^4a^4+\frac{2}{\sigma_0}|m-nq| -(m-nq)^2\right]^{1/2} . \quad (29)$$

Note that the dispersion relation reduces to the well known ideal magnetohydrodynamics dispersion relation when $\mu_\perp=0$ [1,8–11].

From the last equation it follows that there are three states for which $\hat{\Omega}=0$. These states satisfy

$$m=nq , \quad (30)$$

and

$$m = nq_{1,2} \pm \frac{2}{\sigma_0} . \tag{31}$$

The modes given by Eq. (31) have zero perturbed velocity, and, therefore, they cannot lead to convection [1]. However, when resistivity is present these modes have a finite perturbed velocity and have been shown to lead to convection [6].

On the other hand, the modes given by Eq. (30) lead to a finite perturbed velocity [1] and, being both marginal ($Re\Omega=0$) and stationary ($Im\Omega=0$), they can lead to convection [12]. These modes are linear and incompressible solutions of Eqs. (1)–(7), provided that γ in Eq. (3) is infinite [1]. Under some conditions, however, the modes given by Eq. (30) satisfy Eq. (3) for arbitrary finite γ. In Ref. [1], it was shown that this happens for large wave numbers, $ka \gg 1$, and

$$\hat{\mu}_1 \kappa = -\frac{3}{k^4 a} \frac{dp^{(0)}}{dr}\bigg|_{r=a} . \tag{32}$$

From the condition that $ka(\sigma_0^2 - 1)^{1/2}$ be equal to some fixed value, x_0, it follows that

$$\sigma_0^2 = 1 + \frac{x_0^2}{k^2 a^2} , \tag{33}$$

which for large wave numbers reduces to $\sigma_0^2 \rightarrow 1$.

We shall now show that the boundary condition given by Eq. (28) is satisfied for $\sigma_0^2=1$. In fact, from Eq. (28) it follows that for $nq \rightarrow m$, $\sigma_0^2 \rightarrow 1$. This means that $\sigma_0 = \pm 1$, just like in the case when the plasma column is surrounded by perfectly conducting walls [1]. Hence, the properties of stationary convection of a current-carrying cylindrical plasma under the action of a longitudinal magnetic field are the same in both cases. Note that also in the present case, namely, when the plasma is surrounded by vacuum, $v_r(r=a)=v_\theta(r=a)=0$ [1]. In fact, from Eq. (12) it follows that $\xi_r(r=a)=\xi_\theta(r=a)=0$ when

$$1-\sigma = \frac{x_0 J_{m-1}}{m J_m} , \tag{34}$$

and

$$\frac{\sigma - 1}{\sigma} = \frac{x_0 J_{m-1}}{J_m} , \tag{35}$$

respectively. It is clear that the last equations are satisfied for $\sigma = \pm 1$. In particular, for $\sigma=1$, $x_0 = Z_{m-1}$ where Z_{m-1} is the first zero of J_{m-1} [1].

To sum up, we have shown that the set of Eqs. (1)–(7) has stable stationary solutions characterized by $nq=m$. These modes are incompressible and satisfy Eq. (3) for arbitrary finite γ, provided that $\hat{\mu}_1\kappa = -(3/k^4 a) dp^{(0)}/dr|_{r=a}$. The latter condition is fulfilled for large wave numbers, $ka \gg 1$. This situation is identical to the one encountered when the plasma column is bounded by perfectly conducting walls [1]. On the other hand, when the effect of resistivity is considered, the modes given by Eq. (31) have a finite perturbed velocity and when $\eta/\kappa = 8\pi/3$, where η is the resistivity, they have been shown to lead to plasma convection [6]. Finally, it is interesting to note that, as far as convection is concerned, resistivity and viscosity seem to play equivalent roles with one acting as the inverse of the other. Resistivity controls convection at the edge of the unstable spectrum whereas viscosity controls the central region of the spectrum [3].

ACKNOWLEDGMENTS

This work has been supported in part by Fondo Nacional de Ciencia y Technología (FONDECYT) Grant No. 1940360.

[1] L. Gomberoff and M. Hernández, Phys. Rev. A **27**, 1244 (1983).
[2] L. Gomberoff, Phys. Fluids B **10**, 2126 (1989).
[3] L. Gomberoff, Phys. Rev. A **28**, 3125 (1983).
[4] K. Gomberoff and L. Gomberoff, Phys. Fluids B **4**, 3024 (1992).
[5] C. A. M. dos Santos and R. M. O. Galvao, Phys. Fluids B **4**, 4187 (1992).
[6] L. Gomberoff and K. Gomberoff, Phys. Fluids B **4**, 1438 (1992).
[7] S. I. Braginskii, in *Reviews in Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1965), Vol. 1, p. 205.
[8] V. D. Shafranov, *Plasma Physics and the Problem of Controlled Thermonuclear Research* (Pergamon, New York, 1958), Vol. 2, p. 197.
[9] R. J. Tayler, Proc. Phys. Soc. London Sect. B **70**, 1949 (1957).
[10] J. P. Friedberg, Phys. Fluids **13**, 1812 (1970).
[11] J. P. Goedbloed and H. J. L. Hagebeuk, Phys. Fluids **15**, 1090 (1972).
[12] S. Chandrasekhar, *Hydrodynamics and Hydromagnetic Stability* (Oxford University Press, London, 1961).