Langevin equations for continuous time Lévy flights

Hans C. Fogedby
Institute of Physics and Astronomy, University of Aarhus, 8000 Aarhus C, Denmark
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We consider the combined effects of a power law Lévy step distribution characterized by the step index f and a power law waiting time distribution characterized by the time index g on the long time behavior of a random walker. The main point of our analysis is a formulation in terms of coupled Langevin equations which allows in a natural way for the inclusion of external force fields. In the anomalous case for $f < 2$ and $g < 1$ the dynamic exponent z locks onto the ratio f/g. Drawing on recent results on Lévy flights in the presence of a random force field we also find that this result is *independent* of the presence of weak quenched disorder. For d below the critical dimension $d_c = 2f - 2$ the disorder is *relevant*, corresponding to a nontrivial fixed point for the force correlation function.

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I. INTRODUCTION

Anomalous diffusion arising from nontrivial waiting time distributions, so-called continuous time random walks (CTRW's), has been used to model a variety of physical phenomena [2]. For example, the dynamics of carrier diffusion and recombination in disordered media has been described in terms of a CTRW [3–7]. Anomalous diffusion in an intermittent dynamical system [8,9] and in a linear array of convection cells [10,11] has also been analyzed in terms of CTRW's.

Anomalous diffusion is also associated with power law step size distributions, the so-called Lévy distributions [12–15]. The Lévy flights generated by the Lévy step distribution have been used to model a variety of physical processes such as self-diffusion in micelle systems [16] and transport in heterogeneous rocks [14].

In the present paper we consider the combined effect of a power law waiting time distribution and a power law Lévy step size distribution, i.e., the case of continuous time Lévy flights (CTLF's). Since Lévy flights lead to superdiffusive behavior, whereas a power law waiting time distribution entails subdiffusive behavior, the combination of the two yields an interesting description of anomalous diffusion in the general case [2].

The key issue in the paper is the discussion of CTLF's in terms of coupled Langevin equations for the position and the time. This approach is achieved by introducing as an intermediate variable the path or arc length along a particular trajectory. Besides the clearer physical interpretation afforded by Langevin equations as stochastic equations of motion, such an approach also allows in a natural way for the inclusion of external fields, for example, a drift force field.

In Sec. II we discuss the Langevin equation associated with power law step and waiting time distributions. Section III is devoted to the discussion of the associated Fokker-Planck equations. In Sec. IV we discuss the probability distribution for the position of the walker as a function of time. In Sec. V we carry out a scaling analysis in the absence of an external force field. In Sec. VI we give a discussion and comment briefly on the role of an external quenched force field, drawing on recent results.

II. LANGEVIN EQUATIONS

Let us denote the step size distribution by $\pi(\boldsymbol{\eta})$, where $\boldsymbol{\eta}$ is the elementary microscopic step. We assume that $\pi(\boldsymbol{\eta})$ is normalized, i.e., $\int \pi(\boldsymbol{\eta}) d^d\eta = 1$. Likewise, we denote the waiting time distribution by $w(\tau)$, where $w(\tau)$ is the probability of the walker of waiting the time interval τ at a given position before performing the next step. Due to causality $\tau > 0$ and the normalization of $w(\tau)$ reads $\int_0^\infty w(\tau) d\tau = 1$.

Parametrizing the random walk [1] in the continuum limit by means of the path parameter or arc length s along the trajectory we have, for the position of the walker $\mathbf{r}(s)$, after s "steps,"

$$\mathbf{r}(s) = \int_0^s \boldsymbol{\eta}(s') ds', \qquad (1)$$

or the Langevin equation

$$d\mathbf{r}/ds = \boldsymbol{\eta}(s). \qquad (2)$$

In Fig. 1 we have shown the parametrization of a particular trajectory. Equation (2) is readily generalized in the presence of an external drift force field $\mathbf{F}$ depending on the position of the walker. We obtain, in this case, the Langevin equation

$$d\mathbf{r}/ds = \mathbf{F}(\mathbf{r}) + \boldsymbol{\eta}(s). \qquad (3)$$

Similarly, the total elapsed time after s "steps" in the continuum limit is

$$t(s) = \int_0^s \tau(s') ds', \qquad (4)$$

implying the Langevin equation

$$dt/ds = \tau(s). \qquad (5)$$

The coupled Langevin equations (2), (3), and (5) constitute the present formulation of continuous time random walk. We are considering here the separable case where the step distribution $\pi(\boldsymbol{\eta})$ and the waiting time distribution $w(\tau)$ are statistically independent [2,17,18].

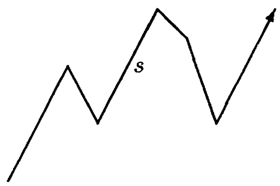

FIG. 1. Plot of a particular random walk parametrized by the arc length s.

In the case of a sharp waiting time distribution $w(\tau) = \delta(\tau - \tau_0)$, corresponding to a fixed hopping rate $1/\tau_0$, Eq. (5) becomes deterministic and can be solved for s, i.e., $t = \tau_0 s$, and we obtain by insertion the usual Langevin equation

$$d\mathbf{r}/dt = (1/\tau_0)\mathbf{F}(\mathbf{r}) + (1/\tau_0)\boldsymbol{\eta}(t). \tag{6}$$

In the general case of a nontrivial waiting time distribution we must, however, discuss the coupled equations.

We shall now focus on power law distributions for the waiting times and steps. For the step $\boldsymbol{\eta}$ we assume an instantly (in terms of s) correlated power law distribution

$$\pi(\boldsymbol{\eta})d^d\eta \propto \eta^{-1-f}d\eta. \tag{7}$$

We assume an isotropic form characterized by the step index f. In order to ensure normalizability we have introduced a lower cut $\eta \sim a$ of the order of a microscopic length a and chosen $f > 0$. For $f > 2$ the second moment $\langle\eta^2\rangle = \int \pi(\boldsymbol{\eta})\eta^2 d^d\eta$ is finite and a characteristic step size is given by the root mean square deviation $\sqrt{\langle\eta^2\rangle}$. For $1 < f < 2$ the second moment diverges, but the mean step $\langle\eta\rangle$ is finite. In the interval $0 < f < 1$ the first moment diverges and even a mean step size is not defined [12]. In a similar way we assume for the waiting time τ the power law distribution

$$w(\tau)d\tau \propto \tau^{-1-g}d\tau. \tag{8}$$

Owing to causality $\tau > 0$. In order to guarantee normalizability we introduce a short time cutoff of the order of a microscopic time scale and choose $g > 0$. The distribution is characterized by the time index g. For $g > 1$ the first moment $\langle\tau\rangle = \int w(\tau)\tau d\tau$ is finite, setting a well defined hopping rate $1/\langle\tau\rangle$. For $0 < g < 1$ the mean value $\langle\tau\rangle$ diverges and we cannot define a characteristic rate or time scale.

In order to eventually discuss the anomalous diffusive characteristics of a random walk driven by the power law noises $\boldsymbol{\eta}$ and τ we must determine the distribution function $P(\mathbf{r}, t)$ and, in particular, the mean square displacement

$$\langle r^2(t)\rangle \propto Dt^{2/z}, \tag{9}$$

where D is the diffusion coefficient and z the dynamic exponent. Since the random walk takes place in physical time t we are thus faced with the issue of eliminating the auxiliary path variable s labeling the walk. Owing to the stochastic nature of the Langevin equation (5) for t it is not possible to invert it and solve it for s and we shall instead turn to the associated deterministic Fokker-Planck equations.

III. FOKKER-PLANCK EQUATIONS

Limiting first the discussion to the force free case, i.e., $\mathbf{F} = \mathbf{0}$, the probability distributions $P_1(\mathbf{r}, s)$ and $P_2(t, s)$ associated with the Langevin equations (2) and (5) are easily inferred [19,20]. From the definition $P_1(\mathbf{r}, s) = \langle\delta(\mathbf{r} - \mathbf{r}(s))\rangle$ and $P_2(t, s) = \langle\delta(t - t(s))\rangle$, Eqs. (1) and (4), and averaging according to the power law noises in Eqs. (7) and (8), we deduce the scaling forms [2]

$$P_1(\mathbf{r}, s) = \int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r} - k^\mu s} = s^{-d/\mu} G_1\left(\frac{r}{s^{1/\mu}}\right), \tag{10}$$

$$P_2(t, s) = \int \frac{d\omega}{(2\pi)} e^{-i\omega t - (-i\omega)^\nu s} = s^{-1/\nu} G_2\left(\frac{t}{s^{1/\nu}}\right). \tag{11}$$

By choosing a suitable scale for t and s we have fixed the coefficients of k^μ and $(-i\omega)^\nu$ to be unity.

The scaling exponents μ and ν depend on the step index f and time index g, respectively, characterizing the power law step size and waiting time distributions in Eqs. (7) and (8). For $f > 2$, i.e., the case of a finite mean square step, μ locks onto the value 2 and the scaling function G_1 takes the Gaussian form characteristic of ordinary Brownian walk, $G_1(x) = \exp(-x^2)$. This is a consequence of the central limit theorem which here leads to universal behavior and defines the universality class of Brownian motion. For $f < 2$ the scaling exponent $\mu = f$ and the scaling function G_1 can only be given explicitly in terms of known functions for $\mu = 1$ and $\mu = 1/2$ (the Cauchy and Smirnov distributions, respectively, [17]). It is, however, easy to show that $G_1 \to \text{const}$ for $x \to 0$ and $G_1 \to 0$ for $x \to \infty$. From the distribution in Eq. (10) we infer the scaling form for the mean square displacement of the walker in terms of the path variable s,

$$\langle r^2(s)\rangle = \int P_1(\mathbf{r}, s) r^2 d^dr \propto s^{2/\mu}. \tag{12}$$

A similar discussion applies to $P_2(t, s)$. For $g > 1$, i.e., the case of a finite first moment, corresponding to a well defined hopping rate, the scaling exponent ν locks onto 1 and the scaling function $G_2(x) = \delta(1 - x)$. This is again a consequence of the central limit theorem which leads to a universal time behavior, i.e., a hopping rate. For $g < 1$ the scaling exponent ν locks onto g, i.e., $\nu = g$, and we obtain a nontrivial scaling behavior. For the mean square displacement in terms of the path variable we have

$$\langle t^2(s)\rangle = \int P_2(t, s) t^2 dt \propto s^{2/\nu}. \tag{13}$$

In Fig. 2 we have depicted the scaling exponents μ and ν as functions of the step index f and the waiting time index g, respectively.

A simple heuristic argument [2] using Eq. (13) to infer that t scales like $s^{1/\nu}$ and eliminating s in Eq. (12) yields the scaling relation

$$\langle r^2(t)\rangle \propto t^{2\nu/\mu} \tag{14}$$

and according to Eq. (9) the dynamic exponent

$$z = \mu/\nu. \tag{15}$$

As we shall discuss in Sec. V this is indeed the correct result following from the scaling law for $P(\mathbf{r},t)$,

$$P(\mathbf{r},t) \equiv t^{-\frac{d\nu}{\mu}} G\left(r/t^{\frac{\nu}{\mu}}\right). \tag{16}$$

We conclude this section by writing down the Fokker-Planck equations following from Eqs. (10) and (11). Introducing the "fractional" nonlocal differential operators

$$\nabla^\mu = -\int \frac{d^dk}{(2\pi)^d} e^{i\mathbf{k}\cdot\mathbf{r}} k^\mu, \tag{17}$$

$$D^\nu = -\int \frac{d\omega}{2\pi} e^{-i\omega t}(-i\omega)^\nu \tag{18}$$

reflecting the long range Lévy steps and waiting times, we have

$$\partial P_1(\mathbf{r},s)/\partial s = \nabla^\mu P_1(\mathbf{r},s), \tag{19}$$

$$\partial P_2(t,s)/\partial s = D^\nu P_2(t,s). \tag{20}$$

In the Brownian case $\mu = 2$ and $\nu = 1$ and ∇^μ reduces to the usual Laplace operator Δ describing ordinary diffusion, whereas D^ν becomes the first order differential operator $-\partial/\partial t$.

In the presence of a force field $\mathbf{F}(\mathbf{r})$ we have, correspondingly, [19]

$$\partial P_1(\mathbf{r},s)/\partial s = -\nabla(\mathbf{F}(\mathbf{r})P_1(\mathbf{r},s)) + \nabla^\mu P_1(\mathbf{r},s). \tag{21}$$

Here the first term on the right hand side of Eq. (21) is the usual drift term due to the motion of the walker in the force field.

IV. THE DISTRIBUTION $P(\mathbf{r},t)$

In order to calculate the probability distribution for the walker as a function of the physical time t we must eliminate the path variable s. In other words, we have to derive the distribution $P_3(s,t)$ since it then follows that $P(\mathbf{r},t)$ is given by the relationship

$$P(\mathbf{r},t) = \int_o^\infty ds P_1(\mathbf{r},s)P_3(s,t), \tag{22}$$

due to the fact that the probability of the walker arriving at $\mathbf{r}$ in time t equals the probability of being at s on the path at time t multiplied by the probability of being at position $\mathbf{r}$ for this path length s, summed over all path lengths.

In order to derive $P_3(s,t)$ we use the general expression for $P(\mathbf{r},t)$ for arbitrary step and waiting time distributions given by Montroll and Weiss [21]. In Fourier space we have

$$P(\mathbf{k},\omega) = (-i\omega)^{-1}\{[1-w(\omega)]/1 - w(\omega)\pi(\mathbf{k})\}, \tag{23}$$

where $\pi(\mathbf{k})$ and $w(\omega)$ are the Fourier transforms of the step and waiting time distributions, i.e.,

$$\pi(\mathbf{k}) = \int e^{-i\mathbf{k}\cdot\boldsymbol{\eta}}\pi(\boldsymbol{\eta})d^d\eta, \tag{24}$$

$$w(\omega) = \int e^{i\omega\tau}w(\tau)d\tau. \tag{25}$$

We notice that since $w(\tau) = 0$ for $\tau < 0$ due to causality Eq. (25) reduces to a Laplace transform and ω is defined along a contour parallel to the x axis in the upper half complex ω plane.

For the power law distributions in Eqs. (7) and (8) we obtain in particular $\pi(\mathbf{k}) = 1 - k^\mu$ for small k and $w(\omega) = 1 - (-i\omega)^\nu$ for small ω, i.e., to leading order in ω and $\mathbf{k}$, the distribution

$$P(\mathbf{k},\omega) = (-i\omega)^{\nu-1}/(-i\omega)^\nu + k^\mu. \tag{26}$$

Inserting Eq. (10) for $P_1(\mathbf{r},s)$ in Eq. (22) and requiring that $P(\mathbf{r},t)$ is the Fourier transform of Eq. (26) it is easy to demonstrate that $P_3(s,t)$ is given by [2]

$$P_3(s,t) = \int (d\omega/2\pi)e^{-i\omega t}(-i\omega)^{\nu-1}e^{-(-i\omega)^\nu s}. \tag{27}$$

We have not found a simple physical argument leading to Eq. (27), i.e., the "inversion" of Eq. (11), but notice that $\partial P_3/\partial t = \partial P_2/\partial s$.

V. SCALING ANALYSIS

In the absence of the force field, i.e., for $\mathbf{F} = 0$, it is an easy task to carry out a scaling analysis. The results are most easily deduced from Eq. (26). We have

$$P(\mathbf{r},t) = \int \frac{d\omega}{2\pi}\frac{d^dk}{(2\pi)^d} e^{-i\omega t + i\mathbf{k}\mathbf{r}} \frac{(-i\omega)^{\nu-1}}{(-i\omega)^\nu + k^\mu}, \tag{28}$$

leading to the scaling form in Eq. (16) with scaling function

$$G(x) = x^{-d}\int \frac{d\omega}{2\pi}\frac{d^dk}{(2\pi)^d}\frac{e^{-i\omega+i\mathbf{k}}}{(-i\omega)[1+x^{-\mu}k^\mu/(-i\omega)^\nu]}. \tag{29}$$

The mean square displacement is given by

$$\langle r^2(t)\rangle = \int d^d r P(\mathbf{r},t)r^2 \propto t^{2/z} \tag{30}$$

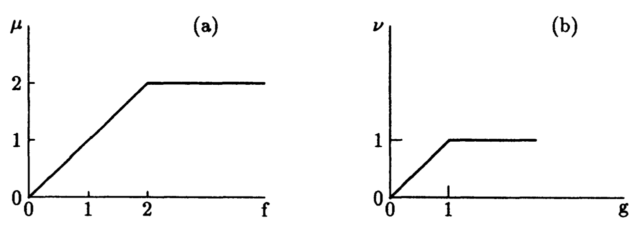

FIG. 2. (a) μ as a function of f. For $f > 2$ the exponent $\mu = 2$ and we have an ordinary Brownian walk; for $f < 2$ the exponent $\mu = f$ and we have Lévy flights, leading to anomalous superdiffusion. (b) ν as a function of g. For $g > 1$ the exponent $\nu = 1$ and we have a well defined hopping rate for the walker; for $g < 1$ we have $\nu = g$ and we obtain a subdiffusive behavior.

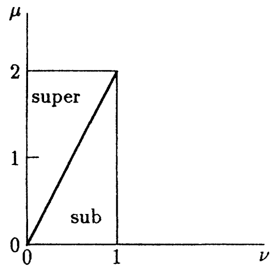

FIG. 3. Plot of μ versus ν. Along the solid line $\mu = 2\nu$ we have the universality class of ordinary Brownian motion with $z = 2$. In the region $\mu > 2\nu$ we have the universality class of anomalous superdiffusion; in the region $\mu < 2\nu$ we have the class of anomalous subdiffusion.

and we find the dynamic exponent $z = \mu/\nu$ in agreement with the heuristic result in Eq. (15).

Along the line $\mu = 2\nu$ we have $z = 2$ and CTLF's have the same scaling characteristics as ordinary Brownian motion. The superdiffusive behavior induced by the long range Lévy steps is precisely balanced by the long waiting times. We notice, however, that the scaling function G depends on the scaling index $\mu = 2\nu$; only in the case $\mu = 2$ and $\nu = 1$ do we obtain the Gaussian distribution.

For $\mu > 2\nu$ we have $z > 2$, the Lévy flights prevail and we obtain superdiffusive behavior; correspondingly, $\mu < 2\nu$ implies $z < 2$, the long waiting times dominate, and we have subdiffusive behavior.

In Fig. 3 we have shown the different universality classes for CTLF. For $z = 2$ ($\mu = 2\nu$) we have the universality class of ordinary Brownian motion. For $\mu > 2\nu$ we obtain the universality class (or classes) of anomalous superdiffusion with an exponent z depending continuously on the ratio of the microscopic exponents f and g. Similarly, for $\mu < 2\nu$ we have the universality class of anomalous subdiffusion.

VI. DISCUSSION AND CONCLUSION

In the present paper we have discussed the combined effects of an algebraic waiting time distribution and an algebraic Lévy type step distribution on the motion of a random walker. In order to include external force fields we have formulated this analysis in terms of a set of coupled Langevin equations. In the absence of force fields a simple scaling analysis shows that the dynamic exponent z characterizing the long time behavior of the mean square displacement is given by the ratio $z = \mu/\nu$, where the scaling exponents μ and ν are related to the microscopic step and waiting time exponents ($\mu = f$ for $f < 2$ and $\nu = g$ for $g < 1$). This dependence defines three universality classes: (i) normal diffusive behavior for $\mu = 2\nu$, (ii) anomalous superdiffusion for $\mu > 2\nu$, and (iii) anomalous subdiffusion for $\mu < 2\nu$.

In the presence of a quenched (time independent) Gaussian random force field and a well defined hopping rate, i.e., for $\nu = 1$, we have recently shown [19] that z locks onto μ for $f < 2$. It is now easy to generalize this result to the case of a nontrivial waiting time distribution with $\nu < 1$. Since the force field is independent of t we can directly apply the discussion in Ref. [19] to Eq. (3), treating the path variable s as the effective time. We conclude that in this case z is also given by μ/ν for weak quenched disorder. The analysis in Ref. [19], leading to the critical dimension $d_c = 2\mu - 2$ below which a nontrivial force correlation fixed point emerges, can also be carried over to the present case. From a physical point of view it is clear that a nontrivial waiting time distribution has no effect since the quenched force field acting at position r simply "waits" until the walker arrives. However, in the case of a time dependent random force field the waiting time distribution will interfere with the temporal force correlations and we have to treat the coupled Langevin equations in order to eliminate the intermediate path variable s. This interesting case will be considered in a forthcoming paper [22].

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