Renormalization-group theory of plasma microturbulence

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The dynamical renormalization-group methods are applied to the gyrokinetic equation describing drift-wave turbulence in plasmas. As in both magnetohydrodynamic and neutral turbulence, small-scale fluctuations appear to act as effective dissipative processes on large-scale phenomena. A linear renormalized gyrokinetic equation is derived. No artificial forcing is introduced into the equations and all the renormalized corrections are expressed in terms of the fluctuating electric potential. The link with the quasilinear limit and the direct interaction approximation is investigated. Simple analytical expressions for the anomalous transport coefficients are derived by using the linear renormalized gyrokinetic equation. Examples show that both quasilinear and Bohm scalings can be recovered depending on the spectral amplitude of the electric potential fluctuations.

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I. INTRODUCTION

Among the theoretical difficulties in the study of the macroscopic plasma behavior in a turbulent state the understanding of transport laws for matter, charge, and energy plays a very important role. Indeed, any new knowledge concerning these transport processes would be a step of outstanding importance in the determination of the experimental conditions needed to perfom controlled thermonuclear fusion in a magnetically confined plasma. This explains the large amount of works devoted to the theory or, more precisely, to the theories of transport in plasmas.

Probably the first self-consistent approach to transport phenomena in plasma is the *classical transport theory* [1,2]. Starting directly from a kinetic equation, it leads to values for the transport coefficients which only depend on the characteristics of the particles composing the plasma and on both ionic and electronic temperatures. It rests on a linear approximation in which the deviation of the local distribution function from the reference state, taken as the Maxwellian velocity distribution, is expanded in series of appropriate moments of the velocity (two common choices are the Laguerre-Sonine polynomials and the irreducible tensorial Hermite polynomials). It has been shown [3] that the results of the classical transport theory are rather insensitive to the choice of the kinetic equation (Boltzmann, Landau, Balescu-Lenard). Moreover, the transport coefficients may be evaluated with a very good accuracy by taking into account only a small number of moments in the expansion (typically 21 in the case of the Hermite polynomials). In the study of confined plasmas, the results of the classical transport theory have been considerably improved by taking into account the toroidal geometry of the magnetic field. The influence of such a geometry on the transport coefficients appears surprisingly important. It has been systematically studied in the *neoclassical transport theory* [4,5].

However, both classical and neoclassical transport theories do not describe the complete physics of transport phenomena in magnetically confined plasmas. The transport in these theories is considered as being entirely described by the binary collisions between plasma particles. Moreover, the distribution functions are assumed to be very close to the Maxwellian distribution. Such assumptions are well adapted to the study of the relaxing collisional modes in the plasmas. Unfortunately, the plasma dynamics is so complex that it can lead to a large number of instabilities usually characterized by wave motions. Such waves, their appearance, and their possible relaxation, cannot be described by binary collisions and are more likely related to collective phenomena. Moreover, the instabilities bring the system far from the equilibrium state in such a way that the basic assumptions of classical and neoclassical transport theories are violated. In this case, the transport originates essentially from the collective phenomena described as the *anomalous transport* [6] and the plasma is said to be in a turbulent state. In the same way that a definitive theory of turbulence in neutral fluids has not yet been achieved, there exists no systematic theory for the anomalous transport in plasmas.

One of the main properties of turbulence in both plasmas and neutral fluids is the wide range of spatial scales characterizing the phenomena. Consequently, the theoretical difficulty in describing turbulent phenomena consists in dealing with a large number of variables. From a practical point of view, not all of these variables are relevant. In many turbulence experiments, some important results are expressed in terms of averaged quantities that do not exhibit very rapid spatial variations. An averaged description of turbulence is thus required, keeping in mind that small-scale phenomena may have a strong influence on that description. Unfortunately, although this problem can be stated quite clearly, it will appear very difficult. This explains the large number of theories and approximations proposed for describing turbulence. Many of these theories have been first developed to study

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neutral fluids before being used as a guide for the investigation of plasma turbulence. For instance, the direct interaction approximation [7,8] (DIA) and the Markovian approximations [9] derived to investigate the statistical properties of solutions of the Navier-Stokes equation, have been applied to the study of plasma turbulence for a long time [10–12].

More recently, the dynamical renormalization-group (RNG) method, developed by Ma and Mazenko [13] for the study of ferromagnetic phase transitions, has been extensively used for describing the large-scale and long-time properties of the turbulent regime of neutral fluids [14]. By using RNG methods, Fournier and Frisch [15] and Yakhot and Orszag [16] have obtained interesting universal results on the statistical properties of forced hydrodynamic turbulence. Quantities such as the energy spectrum and the effective transport coefficients have been evaluated with a certain success. In this paper, our goal is to apply the same method to the drift-wave microturbulence in plasmas.

The instabilities observed in plasma physics are much more diverse than in hydrodynamics. According to the importance of collisional processes, one distinguishes in magnetized plasmas the macroturbulence and the microturbulence. The macroturbulence is generated by the magnetohydrodynamic (MHD) instabilities that appear in the collision-dominated plasmas. The set of equations describing these phenomena consists of the hydrodynamical balance equations coupled with the Maxwell equations. The applications of the RNG to plasma turbulence have usually been restricted to these equations [17–19] and therefore to the description of macroturbulence. Nevertheless, if the electric field generated by the plasma has a significant influence on its dynamics, another kind of turbulence will be observed: the microturbulence generated by the drift-wave instability. The presence of drift waves in a plasma is closely related to the existence of a diamagnetic drift current. Such a current arises as a consequence of both the magnetic field and the spatial dependence of plasma properties like the density and the temperature. Its direction is mutually perpendicular to the magnetic field and the gradient. The drift waves are collective oscillations propagating along the diamagnetic current. If these waves become important, the plasma will reach the so-called drift turbulent state. Such phenomena are usually described by the gyrokinetic equation (GKE).

To our knowledge, the only application of the RNG to microturbulence has been developed by Hamza and Sudan [20] in the study of the simple Hasegawa-Mima model [21]. Such instabilities are usually described by the velocity distribution function. We are then confronted with the very complex problem of the well known Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy [2]. Fortunately, in the case of the weakly collisional plasmas (that is considered here), this hierarchy can be reduced to the Vlasov equation. The velocity dependence of the distribution function leads, however, to some complications when compared to the MHD equations. This could explain why the RNG study of microturbulence has not been widely developed.

Let us stress an important peculiarity of the present application of the RNG. Here, the turbulence is assumed to be driven by the fluctuations of the electric potential. There is no need in the theory of any artificial random noise term. This is a very convenient aspect of the present approach. Indeed, this means that, unlike the previous applications of the RNG to both neutral fluids and plasmas, it is possible to study the universal properties of turbulence without introducing unphysical quantities like artificial random sources into the equations. However, in this RNG application to the GKE, we are treating a non-self-consistent problem in which the statistics of the electric field is assumed to be known.

The final goal of this work is to derive the transport laws in this drift-wave turbulent regime. We start from the renormalized gyrokinetic equation (RGKE). We derive explicit relations between the gradients of temperatures and density, and the macroscopic fluxes of matter and energy. From these relations, the tensor of transport coefficients can be obtained.

In Sec. II, we discuss some approximations which are needed to derive the GKE. This evolution equation for the velocity distribution will be the starting point of our analysis of the drift-wave turbulence in plasmas. In Sec. III, we present the main ideas of the dynamical renormalization group. This theory is applied to the GKE and the renormalization of the linear terms is entered into in some detail. We show in Sec. IV that the general results derived from the renormalization technique are compatible with some other theories. Particularly, both the quasilinear and the DIA limits are investigated. General expressions for both the fluxes and the transport coefficients are given in Sec. V. These expressions can be written in a particularly simple form that is not explicitly dependent on the spectrum of the electric potential fluctuations. Two simple examples are treated in more detail in Sec. VI. We show that both quasilinear and Bohm scaling for the anomalous diffusion can be recovered. Finally, some general consequences of the results are discussed in the last section.

II. THE GYROKINETIC EQUATION

The main motivation of the present work is the investigation of the anomalous transport in strongly magnetized plasmas. In a first approximation, such transport phenomena are assumed to originate essentially from wave interactions in which collisions between particles can be neglected. In the present paper, we consider a plasma consisting of electrons (mass m_e, charge $e_e \equiv -e$) and a single species of singly charged ions (mass m_i, charge $e_i \equiv +e$) in the presence of a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$. In the ordinary kinetic theory, the microscopic state of the plasma is completely specified by the knowledge of the one-particle momentum distribution function of each species $f^\alpha(\mathbf{r},\mathbf{p};t)$ (α takes two possible values, e for the electrons and i for ions). The momentum distribution function gives the density probability to find at a given point $(\mathbf{r},t)$ of the space-time a particle with a momentum $\mathbf{p}$. The Vlasov equation for $f^\alpha(\mathbf{r},\mathbf{p};t)$ has the following general form:

$$\frac{\partial f^\alpha(\mathbf{r},\mathbf{p};t)}{\partial t} = [H^\alpha(\mathbf{r},\mathbf{p};t), f^\alpha(\mathbf{r},\mathbf{p};t)]_p, \quad (1)$$

where $H^\alpha(\mathbf{r},\mathbf{p};t)$ is the Hamiltonian of a particle of species α moving in the self-consistent electromagnetic field created by all the other particles, as well as by the external sources. The bracket $[\ ,\ ]_p$ denotes the Poisson bracket. We are interested here in the more particular but also very important electrostatic drift-wave instabilities. In this case, the GKE is more suitable than the general Vlasov equation. The GKE can be derived from the latter by making a systematic expansion based on the drift approximation in nonrelativistic plasmas [22,23]. It is not the purpose of this paper to describe in detail the derivation of the GKE so we focus on the main ideas of the drift approximation. Let us introduce some notations. The Larmor radius of a particle of species α whose velocity component perpendicular to $\mathbf{B}$, briefly called $v_\perp$, is given by

$$\rho_\alpha = \frac{v_\perp}{\Omega_\alpha}, \quad (2)$$

where $\Omega_\alpha = e_\alpha B/m_\alpha c$ is the Larmor frequency of species α in a magnetic field of intensity B and where c is the speed of light. We also introduce another typical length of the plasma: $L_B = B/\parallel \nabla B \parallel$. It represents the characteristic gradient length associated with the magnetic field geometry. With these definitions, the drift approximation can be expressed in the following forms:

$$\hat{\epsilon}= \frac{\rho_\alpha}{L_B} \ll 1,$$
$$B\sim O(\hat{\epsilon}^{-1}). \quad (3)$$

We now consider that the plasma is perturbed away from a local equilibrium state by an electric potential fluctuation $\delta\phi$, whose statistical properties will be specified below. It is convenient to separate the perturbation of the distribution function into an "adiabatic" part and a "nonadiabatic" part δh^α:

$$f^\alpha(\mathbf{r},\mathbf{v};t) = F_0^\alpha - \frac{e_\alpha}{T_\alpha}F_0^\alpha \delta\phi(\mathbf{r};t) + \delta h^\alpha(\mathbf{r},\mathbf{v};t), \quad (4)$$

where the local equilibrium distribution is given by the Maxwellian

$$F_0^\alpha(x,v) = \left(\frac{1}{V_\alpha\sqrt{\pi}}\right)^3 n(x)\exp\left[-\frac{v^2}{V_\alpha^2}\right]. \quad (5)$$

Here $V_\alpha = \sqrt{2T_\alpha/m_\alpha}$ denotes the thermal velocity of species α. We note that with these definitions the density n and the temperatures T^α appear as moments of the equilibrium distribution function:

$$n(x)= \int d\mathbf{v}\ F_0^\alpha(x,v), \quad (6)$$

$$T^\alpha(x)= \frac{1}{n(x)}\int d\mathbf{v}\frac{m_\alpha v^2}{3}\ F_0^\alpha(x,v). \quad (7)$$

In writing the GKE it is usual to use the velocity rather than the momentum in the distribution function. As the distribution (5) corresponds to the local equilibrium, the macroscopic properties are assumed to slowly vary in space. Following the usual "shearless slab geometry model," we assume that the magnetic field is parallel to $\mathbf{1}_z = \mathbf{b}$ and that n_α and T_α slowly depend on a coordinate x mimicking the radial coordinate in a tokamak. Due to the quasineutrality of the plasma, the ionic and electronic densities are almost equal: $n_i(x) = n_e(x) = n(x)$. In Fourier space the GKE can be written as follows:

$$g(\hat{\mathbf{k}},\mathbf{v})^{-1}\delta\overline{h}^\alpha(\hat{\mathbf{k}},\mathbf{v}) + \Delta^\alpha(\hat{\mathbf{k}},\mathbf{v})\ \delta\overline{\phi}(\hat{\mathbf{k}}) = -\lambda\int d\hat{\mathbf{q}}\ \mathbf{k}\cdot(\mathbf{q}\times\mathbf{b})L_q^\alpha(v_\perp)\delta\overline{\phi}(\hat{\mathbf{q}})\delta\overline{h}^\alpha(\hat{\mathbf{k}}-\hat{\mathbf{q}},\mathbf{v}), \quad (8)$$

where

$$\lambda = \frac{c}{B(2\pi)^4}, \quad (9)$$

and $L_q^\alpha(v_\perp)$ denotes the zero-order Bessel function $J_0(q_\perp v_\perp/\Omega_\alpha)$. The overbar $(\overline{X})$ denotes the gyro average of the quantity X. The gyro-averaged quantities only depend on the parallel component $v_\parallel$ and the length of the perpendicular component $v_\perp$, but not on the gyrophase (cylindrical coordinates). The simple notation $\hat{\mathbf{k}}$ will be used to refer to both the wave vector $\mathbf{k}$ and the frequency ω. Let us note that Eq. (8) describes drift waves for which wavelength and frequency are not totally arbitrary. Indeed, it is usually assumed that the drift waves have a low frequency compared to Ω_α $[\omega/\Omega_\alpha = O(\hat{\epsilon})]$, and a large wavelength in the parallel direction $[k_\parallel\rho_\alpha = O(\hat{\epsilon})]$. On the other hand, the wavelength in the perpendicular direction can be small $[k_\perp\rho_\alpha = O(\hat{\epsilon}^0)]$. The propagator in (6) is given by

$$g(\hat{\mathbf{k}},\mathbf{v}) = \left(-i\omega + ik_\parallel v_\parallel\right)^{-1}, \quad (10)$$

and the so-called vertex term $\Delta^\alpha(\hat{\mathbf{k}},\mathbf{v})$ is given by

$$\Delta^\alpha(\hat{\mathbf{k}},\mathbf{v}) = i[\omega - k_y V_{d\alpha}^T(v)]\ \frac{e_\alpha}{T_\alpha}\ F_0^\alpha(x,v)\ L_k^\alpha(v_\perp), \quad (11)$$

where

$$V_{d\alpha}^T(v) = V_{d\alpha}\left[1+\left(\frac{v^2}{V_\alpha^2}-\frac{3}{2}\right)\eta_\alpha\right]. \quad (12)$$

The diamagnetic drift velocity is given by

$$V_{d\alpha} = \frac{cT_\alpha(x)\partial_x n(x)}{e_\alpha B n(x)}, \quad (13)$$

and $\eta_\alpha = d\log T_\alpha/d\log n$. Let us note that the vertex term depends on the space variable x although Eq. (8) is written in Fourier space. This originates in the multiscale approximation used to derive this equation. The space variable in $T_\alpha(x)$ and $n(x)$ is assumed to refer to the slow spatial dependence in the direction $\mathbf{1}_x$ which is perpendicular to the magnetic surface. On the other hand, the space variable in the nonadiabatic distribution function refers to very rapid spatial variations. The independence of these variables is assumed by making the local approximation [24]. The GKE is a dynamical relation between the nonadiabatic distribution function and the fluctuating electric potential. In a fully self-consistent approach, it should be completed by the Poisson equation which, in the gyrokinetic ordering, degenerates into the quasineutrality equation $\delta n_e = \delta n_i$. In the present approach, we are not using this self-consistent treatment but rather the stochastic acceleration of test particles in a given fluctuating field. In this renormalization-group study of the GKE, we shall therefore consider that the statistical properties of the fluctuating electric potential are completely known. Particularly, we assume that $\delta\phi$ is a Gaussian noise so that its statistics is determined by the knowledge of its average and its variance. As we consider only the fluctuations of the electric potential, the average of $\delta\phi$ clearly vanishes, so we only need to specify the two-point correlations

$$\langle \delta\overline{\phi}(\hat{\mathbf{k}})\delta\overline{\phi}(\hat{\mathbf{q}})\rangle = S(k_{\parallel}, k_{\perp}; \omega)\delta(\hat{\mathbf{k}}+\hat{\mathbf{q}}). \tag{14}$$

The Dirac distribution $\delta(\hat{\mathbf{k}}+\hat{\mathbf{q}})$ in (14) reflects the fact that we are limiting our scope to homogeneous and stationary turbulence. In other words, we are assuming that the correlations of the electric potential $\langle \delta\overline{\phi}(\mathbf{r},t)\delta\overline{\phi}(\mathbf{r}',t')\rangle$ only depend on the differences in space $(\mathbf{r}-\mathbf{r}')$ and in time $(t-t')$. Let us stress that the assumption of stationary and homogeneous turbulence does not preclude time and space variations of the macroscopic properties of the plasma. We only assume that properties like homogeneity and stationarity, possibly broken at large scales, are statistically recovered at small scales. We are showing in the following section how the RNG can be applied to the gyrokinetic equation forced by such an electric potential (14).

III. THE DYNAMICAL RENORMALIZATION GROUP

As mentioned before, the GKE provides an implicit relation between the nonadiabatic distribution function and the fluctuating electric field. As in many other turbulent phenomena, it is impossible to solve this equation in order to derive the precise temporal and spatial dependence of the nonadiabatic distribution function. Another very interesting and apparently easier problem is to transform the GKE into an explicit relation between δh and $\delta\phi$. Due to the nonlinearity in (8) such an explicit relation cannot be exactly obtained. The first widely used approximation consists of simply neglecting the nonlinear term in the GKE. Unfortunately the results deriving from the linear theory [25] are very questionable [26] because of the dominant role of nonlinear interactions in drift-wave turbulence. Much more sophisticated theories have been developed to go farther on in the derivation of nonlinear perturbations. Dupree introduced a general perturbation scheme based on the "test-wave" concept [27] and applied it to the drift-wave turbulence [28]. Weinstock proposed a statistical theory clarifying Dupree's approach [29] which was further extended by Misguich and Balescu [30]. Let us also mention the work of Orszag and Kraichnan [10], and DuBois and Espedal [11] based on the DIA. These perturbative approaches were developed in order to take into account the nonlinear character of the Vlasov equation (or of the GKE). They have been refined in numerous subsequent works among which we only quote two review papers [31,32]. All of them result in modified Vlasov equations renormalized by new "linear" terms. In some sense, the renormalization-group technique presented here belongs to this type of approximation. Nevertheless, in our theory the renormalization of both the propagator and the vertex will appear as a consequence of the small scales elimination. Let us show how this renormalization arises in the study of a generalized GKE, which is written in the following schematic form:

$$g^{-1}\delta h + \Delta\delta\phi = \lambda N[\delta\phi, \delta h]. \tag{15}$$

The parameter λ (9) has been explicitly introduced for further convenience. From now on, we drop the overbar symbol for the gyro average. The possible influence of an additional forcing noise appearing as a consequence of the renormalization scheme could also be investigated. However, as usual in RNG theory of turbulence, we will neglect this self-generated forcing when compared to the fluctuation of the electric potential. This is a nontrivial approximation that could be investigated further in subsequent works. Let us only mention that, in the simple examples presented in Sec. VI, the influence of a possible renormalized forcing seems to be negligible. We define the quantity Λ as the upper limit of wave vectors for which Eq. (15) is valid. This parameter is often called the cutoff wave vector. For the GKE, the cutoff is approximately equal to the Debye wave vector (k_D). In what follows, we assume that the original cutoff is $\Lambda = \infty$. Moreover, we assume that both the propagator and the vertex are renormalized by additional terms, which are proportional to $k_\perp^2 = k_x^2 + k_y^2$:

$$g^{\alpha}(\hat{\mathbf{k}},\mathbf{v}) = \left(-i\omega + ik_{\parallel}v_{\parallel} + \nu_\alpha(\Lambda;\mathbf{v})k_\perp^2\right)^{-1}, \tag{16}$$

$$\Delta^{\alpha}(\hat{\mathbf{k}},\mathbf{v}) = \{i[\omega - k_y V_{d\alpha}^T(v)] - \sigma_\alpha(\Lambda;\mathbf{v})k_\perp^2\} \times \frac{e_\alpha}{T_\alpha} F_0^\alpha(x,v)\, L_k^\alpha(v_\perp). \tag{17}$$

Here, "vertex" refers to the linear term proportional to $\delta\phi$ in the GKE and should not be confused with other uses of the same terminology. For example, in the RNG literature "vertex" sometimes refers to the nonlinear term. We limit the renormalizing terms to those proportional to $k_\perp^2$ by anticipating the properties of the ϵ expansion related to the RNG technique. A more detailed discussion is presented below. The actual GKE is recovered when $\Lambda = \infty$:

$$\begin{aligned}\nu_\alpha(\infty;\mathbf{v}) &= 0,\\ \sigma_\alpha(\infty;\mathbf{v}) &= 0.\end{aligned} \tag{18}$$

The RNG technique is based on the concept of small scales elimination and on perturbative methods. Practically, it will be used to transform the actual GKE with (18) into a new, structurally similar, equation restricted to the domain of wave vectors $|\mathbf{k}| < \Lambda$ but with nonvanishing quantities $\mu(\Lambda) = (\nu_\alpha(\Lambda;\mathbf{v}), \sigma_\alpha(\Lambda;\mathbf{v}))$. Both the vertex and the propagator are then said to be renormalized by the quantities $\mu(\Lambda)$. Particular cases, when these quantities become large, are very important because they may justify a linear approximation of the renormalized equation. We show hereafter how the quantities $\mu(\Lambda)$ can be derived directly from both the structure of the GKE and the statistical properties of the electric potential (14). In Fourier space, the elimination of small scales from the original GKE is expressed by the elimination of the wave vectors in the range $\Lambda < k_\perp$. As the GKE describes wave vectors with small $k_\parallel$ $[k_\parallel \rho_\alpha = O(\hat{\epsilon})]$, this shell corresponds to the largest wave vectors belonging to the definition domain of the equation and consequently to the smallest length scales. Due to the nonlinear character of the GKE, such an elimination cannot be performed exactly. A first approximation consists in decomposing the shell elimination into successive eliminations of the small wave vector domain $\Lambda - \delta\Lambda < k_\perp < \Lambda$. To perform the calculations, the nonadiabatic distribution function is divided into low wave vector and high wave vector components:

$$\delta h(\hat{\mathbf{k}}) = \begin{cases} \delta h^<(\hat{\mathbf{k}}), & k_\perp \le \Lambda - \delta\Lambda \\ \delta h^>(\hat{\mathbf{k}}), & \Lambda - \delta\Lambda < k_\perp \le \Lambda. \end{cases} \tag{19}$$

The same decomposition is used for the electric potential. The equations for $\delta h^<$ and $\delta h^>$ are directly derived from Eq. (15):

$$g^{-1}\delta h^< + \Delta\delta\phi^< = \lambda N[\delta\phi^< + \delta\phi^>, \delta h^< + \delta h^>], \tag{20}$$

$$g^{-1}\delta h^> + \Delta\delta\phi^> = \lambda N[\delta\phi^< + \delta\phi^>, \delta h^< + \delta h^>]. \tag{21}$$

In terms of these new variables, the small spatial scales elimination is equivalent to the derivation of a closed equation for $\delta h^<(\hat{\mathbf{k}})$. The nonlinearity of Eqs. (20) and (21) does not allow a rigorous decoupling between $\delta h^<(\hat{\mathbf{k}})$ and $\delta h^>(\hat{\mathbf{k}})$. It is then necessary to make some approximations. The first one is the λ *expansion.* It simply consists of considering that the nonlinear term is small in comparison with the linear one. Of course, if this assumption were verified for all wave vectors, the λ expansion could be used to obtain an approximate solution of the complete equation (15). The problem is then said to be weakly nonlinear. We already mentioned that such an approximation cannot be used for the GKE due to the importance of nonlinear interactions. Nevertheless, this approximation could be quite accurate in the large wave vector limit. Hence we assume that the λ expansion is convergent for the wave vectors belonging to the shell $\Lambda - \delta\Lambda < k_\perp < \Lambda$. If this assumption is true, an approximate solution for $\delta h^>$ will then be obtained to any order in λ. Unfortunately, we are not able at this stage to prove this convergence. Up to order 1 in λ, $\delta h^>$ is given by

$$\begin{aligned}\delta h^> = {} & -g\Delta\delta\phi^> \\ & + g\lambda N[\delta\phi^< + \delta\phi^>, \delta h^< - g\,\Delta\,\delta\phi^>] + O(\lambda^2).\end{aligned} \tag{22}$$

We can now inject Eq. (22) into Eq. (20) to derive a closed equation for $\delta h^<$. The RNG emerges here because the elimination of the shell of wave vectors $\Lambda - \delta\Lambda < k_\perp < \Lambda$ does not change the structure of the original Eq. (15) but only modifies the quantities $\mu(\Lambda - \delta\Lambda) \neq \mu(\Lambda)$. Consequently, the *structure invariance* of the equation allows an iterative elimination of successive wave vector shells. This leads to a set of differential equations relating the quantities $\mu(\Lambda)$ to the cutoff:

$$\frac{\partial\mu(\Lambda)}{\partial\Lambda} = M(\mu, \Lambda). \tag{23}$$

The equalities (18) play the role of initial conditions for these differential equations. If the changes in μ really increase the linear terms, the λ expansion will become more and more convergent. This could justify the use of this expansion for smaller and smaller wave vectors. Hence, Eq. (23) could be defined in a larger domain of Λ than the convergence domain of the λ expansion for the original GKE. Let us now show explicitly how Eq. (23) can be derived from the GKE. If the approximate solution (22) is injected into the equation for $\delta h^<$ (20), we will obtain an equation where $\delta h^>$ has disappeared:

$$g^{-1}\delta h^< + \Delta\delta\phi^< = \lambda N[\delta\phi^< + \delta\phi^>, \delta h^< - g\,\Delta\,\delta\phi^>] + \lambda^2 N[\delta\phi^< + \delta\phi^>, N[\delta\phi^< + \delta\phi^>, \delta h^< - g\,\Delta\,\delta\phi^>]] + O(\lambda^3). \tag{24}$$

At this stage the structure invariance of the GKE is certainly not obvious for various reasons. Let us first notice that the function $\delta\phi^>$ appears in Eq. (24). This implies that this equation is not yet independent of the small-scale properties of the system. To solve this problem we split all the new terms into an averaged part and a fluctuation, the average being taken over all the possible realizations of $\delta\phi^>$. The fluctuating parts of the new terms are neglected or incorporated in the fluctuating electric field. The average of the new terms leads to renormalization of both ν_α and σ_α.

The structure invariance can also be broken for another reason. Indeed, terms renormalizing the vertex and the propagator have generally a much more complicated form than the corrections proportional to $k_\perp^2$. The exact calculations would exhibit terms proportional to $k_\perp^4, k_\perp^6, \ldots$. A nice feature of RNG techniques is the existence of a well-defined expansion in which all higher order terms in

$k_\perp$ may be neglected. This approximation is known as the ϵ expansion [33] [here ϵ should not be confused with the drift-expansion parameter $\hat{\epsilon}$ introduced in (3)]. It will be developed in Sec. IV B.

Finally, let us note that, strictly speaking, the structure invariance is broken by the appearance of a new cubic nonlinear term $N[\delta\phi^<, gN[\delta\phi^<, \delta h^<]]$. In the renormalization of the Navier-Stokes equation, it has been shown [34] that such cubic nonlinearities do not influence the renormalized linear terms. This result holds in the present case too. This means that the cubic nonlinearity could be kept to order λ^2 in the renormalized equation but that it does not change the propagator or the vertex. However, the final goal of the renormalization is to provide an equation with a renormalized linear term large enough to justify the linear approximation in which this cubic nonlinearity will be finally neglected. For this reason, we neglect this term from the beginning, knowing that this approximation does not influence the linear renormalized GKE.

Let us now present some detailed calculations leading to the differential equation which characterizes the small scales elimination. First, we consider the term renormalizing the coefficient ν_α which appears in the propagator (in the derivation of the equations for ν_α and σ_α we shall systematically omit to write the velocity dependence of these quantities)

$$T_1 = \langle N[\delta\phi^>, gN[\delta\phi^>, \delta h^<]]\rangle. \tag{25}$$

By taking into account that $\delta\phi^>$ is a Gaussian noise, and by injecting the correlations (14) into (25), we obtain

$$T_1 = -\int_V d\hat{\mathbf{q}}\, [\mathbf{k}\cdot(\mathbf{q}\times\mathbf{b})]^2\, (L_q^\alpha)^2 g^\alpha(-\hat{\mathbf{q}}, \mathbf{v}) \times S(\hat{\mathbf{q}})\delta h^{\alpha<}(\hat{\mathbf{k}}, \mathbf{v}). \tag{26}$$

The volume of integration V is defined by the inequality $\Lambda - \delta\Lambda < q_\perp \le \Lambda$ in the four-dimensional space $\{\hat{\mathbf{q}} = (\Omega, \mathbf{q})\}$. In what follows, we also need to introduce the domain of variation $\tilde{V}$ of both the parallel wave vector ($q_\parallel$) and the frequency (Ω). This volume of integration is defined in the two-dimensional space $\{q_\parallel, \Omega\}$ by the approximations made to derive the GKE, i.e., drift waves are assumed to have small frequency as compared to the Larmor frequency and small parallel wave vectors. To lowest order in k, the integration over the perpendicular wave vector can be performed with the following result:

$$T_1 = -k_\perp^2 \delta h^{\alpha<}(\hat{\mathbf{k}}, \mathbf{v}) 2\pi (L_\Lambda^\alpha)^2 \nu_\alpha \Lambda^5 \delta\Lambda \times \int_{\tilde{V}} \frac{dq_\parallel d\Omega S(q_\parallel, \Lambda, \Omega)}{(\Omega - q_\parallel v_\parallel)^2 + \nu_\alpha^2 \Lambda^4} + O(\delta\Lambda^2). \tag{27}$$

At this stage, the influence of T_1 in the GKE becomes clear. Indeed, we note in Eq. (27) that this term is proportional to $k_\perp^2 \delta h^{\alpha<}(\hat{\mathbf{k}}, \mathbf{v})$ which has exactly the same structure as the ν_α term in the propagator. The term T_1 being in the right hand side of Eq. (24), the increase of ν_α will be proportional to T_1 with the opposite sign. The resulting integrodifferential equation, which relates ν_α to the cutoff, is then given by

$$\frac{\partial \nu_\alpha}{\partial \Lambda} = -\pi\lambda^2 (L_\Lambda^\alpha)^2 \nu_\alpha \Lambda^5 \int_{\tilde{V}} \frac{dq_\parallel\, d\Omega\, S(q_\parallel, \Lambda, \Omega)}{(\Omega - q_\parallel v_\parallel)^2 + \nu_\alpha^2 \Lambda^4}. \tag{28}$$

A new minus sign originates from the fact that the parameter ν_α increases when the cutoff decreases from Λ to $\Lambda - \delta\Lambda$. Let us now turn to the renormalization of σ_α, originating from the term

$$T_2 = \langle N[\delta\phi^>, gN[\delta\phi^<, -g\Delta\delta\phi^>]]\rangle. \tag{29}$$

By following the same procedure used in the evaluation of T_1, we inject the correlations (14) into (29). After having performed the integration over the perpendicular wave vector, the resulting integrodifferential equation for σ_α is given by

$$\frac{\partial \sigma_\alpha}{\partial \Lambda} = -\pi\lambda^2 (L_\Lambda^\alpha)^2 \Lambda^5 \int_{\tilde{V}} dq_\parallel\, d\Omega\, S(q_\parallel, \Lambda, \Omega) \times \frac{\sigma_\alpha \nu_\alpha^2 \Lambda^4 - \sigma_\alpha(\Omega - q_\parallel v_\parallel)^2 + 2\nu_\alpha(\Omega - q_\parallel v_\parallel)\Omega}{[(\Omega - q_\parallel v_\parallel)^2 + \nu_\alpha^2\Lambda^4]^2}. \tag{30}$$

Two particular cases will be treated explicitly in the following section. The equations (28) and (30) define the renormalized linear operators that appear in the GKE. Both the propagator and the vertex are then written in terms of the spectrum $S(q_\parallel, \Lambda, \Omega)$ of the fluctuating electric potential (14). Let us stress that the perturbation scheme associated with the RNG does not reduce to an analytic expansion of ν and σ in the coupling parameter λ. Even though the right hand sides of Eqs. (28) and (30) are analytic functions of λ, the solutions of these equations are not necessarily analytic. The integrodifferential equation (28) must be coupled with the "initial" condition $\nu_\alpha(\infty; \mathbf{v}) = 0$. This condition expresses the fact that the parameters ν_α and σ_α vanish in the original equation. By integrating Eq. (28) over Λ from $\Lambda = \infty$ to a new cutoff $\Lambda = k_\perp^*$, one gets

$$\nu_\alpha(k_\perp^*; \mathbf{v}) = \pi\lambda^2 \int_{k_\perp^*}^{\infty} dq_\perp\, (L_q^\alpha)^2 \times \int dq_\parallel\, d\Omega\, \frac{\nu_\alpha(q_\perp; \mathbf{v})\, q_\perp^5\, S(q_\parallel, q_\perp, \Omega)}{(\Omega - q_\parallel\, v_\parallel)^2 + \nu_\alpha(q_\perp; \mathbf{v})^2 q_\perp^4}, \tag{31}$$

where we have replaced the integration variable Λ by $q_\perp$. If we go back to a three-dimensional (3D) spatial integration domain ($2\pi q_\perp dq_\perp dq_\parallel \to d^3q$), we rewrite this equation in the form

$$\nu_\alpha(k_\perp^*; \mathbf{v}) = \lambda^2 \int_{q_\perp > k_\perp^*} d^3q\, d\Omega\, (L_q^\alpha)^2 \times \frac{\nu_\alpha(q_\perp; \mathbf{v}) q_\perp^4 S(q_\parallel, q_\perp, \Omega)}{(\Omega - q_\parallel v_\parallel)^2 + \nu_\alpha(q_\perp; \mathbf{v})^2\, q_\perp^4}. \tag{32}$$

The formulas (31) and (32) will often be used in the following sections. A similar calculation leads to the renormalization of the vertex σ_α:

$$\sigma_\alpha(k_\perp^*;\mathbf{v}) = \lambda^2 \int_{q_\perp > k_\perp^*} d^3q\, d\Omega\, (L_q^\alpha)^2\, S(q_\parallel,\Lambda,\Omega) \frac{\sigma_\alpha \nu_\alpha^2 \Lambda^4 - \sigma_\alpha(\Omega - q_\parallel v_\parallel)^2 + 2\nu_\alpha(\Omega - q_\parallel v_\parallel)\Omega}{[(\Omega - q_\parallel v_\parallel)^2 + \nu_\alpha^2\Lambda^4]^2}, \tag{33}$$

where we have omitted the $(q_\perp,\mathbf{v})$ dependence of ν_α and σ_α in the integral.

IV. COMPARISON WITH OTHER THEORIES

Throughout this paper, we have used λ as an expansion parameter. Formally, we have already explained that such an expansion consists in assuming that the nonlinear term is small when compared to the linear one (in the large wave vector limit). In fact λ is equal to $c/B(2\pi)^4$ and the effective expansion parameter is given by

$$\overline{\lambda}^2 = -\frac{\partial \nu_\alpha}{\partial \Lambda}\, \frac{\Lambda}{\nu_\alpha}. \tag{34}$$

The expressions (28) and (30) have been derived to the lowest order in λ (or $\overline{\lambda}$). At this stage, the smallness of $\overline{\lambda}$ has not been investigated and the expansion procedure has to be justified *a posteriori*. It is interesting to note that two very different limits are compatible with small $\overline{\lambda}$. The first one corresponds to small spectral intensity S while the other is associated with a large value of the renormalized parameter ν.

A. The quasilinear limit

When the intensity of the fluctuation spectrum S is not too large, the wave-wave interactions can be neglected. In this case, the turbulence is driven by wave-particle interactions. This is the quasilinear (QL) limit [35]. It is not the purpose of this section to rederive the expression of QL results. We only mention that the QL effect usually introduces a diffusion coefficient proportional to the spectrum. In the present formulation, the QL limit for the GKE should correspond to a parameter ν_α proportional to S. Due to the presence of ν_α in both sides of Eq. (32), such a result is not obvious. Let us investigate the limit of this equation for small S. We first perform the change of variable $\Omega = q_\parallel v_\parallel + x\nu(q_\perp)q_\perp^2$ to obtain

$$\nu_\alpha(k_\perp^*) = \lambda^2 \int d^3q\, dx\, (L_q^\alpha)^2 q_\perp^2 \times \frac{S(q_\parallel, q_\perp, q_\parallel\, v_\parallel + x\nu_\alpha q_\perp^2)}{x^2+1}. \tag{35}$$

This equation can be schematically rewritten as follows:

$$\nu \sim \hat{S}(qv + \nu). \tag{36}$$

If S is small, ν can be successively approximated by

$$\begin{aligned}
&\nu \sim \hat{S}(qv) + O(\hat{S}^2),\\
&\nu \sim \hat{S}(qv + \hat{S}(qv)) + O(\hat{S}^3),\\
&\vdots
\end{aligned} \tag{37}$$

By using the first-order approximation for ν and the definition of λ (9), we obtain the following explicit solution of Eq. (32):

$$\nu_\alpha = \pi \left(\frac{c}{B(2\pi)^4}\right)^2 \int d^3q[L_q^\alpha(v_\perp)]^2 \times q_\perp^2 S(q_\parallel, q_\perp, q_\parallel v_\parallel) + O(S^2). \tag{38}$$

The expression (38) is compatible with with the QL limit as the renormalized parameter ν is proportional to the spectrum S. As we have considered the non-self-consistent problem, the result (38) differs from the fully self-consistent quasilinear expression for the renormalized propagator. Particularly, the eigenfrequencies of the dispersion function do not appear here. They are replaced by the frequency: $q_\parallel v_\parallel$. To avoid any confusion in the terminology, the result (38) will be said in what follows to recover the QL scaling (rather than the QL results). This is a general result independent of the detailed form of the spectrum. The only condition for recovering the QL scaling is the smallness of the spectral amplitude S. As already mentioned, this also corresponds to a small effective expansion parameter $\overline{\lambda}$. In Eq. (38) we have approximated the integration over x by π even though we know that x cannot take infinite value in the drift ordering. The volume of integration $\tilde{V}$ that appears in the expressions (28) and (30) is the two-dimensional domain of variation of both the parallel wave vector and the frequency. We recall that drift waves are assumed to have small frequency compared to the Larmor frequency and small parallel wave vectors as measured by the Larmor radius. Nevertheless, we will usually consider in what follows that both $q_\parallel$ and Ω vary between $-\infty$ and $+\infty$. This could seem a rather crude approximation but we note that all the integrals to be calculated have integrands dominated by the values of $q_\parallel$ and Ω near the resonance of the propagator ($\Omega \approx q_\parallel v_\parallel$). Consequently, the use of a much larger domain of integration only introduces small errors in the final results.

B. The ϵ expansion

We have shown that the QL scaling is compatible with the general equation for ν derived from the small scale elimination procedure. The iterative small scales elimination presented in Sec. III, like the QL theory, is based on the $\overline{\lambda}$ expansion. However, contrary to the QL theory, this procedure does not assume that $S \to 0$ to justify this expansion. There exists another condition to obtain an effectively small $\bar{\lambda}$. This condition corresponds to large renormalized parameter ν and is usually related to the so called ϵ expansion. Roughly speaking, the latter consists of the search for an ideal system for which the λ expansion leads to exact results. Such a system corresponds to a fixed point in the RNG terminology. The departure between real and ideal systems is then treated as a small parameter ϵ. Let us show by a simple example how this parameter can be defined. We consider a spectrum of potential fluctuations given by

$$S(\kappa) = k_\perp^m \tilde{S}(k_\parallel/k_\perp). \tag{39}$$

This corresponds to a power-law white noise spectrum (independent of the frequency). We introduce the notation

$$\int_{-\infty}^{+\infty} \tilde{S}(x)dx = s_1. \tag{40}$$

The frequency integration can then be easily performed and the solution of Eq. (31) is given by

$$\nu(k_\perp^*) = 2\lambda^2\pi^2 s_1 \frac{(k_\perp^*)^{5+m}}{-5-m}. \tag{41}$$

For a physically meaningful potential spectrum, the parameter s_1 cannot be divergent. Therefore the only way to ensure a large value for ν is to assume an ϵ expansion with $\epsilon = -5 - m$. The parameter ν is then porportional to $(k_\perp^*)^{-\epsilon}/\epsilon$. Let us insist on the following remark: The expression (32) contains both limit cases, QL with small renormalization and RNG with large renormalization. Both these limits are compatible with the $\bar{\lambda}$ expansion. However, the QL limit and the RNG limit can generally not be performed simultaneously.

C. The DIA limit

The renormalization of both propagator and vertex terms has been obtained by various other theories. We show in this section that, under very simple approximations, the parameter ν defined by the expression (32) is the same as the renormalization of the propagator derived from the direct interaction approximation. We first neglect the finite Larmor radius effects in order to simplify the notation, and we assume that $\nu(k_\perp^*)$ is weakly dependent on $k_\perp^*$ for small $k_\perp^*$. In this case, we obtain an integral equation for ν:

$$\nu_\alpha = \nu_\alpha\lambda^2 \int d^3q\, d\Omega\, q_\perp^4 \frac{S(q_\parallel, q_\perp, \Omega)}{(\Omega - q_\parallel v_\parallel)^2 + \nu_\alpha^2 q_\perp^4}. \tag{42}$$

In this equation ν is assumed to be independent of $k_\perp$. The equation (42) is the same as the one derived by Dupree for the ion diffusion coefficient [28]. We also note that Dupree's equation may be derived by using the DIA formalism [36,37]. It is remarkable to note that, in some sense, the RNG theory with the Gaussian assumption on $\delta\phi$ is equivalent to the DIA approach in which deviations from the Gaussianity are taken into account. However, the averaging procedure is performed in a single step in the DIA while it is achieved iteratively in the RNG.

V. ANOMALOUS FLUXES AND TRANSPORT COEFFICIENTS

In this section we determine the relations between the macroscopic fluxes and the thermodynamical forces. The latter originate from the spatial dependence of both the particle densities and the temperatures. Due to the quasineutrality assumption, only three independent forces are available. They are usually denoted

$$\begin{aligned} X_1 &= -\frac{\partial \log[n(x)]}{\partial x}, \\ X_2 &= -\frac{\partial \log[T_e(x)]}{\partial x}, \\ X_3 &= -\frac{\partial \log[T_i(x)]}{\partial x}, \end{aligned} \tag{43}$$

and they represent the radial gradient of the density, the electronic temperature, and the ionic temperature, respectively. The anomalous fluxes are created by the fluctuations of macroscopic quantities such as the electron density and the electron and ion pressures δn_e, δP_e, δP_i. These fluctuations are derived from $\delta\phi$ and δh^α:

$$\begin{aligned} \delta n_\alpha &= \int d\mathbf{v}\ \delta f^\alpha(\hat{\mathbf{k}}, \mathbf{v}) \\ &= -n_\alpha \frac{e_\alpha}{T_\alpha}\ \delta\phi(\hat{\mathbf{k}}) + \int d\mathbf{v}\ L_k^\alpha \delta h^\alpha(\hat{\mathbf{k}}, \mathbf{v}), \end{aligned} \tag{44}$$

$$\begin{aligned} \delta P_\alpha &= \int d\mathbf{v}\ \frac{m_\alpha v^2}{3}\ \delta f^\alpha(\hat{\mathbf{k}}, \mathbf{v}) \\ &= -n_\alpha e_\alpha \delta\phi(\hat{\mathbf{k}}) + \int d\mathbf{v} L_k^\alpha \frac{m_\alpha}{}\frac{v^2}{3} \delta h^\alpha(\hat{\mathbf{k}}, \mathbf{v}). \end{aligned} \tag{45}$$

We have already mentioned in Sec. II that, in a fully self-consistent approach, the GKE should be coupled with the Poisson equation. The latter reduces in the gyrokinetic ordering to the quasineutrality condition $\delta n_i \approx \delta n_e$. In the present approach, such a condition cannot be derived from the basic equations. However, in order to simplify the transport equations, we are assuming that the quasineutrality condition is valid at the macroscopic level. The links between the fluctuations and the anomalous radial fluxes of matter and energy are then given by [6,26,38]

$$\Gamma_\alpha(x) = -\frac{c}{B}\langle \delta n_\alpha(\mathbf{r},t) \nabla_y \delta\phi(\mathbf{r},t)\rangle, \tag{46}$$

$$\frac{Q_\alpha(x)}{T_\alpha} = -\frac{5}{2}\ \frac{c}{T_\alpha B}\langle \delta P_\alpha(\mathbf{r},t) \nabla_y \delta\phi(\mathbf{r},t)\rangle. \tag{47}$$

Being macroscopic quantities, the fluxes are assumed to depend only on the coordinate x. This will be verified *a posteriori* in the transport laws. Moreover, Γ_α and Q_α as defined in (46) and (47) are radial (x) components of vectors. Due to the quasineutrality, only three macroscopic fluxes are independent. They will be denoted

$$J_1 = \Gamma_e, \quad J_2 = Q_e/T_e, \quad J_3 = Q_i/T_i. \tag{48}$$

They respectively represent the radial flux of electrons, electronic energy, and ionic energy. If the notion of anomalous transport coefficients is meaningful, the radial fluxes will be related to the forces by the (linear) transport equations

$$J_r = \sum_s c_{rs} X_s, \quad r,s = 1,2,3, \tag{49}$$

where the c_{rs} are the anomalous transport coefficients. In the quasilinear theory [39,40], the relations between the fluxes (48) and the thermodynamical forces (43) are derived by neglecting the nonlinear term (in the potential) in the GKE. Both δn_α and δP_α are then proportional to $\delta\phi$. As a consequence, the fluxes are proportional to the spectrum $S(\hat{\mathbf{k}})$. Of course, this approximation in which all information from the nonlinear interactions is neglected is only valid for small amplitudes of $\delta\phi$ and δh^α. Here, our goal is to use the linear RGKE instead of the linear GKE in the determination of the transport equations. In this case, a part of the nonlinear interactions is taken into account through the renormalization of both the propagator and the vertex (28)–(30). The linear term is then hopefully large enough to justify the linear approximation for finite amplitude of $\delta\phi$ and δh^α. Let us notice that both ν and σ depend on the spectrum $S(\hat{\mathbf{k}})$. As a consequence, by neglecting the nonlinear term in the RGKE, we obtain an equation which could remain highly nonlinear and even nonanalytical in $\delta\phi$. This will strongly influence the anomalous transport coefficients in some cases.

We note that the expressions (46) and (47) are given in the real space. It is more convenient to use the Fourier space representation of the fluctuations to investigate these fluxes. The spectrum of the electric potential fluctuations (14) can then be used to simplify the formulas. For this reason, we now express the fluxes as the inverse Fourier transform of the Fourier transform of the left hand side of the relations (46) and (47):

$$\Gamma_\alpha(x) = \frac{1}{(2\pi)^4}\int d\hat{\mathbf{q}} e^{i(\Omega t - \mathbf{q}\cdot\mathbf{r})}\, \frac{c}{B}\frac{1}{(2\pi)^4} \times \int d\hat{\mathbf{k}}\; i(\mathbf{k}-\mathbf{q})_y \langle \delta n_\alpha(\hat{\mathbf{k}})\delta\phi(\hat{\mathbf{q}}-\hat{\mathbf{k}})\rangle, \tag{50}$$

$$\frac{Q_\alpha(x)}{T_\alpha} = \frac{1}{(2\pi)^4}\int d\hat{\mathbf{q}}\; e^{i(\Omega t - \mathbf{q}\cdot bfr)}\, \frac{5c}{2BT_\alpha}\frac{1}{(2\pi)^4} \times \int d\hat{\mathbf{k}} i(\mathbf{k}-\mathbf{q})_y \langle \delta P_\alpha(\hat{\mathbf{k}})\delta\phi(\hat{\mathbf{q}}-\hat{\mathbf{k}})\rangle. \tag{51}$$

By using the linear RGKE, we easily derive an expression for δh^α that we inject into Eqs. (44) and (45) and then into Eqs. (50) and (51). Simple algebraic manipulations allow us to reduce the above expressions to:

$$\Gamma_\alpha = -\frac{B\lambda^2}{c}\int d\hat{\mathbf{k}}\; k_y S(\hat{\mathbf{k}}) \times \int dv_\parallel v_\perp dv_\perp L_k^\alpha(v_\perp)\mathrm{Im}[g_R^\alpha \Delta_R^\alpha], \tag{52}$$

$$\frac{Q_\alpha}{T_\alpha} = -\frac{5}{2}\frac{B\lambda^2}{c}\int d\hat{\mathbf{k}} k_y S(\hat{\mathbf{k}}) \times \int\; dv_\parallel v_\perp dv_\perp L_k^\alpha(v_\perp)\frac{m_\alpha}{3}v^2\mathrm{Im}[g_R^\alpha\; \Delta_R^\alpha]. \tag{53}$$

Let us now make use of the expressions (28)–(30) for the renormalized linear terms:

$$\mathrm{Im}[g_R^\alpha\; \Delta_R^\alpha] = \frac{e_\alpha}{T_\alpha}F_\alpha L_k^\alpha(v_\perp)\; \frac{-(\omega - k_\parallel v_\parallel)\sigma_\alpha k_\perp^2 + (\omega - k_y V_{d\alpha}^T)\nu_\alpha k_\perp^2}{(\omega - k_\parallel v_\parallel)^2 + \nu_\alpha^2 k_\perp^4}. \tag{54}$$

Due to the symmetry of the integration domain, the odd powers of $\mathbf{k}$ and ω in the integrands lead to vanishing contributions to the fluxes. From now on, we consider that the spectrum only depends on $\{k_\parallel, k_\perp, |\omega|\}$. In real space this means that

$$\langle \delta\phi(\mathbf{r},t)\delta\phi(\mathbf{r}',t')\rangle = S(r_\parallel - r'_\parallel, |r_\perp - r'_\perp|, |t-t'|). \tag{55}$$

This assumption is very reasonable and leads to important simplifications in the expressions for the fluxes. Indeed, in that case it is quite easy to perform the integration over the angle in the perpendicular plane with respect to $\mathbf{1}_z$ for both the wave vector and the velocity variables, to obtain

$$\Gamma_\alpha = 2n\lambda^2\pi^{1/2}\int_{-\infty}^{+\infty} du_\parallel \int_0^{+\infty} du_\perp u_\perp e^{-u^2}\left[X_1 + (u^2 - \tfrac{3}{2})X_\alpha\right] \times \int_{-\infty}^{+\infty} dk_\parallel \int_0^{+\infty} dk_\perp \nu_\alpha k_\perp^5 \left[L_k^\alpha(v_\perp)\right]^2 \int d\omega \frac{S(\hat{\mathbf{k}})}{(\omega - k_\parallel u_\parallel V_\alpha)^2 + \nu_\alpha^2 k_\perp^4}, \tag{56}$$

$$\frac{Q_\alpha}{T_\alpha} = \frac{10n\lambda^2\pi^{1/2}}{3} \int_{-\infty}^{+\infty} du_\parallel \int_0^{+\infty} du_\perp u_\perp u^2\, e^{-u^2} \left[X_1 + (u^2 - \tfrac{3}{2})X_\alpha\right]$$
$$\times \int_{-\infty}^{+\infty} dk_\parallel \int_0^{+\infty} dk_\perp \nu_\alpha k_\perp^5 \left[L_k^\alpha(v_\perp)\right]^2 \int d\omega \frac{S(\hat{\mathbf{k}})}{(\omega - k_\parallel u_\parallel V_\alpha)^2 + \nu_\alpha^2 k_\perp^4}, \tag{57}$$

where we have used the definitions (12) and (13). We have also introduced the dimensionless velocity $\mathbf{u} = \mathbf{v}/V_\alpha$ and dropped the velocity and wave number dependence of ν_α and L^α. For convenience, we use the notation $X_e = X_2$ and $X_i = X_3$. The comparison between these expressions and Eq. (49) leads to the determination of the anomalous transport coefficients. The X_i dependence in the fluxes arises from the drift velocity defined in (13) that appears in the vertex term. Let us notice that the parameter σ_α, which renormalizes the vertex, does not influence the fluxes when the spectrum has the properties described by Eq. (55). Such a property, which is quite general, appears here more clearly than in some other approximations where σ_α is usually set equal to ν_α (as in the coherent approximation).

It must be noted that, strictly speaking, the expressions (56) and (57) for the fluxes depend on the cutoff wave vector Λ introduced in the renormalization procedure. Indeed, the renormalized parameter ν_α is explicitly Λ dependent. As usual in the evaluation of both the transport coefficients and the energy spectra [16], the cutoff dependence in the renormalized equation is considered to be a wave vector dependence

$$\nu_\alpha(\Lambda, \mathbf{v}) \approx \nu_\alpha(k_\perp, \mathbf{v}). \tag{58}$$

This can be justified as follows. We recall that the renormalized GKE after the elimination of wave vectors larger than Λ can be written as

$$g^{-1}(\hat{\mathbf{k}}, \Lambda)\ \delta h(\hat{\mathbf{k}}) + \Delta(\hat{\mathbf{k}}, \Lambda)\ \delta\phi(\hat{\mathbf{k}}) = \lambda N[\delta\phi, \delta h]. \tag{59}$$

In the procedure used to derive the fluxes (56) and (57), the linearized version of this equation has been used. We have already mentioned that the use of the linear RGKE is justified by the hope that the renormalizing parameters (ν and σ) are large enough to make the linear terms larger than the nonlinear ones. This is the main idea of the renormalization procedure. However, the nonlinearity has not disappeared from Eq. (59). Let us consider a given Fourier mode of the nonadiabatic velocity distribution $\delta h^\alpha(\hat{\mathbf{k}}, \mathbf{v})$. It is submitted to a large number of interactions (represented by the nonlinear term). In the RGKE, all interactions with modes in the range $q_\perp > \Lambda$ have been perturbatively eliminated by the renormalization scheme and are taken into account through the parameters $\nu_\alpha(\Lambda, \mathbf{v})$ and $\sigma_\alpha(\Lambda, \mathbf{v})$. The interactions with modes $q_\perp < \Lambda$ have not been eliminated and are represented by the remaining nonlinear term. Consequently, the best linear RGKE that can be used to describe the evolution of the mode $\delta h^\alpha(\mathbf{k}, \mathbf{v})$ should correspond to the equation in which all interactions with modes $q_\perp > k_\perp$ have been perturbatively removed. This linear RGKE can then be rewritten as

$$g^{-1}(\hat{\mathbf{k}}, \nu(k_\perp))\ \delta h(\hat{\mathbf{k}}) + \Delta(\hat{\mathbf{k}}, \sigma(k_\perp))\delta\phi(\hat{\mathbf{k}}) = 0. \tag{60}$$

This justifies the use of the relation (58). In that case, the expressions (56) and (57) for the fluxes can be simplified by taking into acount the relation (31) with $k_\perp^* = 0$. This leads to

$$\Gamma_\alpha = \frac{n}{\sqrt{\pi}} \int du_\parallel du_\perp u_\perp e^{-u^2} \times D^\alpha(\mathbf{u})\left[X_1 + (u^2 - \tfrac{3}{2})X_\alpha\right], \tag{61}$$

$$\frac{Q_\alpha}{T_\alpha} = \frac{n}{\sqrt{\pi}} \int du_\parallel du_\perp u_\perp u^2\ e^{-u^2} \times D^\alpha(\mathbf{u})\left[X_1 + (u^2 - \tfrac{3}{2})X_\alpha\right], \tag{62}$$

where

$$D^\alpha(\mathbf{u}) = \nu_\alpha(k_\perp^* = 0, \mathbf{u}V_\alpha). \tag{63}$$

These very simple formulas for the transport equations actually hide the difficulty in the determination of the diffusion coefficient $D^\alpha(\mathbf{u})$. Hence, we have to couple the relations (61) and (62) with the renormalization equation for the parameter ν_α.

VI. EXAMPLES

As explained in the Introduction, the final goal of this work is to give a theoretical evaluation of the anomalous transport coefficients. In the previous section, we have shown that the entire difficulty of this evaluation can be reduced to the calculation of the parameter $D^\alpha(\mathbf{u})$ or equivalently to the evaluation of ν_α. Of course, this evaluation cannot be obtained if the spectrum $S(\hat{\mathbf{k}})$ is not known. Hence we consider two examples in which we specify the form of the spectrum. In some sense the following examples could appear as rather academic but they supply us with a suitable tool for investigating the link between several assumptions made on the electric potential and the final form of the renormalized GKE. In both examples, we are neglecting the finite Larmor radius effects in order to simplify the calculations.

A. White noise potential fluctuations

We first consider the particular situation of uncorrelated potential fluctuations for different times

$$\langle \delta\phi_t \delta\phi_{t'} \rangle \sim \delta(t - t'). \tag{64}$$

Such fluctuating processes are usually referred to as white noises. This assumption is usually made in the study of the renormalization of the Navier-Stokes equation [14–16]. Let us however mention two papers where "colored noises" have been studied in the application of RNG theory to neutral fluid evolution. In the first one Carati [41] studied the influence of the color of the noise on the convergence properties of the ϵ expansion. In another approach based on the path integral formalism [8,42] Yuan and Ronis [43] tried to describe different aspects of turbulence, including both intermittency and Kolmogorov regimes, using more general noises. In Fourier space the correlation (14) can now be written

$$\langle \delta\phi(\hat{\mathbf{k}})\delta\phi(\hat{\mathbf{q}})\rangle = S(k_{\parallel}, k_{\perp})\delta(\hat{\mathbf{k}} + \hat{\mathbf{q}}). \tag{65}$$

When the correlation (65) is injected into Eqs. (28) and (30), the frequency integration can be performed exactly and one obtains the following relations:

$$\nu_\alpha(k_\perp^*; \mathbf{v}) = \sigma_\alpha(k_\perp^*; \mathbf{v}) = \pi\lambda^2 \int_{q>k_\perp^*} d^3q\ q_\perp^2\ S(\mathbf{q}). \tag{66}$$

It is remarkable that this result is identical to the QL scaling for ν given in Eq. (38). From the relation (66), it is now very simple to derive the value of the parameter D defined in (63):

$$D = \pi\lambda^2 \int d^3q\ \mathcal{E}_\perp(q), \tag{67}$$

where $\mathcal{E}_\perp(q)$ is the fluctuation spectrum of the transverse electric field. The relation (67) is valid as long as the potential fluctuations behave in time like a white noise. However, to be meaningful this relation requires that the wave vector integration converges. This will be the case in any realistic physical situation but would not be true for some academic spectra like a pure power law. Hereafter, we are assuming that the integration in Eq. (67) is well defined. It appears that D is now completely independent of both the velocity and the species index. In this case, the velocity integrations in the relations (61) and (62) can be explicitly performed:

$$\Gamma_\alpha = nDX_1, \tag{68}$$

$$\frac{Q_\alpha}{T_\alpha} = \frac{5nD}{2}(X_1 + X_\alpha). \tag{69}$$

The transport coefficients are then given by the tensor

$$c_{rs} = \begin{pmatrix} 1 & 0 & 0 \\ 5/2 & 5/2 & 0 \\ 5/2 & 0 & 5/2 \end{pmatrix} Dn. \tag{70}$$

The matter fluxes being independent of the species index, it is an easy matter to verify that the ambipolarity condition is recovered for the fluxes

$$\sum_\alpha \Gamma_\alpha e_\alpha = 0. \tag{71}$$

We finally remark that the transport coefficients are here proportional to the electric field spectrum and inversely proportional to the square of the magnetic field intensity (QL scaling).

B. Frequency dependent spectrum

Let us now consider another example with a more realistic spectrum

$$S(k_\perp) = \frac{\tilde{S}(k_\perp)}{[1 + (\omega/\omega^c)^2]\,[1 + (k_\parallel/k_\parallel^c)^2]}, \tag{72}$$

where ω^c and $k_\parallel^c$ represent a characteristic frequency and a characteristic parallel wave vector, respectively. In this case, Eq. (31) reduces to

$$\nu_\alpha(k_\perp^*) = \lambda^2\pi^3 \int_{k_\perp^*}^{\infty} dq_\perp \frac{q_\parallel^c \omega^c q_\perp^3 \tilde{S}(q_\perp)}{q_\parallel^c v_\parallel + \omega_c + \nu_\alpha(q_\perp) q_\perp^2}. \tag{73}$$

Of course, this integral equation for ν is too complicated to be solved exactly. We have already shown in Sec. IV A that the parameter ν_α is proportional to the spectral amplitude if the latter is small. Let us here consider the opposite limit. In this case the quantity ν_α becomes large and, more precisely,

$$\nu_\alpha(q_\perp) q_\perp^2 \gg \omega^c + q_\parallel^c v_\parallel. \tag{74}$$

Of course, this approximation cannot be valid for small $q_\perp$. However, this domain in the integration can be neglected because of the factor $q_\perp^3$. Moreover, the presence of this factor has the following consequence:

$$\nu(k_\perp^*) \approx \nu(0) \text{ for small } k_\perp^*. \tag{75}$$

Indeed, the difference between $\nu(k_\perp^*)$ and $\nu(0)$ is given by the integral appearing in (73) with the range of integration ($0 < k_\perp < k_\perp^*$), precisely the domain we have considered as negligible. Thus the function $\nu_\alpha(q_\perp)$ in the integral in (73) can be approximated by $\nu_\alpha(0) \equiv \nu_\alpha$. This parameter is then determined by

$$\nu_\alpha^2 \approx \lambda^2\pi^3\omega_c q_\parallel^c \int_0^\infty dq_\perp q_\perp \tilde{S}(q_\perp). \tag{76}$$

Let us denote by $\tilde{s}$ the integral of the wave vector spectrum $\tilde{S}(q_\perp)$. The parameter ν_α is then simply given by

$$\nu_\alpha \approx \sqrt{\pi^3\omega_c q_\parallel^c \tilde{s}}\ \frac{c}{B(2\pi)^4}. \tag{77}$$

The matrix of transport coefficient has exactly the same strucure as in the previous example. However, the transport coefficients amplitude is very different. It is now proportional to the square root of the spectrum S and inversely proportional to the magnetic field intensity. This is exactly the Bohm scaling for the diffusion coefficient. It can be seen in Eq. (31) that this is the normal scaling for the quantity ν_α if the inequality (74) is verified in a large part of the spectrum.

VII. DISCUSSION

In some sense our work may be considered as an extension of the application of the RNG to fluid turbulence.

Indeed, the velocity plays here the role of a parameter in the gyrokinetic equation (and is not a dynamical variable as in the Vlasov equation). As suggested by the first paragraphs of our introduction, we are more interested in using the RNG as a tool to derive some information concerning transport laws in turbulent plasmas than in improving the underlying theoretical basis of the RNG. We have combined the RNG methods with the determination of the transport laws in the presence of macroscopic gradients of temperature and density. Although none of these parts should be considered as entirely new, their combination leads to some nontrivial consequences. In particular, the expressions of the transport laws, obtained by coupling the classical expression for the fluxes (46) and (47) and the renormalization equation for the parameter ν, are very simple and represent, in our opinion, interesting results.

The only point in our work which could be considered as new in the RNG context is the use of a simultaneously additive and multiplicative random source with a well-defined physical meaning. Indeed, the particular case of purely additive random sources of energy has been widely studied in the application of the RNG to the Navier-Stokes equation and to MHD equations. On the other hand, the RNG study of passive scalar diffusion is more closely related to multiplicative noise. The renormalized diffusivity is then generated by the velocity fluctuations that appear as a multiplicative factor in the convective term. The present work shows that more complex forcings do not introduce additional difficulties in the renormalization scheme. We also notice that the turbulence is kept in a stationary state by assuming that the statistics of $\delta\phi$ is itself stationary. To our knowledge, in all the previous applications of the RNG to the study of a turbulent state, an external and artificial forcing noise is systematically added into the equation. The equations are then modified by the addition of a new stochastic term. This is the case for the Navier-Stokes equation [14–16] and for the MHD equations [17–19] as well as in the renormalization of the Hasegawa-Mima model [20]. The reason for that is simple: In all these examples, the dynamical renormalization group leads to extremely complicated and actually untractable calculations if the source of energy is explicitly specified. Indeed, such sources as pressure gradients or external fields explicitly introduce anisotropy or inhomogeneity. Unfortunately, the physical meaning of the artificial noises is never clear: this represents one of the major drawbacks of these theories. The turbulence generated by the actual experimental conditions has to be assumed statistically equivalent to the one generated by an artificial random forcing. However, this assumption is questionable even if it has been partially supported by numerical simulations [44]. Let us stress that the present situation is quite different. First, the GKE is by definition a stochastic equation relating the fluctuation of the velocity distribution function to the fluctuations of the electric potential. Secondly, the source of energy has now a simple physical meaning: it is caused by the fluctuations of the non-self-consistent electric potential. In this sense, the present application of the RNG appears much more satisfactory. The next step in the application of the RNG to the GKE would now be the self-consistent treatment of the electric field.

Let us now give some arguments in favor of the approximations used to obtain the results (31) and (33) (the ϵ expansion or the nonlocal expansion consisting in keeping only the contribution proportional to $k_\perp^2$ in the renormalized linear terms).

Following Eq. (41), spectra satisfying $\epsilon = 0$ in the GKE should decrease for large wave vectors as k^{-5}. Experimental results [45] as well as gyrokinetic simulations [46] (see also Ref. [47]) present the same main results: Turbulence exhibits a clear exponential decay for the smallest wavelengths and broad spectra for wavelengths larger than a few gyroradii. In the latter region, the precise determination of scaling laws from these data is quite hazardous considering their accuracy. The presence of this broad spectrum clearly stresses the necessity of using nonlinear approaches of drift-wave turbulence although this property is not a sufficient condition for the applicability of RNG methods, nor does it allow one to make a definite choice between RNG and other theories. It should be noted that the equality $\epsilon = 0$ is not by itself a prerequisite of the RNG method even though the convergence properties of the ϵ expansion are not fully known. It is clear that any expansion should be justified by a small value for the expansion parameter. However, this is rarely the case for the ϵ expansion. In most of the phase transition problems treated by the RNG, ϵ takes values like 1, 2, and even 3. The use of the ϵ expansion in fluid turbulence at high Reynolds numbers has been shown to be compatible with the Kolmogorov spectrum only if $\epsilon = 4$. Despite this difficulty and its somewhat unclear approximations, the RNG methods in fluid turbulence have been applied with some success.

We also note that in fluid turbulence the nonlocal approximation does not seem to play a crucial role. Indeed, two distinct approaches have been used to tackle turbulence starting from RNG ideas. The so-called Yakhot-Orszag [16] theory is based on the Forster-Nelson-Stephen [14] (FNS) paper which explicitly used the ϵ expansion and a nonlocal (in Fourier space) expansion. We have followed this approach in the present paper. Rose [48] has also proposed an iterative small scales elimination, based on similar arguments but without the ϵ and the nonlocal expansions. His presentation has the advantage to be easier to justify from a physical point of view but leads to a more complicated mathematical formulation. In this case, the expressions (31) and (33) for the renormalized parameters would not be easy to derive and probably more complex.

It is interesting to note that the results derived from both Rose's and the FNS approach are qualitatively and quantitatively similar. The derivation of the present results following Rose's approach could be an interesting but probably arduous prolongation of this work. It must be stressed that the nonlocal expansion and the Markovian approximation are also invoked to simplify other nonlinear theories like the DIA. We would close this discussion on the nonlocal expansion by mentioning two papers emphasizing the persisting discussion on the role of local and nonlocal interactions in turbulence. First, Teodorovich [49] has recently presented some arguments, based on a field-theory formalism, which support the idea that RNG methods in conjunction with ϵ expansion mean taking into account the local interactions in turbulence. Secondly, although the popular Kolmogorov-like derivation of turbulent spectra is based on purely local interactions, some opposite theories exist. For instance, Balk *et al.* [50] have recently developed theoretical arguments in favor of nonlocal drift-wave turbulence.

We now discuss some general results concerning the evaluation of the anomalous transport coefficient by using the RGKE. First, we have shown in Sec. IV that the anomalous transport coefficients reduce to their quasilinear scaling when the expansion parameter λ is small. This scaling is also recovered if the electric potential fluctuation is a white noise process [51].

Secondly, we deduce from the tensor of transport coefficients that the Onsager symmetry is broken for the anomalous flux-force relations. A similar result has been recently obtained in a totally different way by Balescu [52]. This result has also been obtained in the study of turbulence in weakly anisotropic neutral media [53].

Let us finally stress the great similarity between the present results and the other renormalization schemes of the gyrokinetic equation. For example, as in the DIA [10,11,30–32,36], the propagator, the vertex, and the source are renormalized by terms proportional to $k_\perp^2$. Another peculiarity of the white-noise potentials treated as the first example is the equality between ν and σ. Such a result can be seen as a frequency broadening ($\omega \rightarrow \hat{\omega} = \omega + ik_\perp^2 \nu$) in the renormalized GKE:

$$-i(\hat{\omega}-k_{\parallel}v_{\parallel})\delta h^{\alpha}(\hat{\mathbf{k}},\mathbf{v}) + i[\hat{\omega} - k_y V_{d\alpha}^T(v)]\frac{e_\alpha}{T_\alpha}F_0^\alpha(x,v)\delta\phi(\hat{\mathbf{k}}) = O(\epsilon). \tag{78}$$

Contrary to the work of Weinstock [54], no real frequency shift arises in our renormalized GKE. This is a simple consequence of the assumption of vanishing mean electric field. Let us also mention the crucial role played by the diamagnetic drift velocity in the second example. Indeed, the renormalized random source is in this case directly proportional to this velocity.

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