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Current-current correlation function in a driven diffusive system with nonconserving noise

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By field-theoretic renormalization group methods we study a diffusive system subjected to an external driving force with a noise which does not conserve the particle number. Especially, for dimensions $d<4$ we obtain the universal shape of the current-current correlation function containing both exact exponents and universal amplitudes.

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I. INTRODUCTION

Over the past years the long-time and critical behavior of diffusive system subjected to a driving force has attracted rapidly growing interest [1]. This is mainly caused by the richness of their highly nontrivial features, which generically result from the fact that, due to the driving force, in general these systems are in a state far from thermal equilibrium. Besides, driven diffusive systems might be suitable models for fast ionic conductors, which was first suggested by Katz, Lebowitz, and Spohn [2].

In the preceding papers of our group [3–5] the diffusional motion of driven interacting particles with various kinds of (particle number-) conserving noise has been investigated. In the present work, however, we study a system with a noise that locally violates particle number conservation. Such a model was originally introduced by Hwa and Kardar [6] as a continuum description of running sandpiles, but instead of the particle density here their equations involve a height variable, which is the deviation from the flat steady-state sand profile.

II. MODEL, RENORMALIZATION GROUP, AND SCALING RELATIONS

The general features of modeling and analyzing driven diffusive systems are described in [3,4]. Without noise, the density fluctuation $s(\mathbf{r},t)$ defined as the deviation of the actual density $c(\mathbf{r},t)$ from its uniform average $\bar{c}$ satisfies a continuity equation with a current $\mathbf{j}$ generated by the diffusional jumps of the particles and by the nonlinear drift in the direction of the driving force $j_\parallel(c)=j_\parallel(\bar{c})+j'_\parallel(\bar{c})s+\frac{1}{2}j''_\parallel(\bar{c})s^2+\cdots$ (henceforth the indices $\parallel$ and $\perp$ distinguish the spatial directions parallel and perpendicular to this direction). With a noise $\zeta(\mathbf{r},t)$ taken into account the equation of motion for s reads

$$\partial_t s(\mathbf{r},t)=\lambda(\Delta_\perp+\rho\Delta_\parallel)s(\mathbf{r},t)+\lambda\nabla_\parallel[\tilde{k}_s(\mathbf{r},t)+\tfrac{1}{2}gs(\mathbf{r},t)^2]+\zeta(\mathbf{r},t)\ . \quad (2.1)$$

The kind of noise ζ considered here is quite different from that in the previous works of our group [3–5]. So far, the noise has always been assumed to be particle number conserving, i.e., it was a derivative of a random current, whereas the noise in the system studied here violates this conservation locally, which means there are random particle sources and drains in this system only conserving the particle number in the mean.

Dimensional analysis (power counting) shows that—in the renormalization group sense—the relevant part of the noise is Gaussian with zero mean,

$$\langle\zeta(\mathbf{r},t)\rangle=0\ ,\qquad \langle\zeta(\mathbf{r},t)\zeta(\mathbf{r}',t')\rangle=2\lambda\delta(\mathbf{r}-\mathbf{r}')\delta(t-t')\ . \quad (2.2)$$

By a suitable scale change of s the kinetic coefficient λ in (2.2) is the same one as in (2.1). The term proportional to $\tilde{k}$ can be eliminated by a Galilei transformation

$$\mathbf{r}\to\mathbf{r}+\lambda\tilde{k}\mathbf{e}_\parallel t,\quad t\to t\ , \quad (2.3)$$

with $\mathbf{e}_\parallel$ being the unit vector in the longitudinal direction.

Note that Eqs. (2.1) and (2.2) constitute the fundamental equation of motion for noncritical driven diffusive systems with nonconserving noise and contain all relevant terms in the renormalization group sense, which can be proved by power counting near $d_c=4$. Up to modified notation for fields and coupling constants (2.1) and (2.2) is exactly the model Hwa and Kardar (henceforth abbreviated HK) recently suggested to describe running sandpiles [6,7]. Concerning this model we wish to remark the following.

(i) In the simulations of HK, particles are only randomly added to the sandpile, but not removed. Notice, however, that a continuum model that is naively derived by adding a random deposition process to a conserved density dynamics does not describe the running sandpile properly, since such a model involves a noise $\zeta(\mathbf{r},t)=\bar{\zeta}+\delta\zeta(\mathbf{r},t)$ that consists of an overall average value $\bar{\zeta}>0$ and fluctuations $\delta\zeta(\mathbf{r},t)$ about this value, where $\delta\zeta(\mathbf{r},t)$ obeys (2.2). Then, $\bar{\zeta}$ is a relevant parameter, which cannot be set to zero as long as $\lambda\neq0$ in (2.2) in order to hold the particle source density $\zeta(\mathbf{r},t)$ positive. As a consequence the averaged stationary solution yields a current density, which linearly increases with the coordinate in the longitudinal direction due to the constant amount of the noise, whereas in the simulations of HK a constant current density is expected.

(ii) HK used a dynamical renormalization group analysis to study their model defined by (2.1) and (2.2). Below the critical dimension $d_c=4$ they find the exact exponents that describe the scaling of various quantities at the nontrivial stable fixed point of the system. It is worth mentioning that one of these scaling relations says that density fluctuations spread faster than diffusively, which is a well known scenario of driven diffusive systems [1–5].

Moreover, in order to compare their calculation with their simulations they investigated the scaling properties of the current-current correlation function (from now on abbreviated CCC function), where, however, first they completely ignored the diffusive part of the current (proportional to $\nabla_\parallel s$) and second factorized the remaining $\langle s^2 s^2\rangle$ correlation to $\langle ss\rangle^2$. Here this procedure leads to the correct scaling properties. But this factorization fails if the composite field s^2 needs a multiplicative renormalization (as, e.g., in the Φ^4 theory).

(iii) As the CCC function should be an easily accessible quantity also in the simulations of a noncritical driven diffusive system with nonconserving noise, we study this function in Secs. III and IV in a systematic field-theoretic approach. Eventually we obtain the exact scaling form in the asymptotic long wavelength limit with universal amplitudes.

We proceed investigating the driven diffusive system with nonconserving noise based on (2.1) and (2.2).

To set up a renormalized field theory, it is convenient to recast the model in terms of a dynamic functional [8–11]

$$\mathcal{J}[s,\tilde{s}]=\int dt\, d^dx\{\tilde{s}[\dot{s}-\lambda(\Delta_\perp+\rho\Delta_\parallel)s]+\tfrac{1}{2}\lambda g(\nabla_\parallel\tilde{s})s^2-\lambda\tilde{s}^2\}\ , \tag{2.4}$$

where $\tilde{s}(\mathbf{r},t)$ is a Martin-Siggia-Rose [12] response field. Correlation and response functions can now be expressed as functional averages with weight $\exp(-\mathcal{J})$. The functional $\mathcal{J}$ is invariant under the scale transformation

$$\begin{aligned} &x_\parallel\to\alpha x_\parallel,\quad x_\perp\to x_\perp,\quad s\to\alpha^{-1/2}s\ ,\\ &\tilde{s}\to\alpha^{-1/2}\tilde{s},\quad \rho\to\alpha^2\rho,\quad g\to\alpha^{3/2}g\ . \end{aligned} \tag{2.5}$$

Hence the parameter

$$u=A_\epsilon\mu^{-\epsilon}\rho^{-3/2}g^2 \tag{2.6}$$

is the true invariant dimensionless expansion parameter of perturbation theory, where μ^{-1} is an arbitrary external length scale and A_ϵ is a convenient numerical factor depending on $\epsilon=d_c-d$, which will be determined below.

Additionally, $\mathcal{J}$ is Galilei invariant under

$$\begin{aligned} &s(\mathbf{r},t)\to s(\mathbf{r}+\lambda g a\mathbf{e}_\parallel t,t)+a\ ,\\ &\tilde{s}(\mathbf{r},t)\to\tilde{s}(\mathbf{r}+\lambda g a\mathbf{e}_\parallel t,t)\ . \end{aligned} \tag{2.7}$$

To study the infrared properties we apply standard renormalization group methods [13–15]. We use dimensional regularization in $d=4-\epsilon$ followed by minimal subtraction. Denoting by $\Gamma_{\tilde{n},n}(\{\mathbf{q},\omega\})$ the one-line-irreducible vertex functions with $\tilde{n}$ $\tilde{s}$ legs and n s legs, only $\Gamma_{1,1}$ and $\Gamma_{1,2}$ are primitively divergent. Note that the nontrivial diagrams contributing to $\Gamma_{\tilde{n},n}(\{\mathbf{q},\omega\})$ cast out at least a factor $|\mathbf{q}|^{\tilde{n}}$, because the interaction vertex is conserving.

As a consequence of Galilean invariance the ultraviolet divergent graphs associated with $\Gamma_{1,2}$ sum to zero order by order in perturbation theory [3,4]. Thus the coupling constant $\mathring{g}=g$ needs no renormalization. The divergence of $\Gamma_{1,1}$ is cured by a multiplicative redefinition of the parameter $\rho\to\mathring{\rho}=Z_\rho\rho$. A one-loop calculation in dimensional regularization yields to order $q_\parallel^2$:

$$\Gamma_{1,1}(\mathbf{q},\omega)=i\omega+\lambda q_\perp^2+\lambda q_\parallel^2 Z_\rho\rho\left[1+\frac{3}{4\epsilon}\frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{6}\right)}{(4\pi)^{d/2}\left(1-\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{4}\right)A_\epsilon}u\left(\frac{2\lambda\mu^2}{i\omega}\right)^{\epsilon/2}+O(u^2)\right]\ . \tag{2.8}$$

We set

$$A_\epsilon=\frac{\Gamma\left(1+\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{6}\right)}{(4\pi)^{d/2}\left(1-\frac{\epsilon}{2}\right)\left(1-\frac{\epsilon}{4}\right)}\ , \tag{2.9}$$

where $\Gamma(z)$ denotes Euler's Γ function, and find by minimal subtraction

$$Z_\rho=1-\frac{3}{4\epsilon}u+O(u^2)\ . \tag{2.10}$$

The unrenormalized theory is independent of the parameter μ. This leads to the renormalization group equation

$$\{\mu\partial_\mu+\beta\partial_u+\rho\zeta\partial_\rho\}\Gamma_{\tilde{n},n}(\{\mathbf{q},\omega\},\rho,u,\mu,\lambda)=0\ . \tag{2.11}$$

The parameter functions are given by

$$\begin{aligned} &\beta(u)=-[\epsilon+\tfrac{3}{2}\zeta(u)]u\ ,\\ &\zeta(u)=-\tfrac{3}{4}u+O(u^2)\ , \end{aligned} \tag{2.12}$$

where the former equation holds exactly due to the nonrenormalization of g. Taking l as a flow parameter and defining the trajectories $\bar{u}(l)$, $\bar{\rho}(l)$, and $\bar{\mu}(l)=\mu\, l$ via

$$l\frac{d}{dl}\bar{u}(l)=\beta[\bar{u}(l)],\quad l\frac{d}{dl}\ln\bar{\rho}(l)=\zeta[\bar{u}(l)]\ , \tag{2.13}$$

yields the solution of (2.11):

$$\Gamma_{\tilde{n},n}(\{\mathbf{q},\omega\},\rho,u,\mu,\lambda)=\Gamma_{\tilde{n},n}[\{\mathbf{q},\omega\},\bar{\rho}(l),\bar{u}(l),\mu l,\lambda] \; . \tag{2.14}$$

In the scaling limit $l \ll 1$ corresponding to $|q_\parallel/\mu|,|q_\perp/\mu|,|\omega/\lambda\mu^2| \ll 1$, $\bar{u}(l)$ flows to an infrared stable fixed point $u_* \neq 0$ for $\epsilon>0$, which, according to Eq. (2.12), is exactly given by

$$\zeta(u_*)=-\tfrac{2}{3}\epsilon \; . \tag{2.15}$$

Using the one-loop result we obtain

$$u_*=\tfrac{8}{9}\epsilon+O(\epsilon^2) \; . \tag{2.16}$$

A two-loop calculation, which is shown in Appendix A, yields

$$\zeta=-[\tfrac{3}{4}+(\tfrac{15}{64}\ln\tfrac{4}{3}+\tfrac{11}{384})u+O(u^2)]u \; , \tag{2.17}$$

giving the fixed point to order ϵ^2:

$$u_*=\tfrac{8}{9}\epsilon[1-(\tfrac{11}{324}+\tfrac{5}{18}\ln\tfrac{4}{3})\epsilon+O(\epsilon^2)] \; . \tag{2.18}$$

As g needs no renormalization,

$$A_\epsilon g^2=\bar{\mu}(l)^\epsilon\bar{u}(l)\bar{\rho}(l)^{3/2} \tag{2.19}$$

is invariant under the flow of l. A comparison of the right-hand side (rhs) for $l=1$ and $l \ll 1$ directly yields the trajectory $\bar{\rho}(l)$ for $l \ll 1$:

$$\bar{\rho}(l)=\rho\left[\frac{u}{u_*}\right]^{2/3} l^{-2\epsilon/3} \; . \tag{2.20}$$

Thus the renormalization group equation (2.11), in conjunction with dimensional analysis and the scale transformation (2.5), is exploited to give the exact scaling form of the vertex functions for $l \ll 1$:

$$\Gamma_{\tilde{n},n}(\{\omega,q_\parallel,q_\perp\},u,\rho,\lambda,\mu)=l^{\delta(\tilde{n},n)}\Gamma_{\tilde{n},n}\left[\left\{\frac{\omega}{l^2},\frac{q_\parallel}{l^{1+\epsilon/3}},\frac{q_\perp}{l}\right\},u_*,\left[\frac{u}{u_*}\right]^{2/3}\rho,\lambda,\mu\right] , \tag{2.21}$$

where

$$\delta(\tilde{n},n)=\tfrac{1}{3}[d(2-\tilde{n}-n)+10-5\tilde{n}+n] \; . \tag{2.22}$$

This relation easily provides the exact exponents calculated by HK.

III. RENORMALIZATION OF $j_\parallel$ AND s^2

Before investigating the current-current correlation function we have to take a closer look into its single components, namely the current. As anomalous properties arise only in the driving field direction we restrict ourselves to the longitudinal current $j_\parallel$, which is [Eq. (2.1)]

$$-j_\parallel(\mathbf{r},t)=\lambda\rho\nabla_\parallel s(\mathbf{r},t)+\tfrac{1}{2}\lambda g s^2(\mathbf{r},t) \; . \tag{3.1}$$

In terms of renormalized field theory, s^2 is a composite operator [13,16], which consists of the product of two fundamental fields s taken at the same point $(\mathbf{r},t)$. This is why an insertion of a composite operator into correlation or response or vertex functions generally leads to new divergencies that must be removed in a renormalization procedure beyond that in Sec. II.

Adding another source term to the dynamic functional

$$\mathcal{J}\to\mathcal{J}-\tfrac{1}{2}\int dt\, d^drk(\mathbf{r},t)s^2(\mathbf{r},t) \; , \tag{3.2}$$

where the coupling k is assumed to be local and time dependent for the purpose of functional differentiation, we are now able to generate response and correlation functions with insertion of the composite operator $\tfrac{1}{2}[s^2]$ by differentiating the generating functional

$$Z[h,\tilde{h},k]=\int\mathcal{D}[s]\mathcal{D}[i\tilde{s}]\exp\left\{-\mathcal{J}[s,\tilde{s}]+\int d^dr\,dt(\tilde{h}\tilde{s}+hs)\right\} \tag{3.3}$$

with respect to h, $\tilde{h}$, and k:

$$\left\langle\prod_{j=1}^{\tilde{n}}\tilde{s}(\tilde{\mathbf{r}}_j,\tilde{t}_j)\prod_{i=1}^{n}s(\mathbf{r}_i,t_i)\tfrac{1}{2}[s^2(\mathbf{r},t)]\right\rangle=\prod_{j=1}^{\tilde{n}}\frac{\delta}{\delta\tilde{h}(\tilde{\mathbf{r}}_j,\tilde{t}_j)}\prod_{i=1}^{n}\frac{\delta}{\delta h(\mathbf{r}_i,t_i)}\frac{\delta}{\delta k(\mathbf{r},t)}Z[h,\tilde{h},k]|_{h=\tilde{h}=k=0} \; . \tag{3.4}$$

The fact that in our bulk theory the integral over a total derivative with respect to either s or $\tilde{s}$ vanishes is expressed in the so-called "equation of motion":

$$0=\int\mathcal{D}[s]\mathcal{D}[i\tilde{s}]\frac{\delta}{\delta\tilde{s}}\exp\left\{-\mathcal{J}[s,\tilde{s}]+\int d^dr\,dt(\tilde{h}\tilde{s}+hs)\right\} . \tag{3.5}$$

The functional differentiation of the integrand and the usual Legendre transformation from Z to the generating functional Γ of the vertex functions via

$$\ln Z[h,\tilde{h}]+\Gamma[s,\tilde{s}]=\int d^dr\,dt(\tilde{h}\tilde{s}+hs)$$

$$\text{with } \tilde{s}:=\frac{\delta\ln Z}{\delta\tilde{h}},\; s:=\frac{\delta\ln Z}{\delta h} \tag{3.6}$$

produce the following equation:

$$\frac{\delta\mathring{\Gamma}}{\delta\mathring{\tilde{s}}}+[-\partial_t+\lambda(\Delta_\perp+\mathring{\rho}\Delta_\parallel)]\tilde{s} \\ -\lambda\mathring{g}\nabla_\parallel\frac{\delta\mathring{\Gamma}}{\delta\mathring{k}}+2\lambda\mathring{\tilde{s}}|_{s=\tilde{s}=\mathring{k}=0}=0\ , \quad (3.7)$$

where the superscript ° means the bare unrenormalized quantities. From this equation, by differentiation with respect to $\tilde{s}$ and s, we can deduce a series of Ward identities, which connect vertex functions with and without insertion of $[s^2]/2$. Especially, single differentiation with respect to s yields the (Fourier-transformed) relation

$$\mathring{\Gamma}_{1,1}(\mathbf{q},\omega)=[i\omega+\lambda(q_\perp^2+\mathring{\rho}q_\parallel^2)] \\ -i\lambda\mathring{g}q_\parallel\mathring{\Gamma}_{0,1;[s^2]/2}(\mathbf{q},\omega)\ , \quad (3.8)$$

where $\Gamma_{\tilde{n},n;[s^2]/2}$ denotes the vertex functions with $\tilde{n}$ $\tilde{s}$ legs, n s legs, and one insertion of the composite operator $[s^2]/2$. After transition to renormalized quantities according to $\mathring{\rho}=Z_\rho\rho, \mathring{g}=g, \mathring{\Gamma}=\Gamma$, the previous equation reads

$$\Gamma_{1,1}(\mathbf{q},\omega)=i\omega+\lambda(q_\perp^2+Z_\rho\rho q_\parallel^2) \\ -i\lambda g q_\parallel\Gamma_{0,1;[s^2]/2}(\mathbf{q},\omega)\ . \quad (3.9)$$

As $\Gamma_{1,1}(\mathbf{q},\omega)$ is finite in the renormalized theory we immediately see from the singular parts of the previous equation that the divergence, which is produced by the insertion of $[s^2]/2$ into $\Gamma_{0,1;[s^2]/2}$, can be cured by an additive minimal renormalization,

$$\Gamma_{0,1;[s^2]_R/2}=\Gamma_{0,1;[s^2]/2}+\frac{Z_\rho-1}{g}\rho i q_\parallel \quad (3.10)$$

or, explicitly, the composite field $[s^2]$ needs an additive renormalization (in real space):

$$[s^2]_R=[s^2]+2\frac{Z_\rho-1}{g}\rho\nabla_\parallel s\ . \quad (3.11)$$

Notice that all other $\Gamma_{\tilde{n},n;[s^2]/2}$ are finite, which directly follows from Eq. (3.7). Turning now back to the full longitudinal current [Eq. (3.1)],

$$-j_\parallel=\lambda\frac{g}{2}s^2+\lambda\mathring{\rho}\nabla_\parallel s \\ =\lambda\frac{g}{2}\left[s^2+2\frac{Z_\rho-1}{g}\rho\nabla_\parallel s\right]+\lambda\rho\nabla_\parallel s \\ =\lambda\frac{g}{2}[s^2]_R+\lambda\rho\nabla_\parallel s \quad (3.12)$$

demonstrates that the gradient term of $j_\parallel$ provides the additive renormalization for the composite field $s^2/2$ so that only the complete longitudinal current is a well defined renormalized quantity since ρ is renormalized. Therefore, when investigating correlation functions with $j_\parallel$ the gradient term cannot be neglected.

IV. CURRENT-CURRENT CORRELATION FUNCTION

The scaling properties of the CCC function $C_{jj}(\mathbf{r},t):=\langle j_\parallel(\mathbf{r},t)j_\parallel(\mathbf{0},0)\rangle_c$ (the subscript c denotes the cumulant or connected part of the correlation function, i.e., C measures fluctuations around the average) are directly obtained by combining scale transformation ($j_\parallel\to\alpha^{1/2}j_\parallel$), dimensional analysis ($j\sim\lambda\mu^{d/2}$), and the renormalization group equation (2.11):

$$C_{jj}(\{x_\parallel,x_\perp,t\},u,\rho,\lambda,\mu)=l^{4(1-\epsilon/3)}C_{jj}\left[\{l^{1+\epsilon/3}x_\parallel,lx_\perp,l^2t\},u_*,\left[\frac{u}{u_*}\right]^{2/3}\rho,\lambda,\mu\right]\ . \quad (4.1)$$

Its Fourier transform reads accordingly, after choosing the flow parameter $l^2=|\omega/\lambda\mu^2|$,

$$C_{jj}(\{q_\parallel,q_\perp,\omega\})=\frac{1}{\omega^{1+\epsilon/3}}C_{jj}\left[\left\{\frac{q_\parallel}{\omega^{(1+\epsilon/3)/2}},\frac{q_\perp}{\omega^{1/2}}\right\}\right]\ . \quad (4.2)$$

This agrees with the scaling form given by HK, but we remark that both terms of $j_\parallel$ lead to this scaling relation, not only the s^2 term. Now we shall explicitly calculate the CCC function. As pointed out in the preceding section, we must take both terms of $j_\parallel$ into account so that

$$C_{jj}(1,2)=\left[\frac{\lambda g}{2}\right]^2\langle s^2(1)s^2(2)\rangle_c \\ +\frac{\lambda^2\rho g}{2}\nabla_\parallel^{(1)}\langle s(1)s^2(2)\rangle_c \\ +\frac{\lambda^2\rho g}{2}\nabla_\parallel^{2}\langle s(2)s^2(1)\rangle_c \\ +(\lambda\rho)^2\nabla_\parallel^{(1)}\nabla_\parallel^{(2)}\langle s(1)s(2)\rangle_c\ , \quad (4.3)$$

where $\nabla_\parallel^{(i)}$ means the derivative with respect to the ith argument and i as argument is the abbreviation for $(\mathbf{r}_i,t_i)$ from now on.

It is useful to reduce these cumulants to two functions $E(1-2)$ and $D(1-2)$, which can be generated by functional differentiation of an extended dynamic functional [cf. Eq. (3.2)]

$$\mathcal{J}\to\mathcal{J}-\int dt\, d^dr[\tfrac{1}{2}ks^2+\tilde{k}(\nabla_\parallel\tilde{s})s] \tag{4.4}$$

and are defined by

$$D(1-2):=-4\frac{\delta^2\Gamma}{\delta k(1)\delta k(2)}\ ,$$
$$\nabla_\parallel^{(1)}E(1-2):=2\frac{\delta^2\Gamma}{\delta k(1)\delta\tilde{k}(2)}\ , \tag{4.5}$$

where an overall factor $\nabla_\parallel^{(1)}$ coming from the insertion of $(\nabla_\parallel\tilde{s})s$ is explicitly factored out in the definition of E. A graphical representation of E and D is given in Fig. 1. The connection between the four terms on the rhs of (4.3) and E,D as well as the propagator G and the correlator C is developed in detail in Appendix B.

A further reduction can be achieved by the renormalized (i.e., finite) parts of Eq. (3.8),

$$\Gamma_{1,1}(1-2)=[\partial_t-\lambda(\Delta_\perp+\rho\Delta_\parallel)]\delta(1-2)-\tfrac{1}{2}(\lambda g)^2\Delta_\parallel^{(1)}E(1-2)\ , \tag{4.6}$$

and by another differentiation of the equation of motion (3.7) with respect to $\tilde{s}$:

$$\Gamma_{2,0}(1-2)=-2\delta(1-2)+\left[\frac{\lambda g}{2}\right]^2\Delta_\parallel^{(1)}D(1-2)\ . \tag{4.7}$$

Turning now to Fourier-transformed quantities we must take their time dependence into consideration:

$$C(t_1-t_2)=C(t_2-t_1),\quad D(t_1-t_2)=D(t_2-t_1)\ , \tag{4.8}$$

but

$$G(t_1-t_2)\sim\Theta(t_1-t_2),\quad E(t_1-t_2)\sim\Theta(t_1-t_2)\ . \tag{4.9}$$

Combining the results of Appendix B [(B.5), (B.8), and (B.9)] with Eqs. (4.6)–(4.8) and using the well known relations

$$G(\mathbf{q},\omega)=[\Gamma_{1,1}(\mathbf{q},\omega)]^{-1}\ ,$$
$$C(\mathbf{q},\omega)=-\Gamma_{2,0}(\mathbf{q},\omega)G(\mathbf{q},\omega)G(\mathbf{q},-\omega)\ , \tag{4.10}$$

we obtain C_{jj} in a compact form:

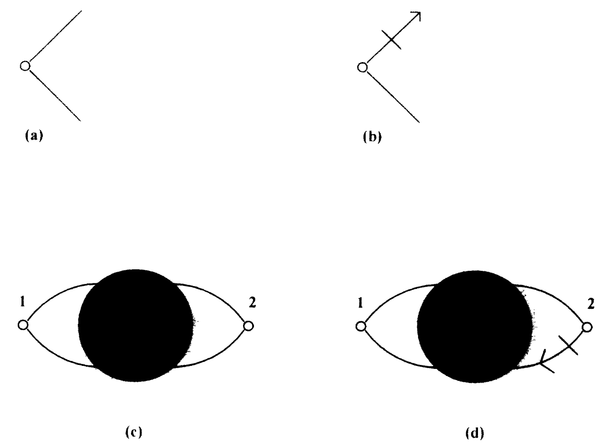

FIG. 1. In addition to the graphical elements shown in Appendix A there are (a) an $[s^2/2]$ insertion and (b) an $[s\nabla_\parallel\tilde{s}]$ insertion. Thus (c) represents $D(1-2)/4$ [Eq. (4.5)] and (d) $-\nabla_\parallel^{(1)}E(1-2)/2$ [Eq. (4.5)]. The shaded areas means the connected, one-line-irreducible part of a vertex function. A close look at (c) and (d) directly provides Eqs. (4.6) and (4.7).

$$C_{j,j}(q_\parallel,q_\perp,\omega)=[\Gamma_{1,1}(\mathbf{q},\omega)\Gamma_{1,1}(\mathbf{q},-\omega)]^{-1}\left\{[\omega^2+(\lambda q_\perp^2)^2]\frac{(\lambda g)^2}{4}D(\mathbf{q},\omega)+2\lambda q_\parallel^2\left[\lambda\rho+\frac{(\lambda g)^2}{2}E(\mathbf{q},\omega)\right]\left[\lambda\rho+\frac{(\lambda g)^2}{2}E(\mathbf{q},-\omega)\right]\right\}\ . \tag{4.11}$$

Notice that the previous equation holds exactly, whereas we are now forced to compute the three functions $\Gamma_{1,2}(\mathbf{q},\omega)$, $E(\mathbf{q},\omega)$, and $D(\mathbf{q},\omega)$ perturbatively.

(i) A one-loop calculation of $\Gamma_{1,1}(\mathbf{q},\omega)$ was already performed in Sec. II [Eq. (2.8)] and its renormalized version reads

$$\Gamma_{1,1}(q_\parallel,q_\perp,\omega;u,\rho,\lambda,\mu)=i\omega+\lambda q_\perp^2+\lambda\rho q_\parallel^2\left[1-\frac{3u}{8}\ln\left[\frac{i\omega}{2\lambda\mu^2}\right]\right]+O(u^2,q^4)\ , \tag{4.12}$$

which by applying the scaling relation (2.21) becomes

$$\Gamma_{1,1}(q_\parallel,q_\perp,\omega;u,\rho,\lambda,\mu)=l^2\left\{\frac{i\omega}{l^2}+\lambda\frac{q_\perp^2}{l^2}+\lambda\left[\frac{u}{u_*}\right]^{2/3}\rho\frac{q_\parallel^2}{l^{2+2\epsilon/3}}\left[1-\frac{3u_*}{8}\ln\left[\frac{i\omega}{2\lambda\mu^2l^2}\right]+O(u_*^2,q^2)\right]\right\}\ . \tag{4.13}$$

By an adequate choice of $l^2=i\omega/2\lambda\mu^2\ll 1$ (we have to consider $i\omega$ as a positive real quantity first and then continue it analytically) and a subsequent elimination of the length scale μ^{-1} via Eq. (2.6) we obtain $\Gamma_{1,1}$, expressed in the dimensionless variables

$$\hat{\omega}=\frac{\omega}{2g^{4/\epsilon}\lambda},\quad \hat{q}=\frac{q}{g^{2/\epsilon}}\ , \tag{4.14}$$

as

$$\Gamma_{1,1}(q_\parallel,q_\perp,\omega)=2\lambda g^{4/\epsilon}\left[i\hat\omega+\tfrac{1}{2}\hat q_\perp^2+\tfrac{1}{2}C_1\frac{\hat q_\parallel^2}{(i\hat\omega)^{\epsilon/3}}\right]+O(\hat q^4)\;, \tag{4.15}$$

where $C_1=(A_\epsilon/u_*)^{2/3}$ is a universal constant.

(ii) $E(\mathbf{q},\omega)$ is closely related to $\Gamma_{1,1}(\mathbf{q},\omega)$ via Eq. (4.6). As we are interested in a q expansion of C_{jj}, it is sufficient to take $E(\mathbf{q},\omega)$ at $q=0$; then its scaled version reads

$$\rho+\tfrac{1}{2}\lambda g^2E(\mathbf{q}=0,\omega)=C_1\left[\frac{1}{i\hat\omega}\right]^{\epsilon/3}\;. \tag{4.16}$$

(iii) $D(\mathbf{q},\omega)$ remains to be computed and in a one-loop approximation the integral

$$D(\mathbf{q},\omega)^{1-l}=\frac{2}{(2\pi)^d}\int d^dp\left[p+\frac{q}{2}\right]_\rho^{-2}\left[p-\frac{q}{2}\right]_\rho^{-2}\left[\left\{i\omega+\lambda\left[\left[p+\frac{q}{2}\right]_\rho^2+\left[p-\frac{q}{2}\right]_\rho^2\right]\right\}^{-1}+[\omega\to-\omega]^{-1}\right] \tag{4.17}$$

is to be evaluated, where $q_\rho^2:=q_\perp^2+\rho q_\parallel^2$ and the frequency integration is already performed. The fact that the integrand consists of two complex conjugated parts originates from the two possible time orders of the s^2 insertions in the one-loop graph. We calculate it on the one hand for $\omega=0$, finding

$$D(\mathbf{q},\omega=0)=\frac{2^{3+\epsilon}\Gamma(1+\epsilon/2)\ln 2}{(4\pi)^{d/2}\lambda\rho^{1/2}}(q_\rho^2)^{-1-\epsilon/2}[1+O(u,\epsilon,q^2)]\;, \tag{4.18}$$

and on the other hand for $q_\rho^2\ll 2|\omega|/\lambda$, obtaining

$$D(\mathbf{q},\omega)=\frac{2^{2+\epsilon}\Gamma(1+\epsilon/2)}{(4\pi)^{d/2}\lambda\rho^{1/2}}\left\{q_\rho^{-\epsilon}\frac{2\Gamma(1/2)\Gamma(2-\epsilon/2)}{\epsilon(2-\epsilon)\Gamma(3/2-\epsilon/2)}\left[\frac{\lambda}{2i\omega}\right]\right.$$
$$\left.-\frac{4}{\epsilon(2-\epsilon)}\left[\frac{\lambda}{2i\omega}\right]^{1+\epsilon/2}\left[1-\frac{\epsilon(6-\epsilon)}{2(4-\epsilon)}\frac{\lambda q_\rho^2}{2i\omega}+O\left[\left[\frac{q_\rho^2}{2i\omega}\right]^2\right]\right]+\omega\to-\omega\right\}\;. \tag{4.19}$$

Herein the first term is cancelled by its complex conjugated part and therefore

$$D(\mathbf{q}=0,\omega)=\frac{2^{5+\epsilon}\Gamma(1+\epsilon/2)}{(4\pi)^{d/2}\lambda\rho^{1/2}(2-\epsilon)}\frac{\sin(\pi\epsilon/4)}{\epsilon}\times\left[\frac{\lambda}{2|\omega|}\right]^{1+\epsilon/2}\;. \tag{4.20}$$

Notice that $D(\mathbf{q},\omega)$ does not show any divergence, neither in limit $|\mathbf{q}|\to 0,\epsilon\to 0$ of (4.20), because the ϵ pole is compensated by the $\sin(\pi\epsilon/4)$, nor at $\omega=0$ (4.18). This statement is corroborated by Eq. (4.7), which only involves finite quantities. To find the scaled shape of $D(\mathbf{q}=0,\omega)$, the scaling equation (2.21) is employed for $\Gamma_{2,0}$, which is referred to $D(\mathbf{q},\omega)$ via Eq. (4.7) and gives

$$\left[\frac{\lambda g}{2}\right]^2D(\mathbf{q}=0,\omega)=\frac{2\lambda\sin(\pi\epsilon/4)\Gamma(1+\epsilon/2)}{g^{4/\epsilon}(4\pi)^{d/2}\epsilon(2-\epsilon)}\times\left[\frac{u_*}{A_\epsilon}\right]^{1/3}\frac{1}{|\hat\omega|^{1+\epsilon/3}}\;. \tag{4.21}$$

Setting now (4.15), (4.16), and (4.21) into (4.11), where we point out that ω^2, $q_\perp^2$, and $q_\parallel^2$ as coefficients of $D(\mathbf{q},\omega)$ and $E(\mathbf{q},\omega)$ remain unscaled, we finally arrive at the universal form of the CCC function (up to higher orders in q)

$$C_{jj}(q_\parallel,q_\perp,\omega)=\frac{\lambda}{g^{4/\epsilon}}\left[\frac{1}{|\hat\omega|}\right]^{1+\epsilon/3}\times\frac{C_2\left[1+\dfrac{1}{4}\left[\dfrac{\hat q_\perp^2}{|\hat\omega|}\right]^2\right]+\dfrac{1}{2}C_1^2\dfrac{\hat q_\parallel^2}{|\hat\omega|^{1+\epsilon/3}}}{\left|1+\dfrac{1}{2}\dfrac{\hat q_\perp^2}{i\hat\omega}+\dfrac{1}{2}C_1\dfrac{\hat q_\parallel^2}{(i\hat\omega)^{1+\epsilon/3}}\right|^2}\;. \tag{4.22}$$

We stress that, besides the universal and even exact exponents, C_{jj} also contains the universal amplitudes

$$C_1=A_\epsilon^{2/3}u_*^{-2/3}\;,$$
$$C_2=\frac{2\sin(\pi\epsilon/4)\Gamma(1+\epsilon/2)}{\epsilon(2-\epsilon)(4\pi)^{d/2}A_\epsilon^{1/3}}u_*^{1/3}\;. \tag{4.23}$$

In order to make these results comparable to Monte Carlo simulations, which have not been our topic here and remain to be performed, we give the numerical values for C_1 and C_2 in $d=3$, i.e., $\epsilon=1$.

It is advantageous to extract the purely geometric factor $S_d=O_d/(2\pi)^d=2/(4\pi)^{d/2}\Gamma(d/2)$, where O_d is the surface of the unit sphere in d dimensions, from A_ϵ (2.9):

$$A_\epsilon=\frac{1-\epsilon/6}{1-\epsilon/4}\tfrac{1}{2}S_d[1+O(\epsilon^2)]\;. \tag{4.24}$$

Then, with the two-loop value of u_* (2.18) the ϵ expansion of C_1 and C_2 reads

$$C_1=\left[\frac{9}{16}\frac{S_d}{\epsilon}\right]^{2/3}[1+\epsilon(\tfrac{19}{243}+\tfrac{5}{27}\ln\tfrac{4}{3})+O(\epsilon^2)]\ ,$$
$$C_2=\frac{\pi}{4}\left[\frac{2}{9}\epsilon S_d^2\right]^{1/3}[1-\tfrac{1}{2}\epsilon(\tfrac{19}{243}+\tfrac{5}{27}\ln\tfrac{4}{3})+O(\epsilon^2)]\ , \tag{4.25}$$

where S_d remains unexpanded.

Setting now $\epsilon=1$ and $d=3$, respectively, we obtain

$$C_1(d=3)=0.107,\quad C_2(d=3)=0.0609\ . \tag{4.26}$$

In addition, we consider the combination

$$C_1C_2^2=\left[\frac{\pi S_d}{8}\right]^2[1+O(\epsilon^2)] \tag{4.27}$$

that might be the numerically stablest value, because it is independent both of u_* and of A_ϵ. In three spatial dimensions it is

$$C_1C_2^2(d=3)=3.96\times10^{-4}\ . \tag{4.28}$$

V. CONCLUDING REMARKS

Equations (4.22) and (4.15) contain very detailed information on the current-current correlation function and on the response function $G_{1,1}(\mathbf{q},\omega)=\Gamma_{1,1}^{-1}(\mathbf{q},\omega)$, respectively. It is desirable to confirm these results by Monte Carlo simulations, which have not been done yet. Both functions should be easily accessible in such simulations. Notice that these simulations of a driven diffusive system with nonconserving noise are not redundant, for it is an independent model, which is distinct from the running sandpile.

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APPENDIX A: TWO-LOOP CALCULATION OF $\Gamma_{1,1}$

In this section we calculate the singular parts of $\Gamma_{1,1}(\mathbf{q},\omega)$ in a two-loop approximation. The elements of our perturbation expansion are the Gaussian propagator

$$G_0(\mathbf{q},\omega)=[i\omega+\lambda(q_\perp^2+\rho q_\parallel^2)]^{-1}\ , \tag{A1}$$

the Gaussian correlator

$$C_0(\mathbf{q},\omega)=2\lambda[\omega^2+\lambda^2(q_\perp^2+\rho q_\parallel^2)^2]^{-1}\ , \tag{A2}$$

and the conserving vertex

$$v(\mathbf{q})=-i\lambda g q_\parallel\ . \tag{A3}$$

Their graphical representation is shown in Fig. 2, where the $\tilde{s}$ legs are indicated by an arrow and the $q_\parallel$ of the vertex by a dash perpendicular to the propagator line.

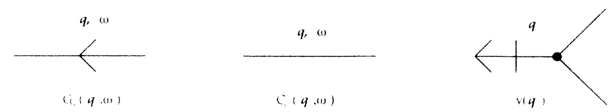

FIG. 2. Elements of perturbation theory: Gaussian propagator $G_0(\mathbf{q},\omega)$, Gaussian correlator $C_0(\mathbf{q},\omega)$, and vertex $v(\mathbf{q})$.

In the wave-vector-frequency picture there are eight different two-loop diagrams (Fig. 3) contributing to $\Gamma_{1,1}$ and obeying causality, which forbids closed propagator loops. Energy and momentum conservation demand that at each vertex the sum both over all frequencies and over all wave vectors is zero.

Evaluation of these diagrams means integration over all internal frequencies and wave vectors. The integration over the internal frequencies can be easily performed with the help of residues.

As $\Gamma_{1,1}$ is quadratically divergent and the external $\tilde{s}$ leg already provides a factor $q_\parallel$, the parts of the integrands, which are proportional to $q_\parallel$, contain all singularities. Therefore, before integrating over the internal momenta the integrands are expanded with respect to the external momentum $q_\parallel$.

The subsequent integration over the internal momenta for all two-loop diagrams together yields, up to convergent parts,

$$\frac{\lambda g^4 q_\parallel^2}{8\rho^2}G_\epsilon^2\left[\frac{i\omega}{2\lambda}\right]^{-\epsilon}\left[\frac{9}{8}\frac{1}{\epsilon^2}+\frac{15}{16}\frac{1}{\epsilon}\ln\frac{3}{4}+\frac{115}{96}\frac{1}{\epsilon}\right]\ . \tag{A4}$$

Thus the perturbation expansion of $\Gamma_{1,1}$ to two-loop order reads

$$\Gamma_{1,1}=i\omega+\lambda q_\perp^2+\lambda q_\parallel^2\times\left\{\mathring{\rho}+\left[\frac{3}{4}+\frac{7}{16}\epsilon\right]\frac{\mathring{g}^2}{\mathring{\rho}^{1/2}}\frac{G_\epsilon}{\epsilon}\left[\frac{i\omega}{2\lambda}\right]^{-\epsilon/2}-\left[\frac{9}{64}+\left[\frac{15}{128}\ln\frac{3}{4}+\frac{115}{768}\right]\epsilon\right]\times\frac{\mathring{g}^4}{\mathring{\rho}^2}\frac{G_\epsilon^2}{\epsilon^2}\left[\frac{i\omega}{2\lambda}\right]^{-\epsilon}\right\}\ . \tag{A5}$$

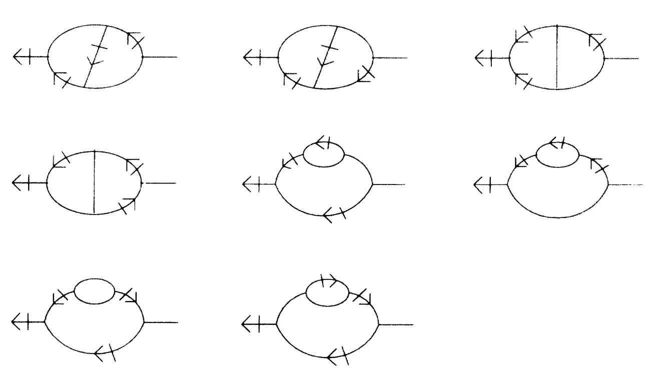

FIG. 3. All two-loop diagrams of $\Gamma_{1,1}(\mathbf{q},\omega)$.

The transition from bare to renormalized quantities via $\mathring{\rho}=Z_\rho \rho, \mathring{g}=g$ and the ϵ expansion of the last equation provides in minimal subtraction (i.e., the Z factor exactly absorbs the ϵ poles from the vertex function

$$Z_\rho = 1-\frac{3}{4}\frac{u}{\epsilon}-\frac{u^2}{\epsilon}\left[\frac{9}{64}\frac{1}{\epsilon}-\frac{15}{128}\ln\frac{3}{4}+\frac{11}{768}\right]+O(u^3) . \quad \text{(A6)}$$

The parameter function ζ given in (2.17) is now directly obtained by

$$\zeta(u)=-\epsilon u\frac{d\ln Z_\rho}{du} . \quad \text{(A7)}$$

APPENDIX B: REDUCTION OF C_{jj}

Treating the dynamic functional $\mathcal{J}$ that is extended by two source terms (4.4) in analogy to the procedure in Sec. III [(3.3)–(3.7)], where only one source term was introduced in $\mathcal{J}$ (3.2), we obtain on the one hand that Eq. (3.7) still holds (for this equation is restricted to $k=\tilde{k}=0$) and on the other hand from differentiation of (3.5) with respect to s instead of $\tilde{s}$,

$$\left.\frac{\delta\mathring{\Gamma}}{\delta\mathring{s}}+[\partial_t+\lambda(\Delta_\perp+\mathring{\rho}\Delta_\parallel)]\mathring{\tilde{s}}+\lambda\mathring{g}\frac{\delta\mathring{\Gamma}}{\delta\mathring{k}}\right|_{\mathring{s}=\mathring{\tilde{s}}=\mathring{k}=0}=0 . \quad \text{(B1)}$$

Both equations serve as a tool to simplify the four cumulants of the CCC function (4.3). The first term of Eq. (4.3),

$$\langle s^2(1)s^2(2)\rangle_c=4\frac{\delta^2\ln Z}{\delta k(1)\delta k(2)} \quad \text{(B2)}$$

[cumulants are generated by functional differentiation of $\ln Z$ in contrast to full correlation functions where only Z is used (3.4)], reads after transition to vertex functions (3.6)

$$\begin{aligned}\langle s^2(1)s^2(2)\rangle_c=-4\frac{\delta^2\Gamma}{\delta k(1)\delta k(2)}+4\int d^dr_3dt_3d^dr_4dt_4\Bigg\{&\frac{\delta^2\Gamma}{\delta k(1)\delta s(3)}C(3-4)\frac{\delta^2\Gamma}{\delta s(4)\delta k(2)}\\&+\frac{\delta^2\Gamma}{\delta k(1)\delta s(3)}G(3-4)\frac{\delta^2\Gamma}{\delta\tilde{s}(4)\delta k(2)}\\&+\frac{\delta^2\Gamma}{\delta k(1)\delta\tilde{s}(3)}G(4-3)\frac{\delta^2\Gamma}{\delta s(4)\delta k(2)}\Bigg\} ,\end{aligned} \quad \text{(B3)}$$

with C and G being the cumulant of the correlation and response function. To relate the unknown expressions in (B3) to the functions E and D (4.5), we differentiate Eq. (3.7) and (B1) with respect to k,

$$\begin{aligned}\frac{\delta^2\Gamma}{\delta\tilde{s}(1)\delta k(2)}&=\lambda g\nabla_\parallel^{(1)}\frac{\delta^2\Gamma}{\delta k(1)\delta k(2)}=-\frac{\lambda g}{4}\nabla_\parallel^{(1)}D(1-2) ,\\ \frac{\delta^2\Gamma}{\delta k(1)\delta s(2)}&=-\lambda g\frac{\delta^2\Gamma}{\delta k(1)\delta\tilde{k}(2)}=-\frac{\lambda g}{2}\nabla_\parallel^{(1)}E(1-2) ,\end{aligned} \quad \text{(B4)}$$

and eventually obtain

$$\begin{aligned}\langle s^2(1)s^2(2)\rangle_c=D(1-2)+\frac{(\lambda g)^2}{2}\Delta_\parallel^{(1)}\int_{3,4}\{&E(1-3)G(3-4)D(4-2)\\&+E(2-3)G(3-4)D(4-1)-2E(1-3)C(3-4)E(2-4)\} .\end{aligned} \quad \text{(B5)}$$

The second term,

$$\nabla_\parallel^{(1)}\langle s(1)s^2(2)\rangle_c=2\nabla_\parallel^{(1)}\frac{\delta^2\ln Z}{\delta h(1)\delta k(2)} , \quad \text{(B6)}$$

reads after transition to vertex functions

$$\begin{aligned}\nabla_\parallel^{(1)}&\langle s(1)s^2(2)\rangle_c\\&=-2\nabla_\parallel^{(1)}\int_3\left[\frac{\delta s(3)}{\delta h(1)}\frac{\delta^2\Gamma}{\delta s(3)\delta k(2)}+\frac{\delta\tilde{s}(3)}{\delta h(1)}\frac{\delta^2\Gamma}{\delta\tilde{s}(3)\delta k(2)}\right] ,\end{aligned} \quad \text{(B7)}$$

and eventually gives with (B4)

$$\begin{aligned}\nabla_\parallel^{(1)}&\langle s(1)s^2(2)\rangle_c\\&=\lambda g\Delta_\parallel^{(1)}\int_3[\tfrac{1}{2}G(1-3)D(3-2)\\&\qquad-C(1-3)E(2-3)] .\end{aligned} \quad \text{(B8)}$$

The third term $\nabla_\parallel^{(2)}\langle s(2)s^2(1)\rangle_c$ is deduced from the second one by interchanging the arguments 1 and 2. The fourth term is simply

$$\begin{aligned}\nabla_\parallel^{(1)}\nabla_\parallel^{(2)}\langle s(1)s(2)\rangle_c&=\nabla_\parallel^{(1)}\nabla_\parallel^{(2)}C(1-2)\\&=-\Delta_\parallel^{(1)}C(1-2) .\end{aligned} \quad \text{(B9)}$$

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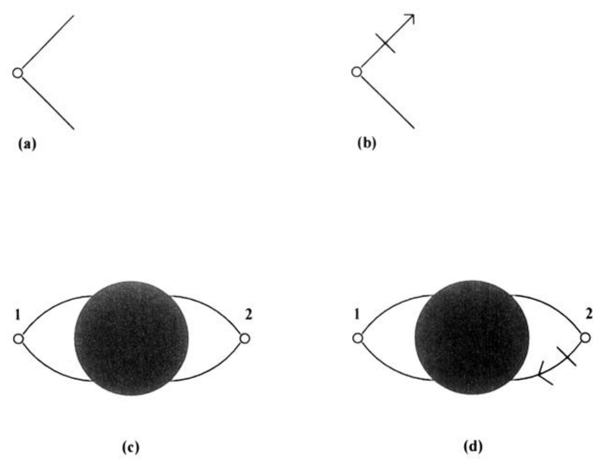

FIG. 1. In addition to the graphical elements shown in Appendix A there are (a) an $[s^2/2]$ insertion and (b) an $[s\nabla_{\|}\tilde{s}]$ insertion. Thus (c) represents $D(1-2)/4$ [Eq. (4.5)] and (d) $-\nabla_{\|}^{(1)}E(1-2)/2$ [Eq. (4.5)]. The shaded areas means the connected, one-line-irreducible part of a vertex function. A close look at (c) and (d) directly provides Eqs. (4.6) and (4.7).