Scaling and crossover in the large-N model for growth kinetics

A. Coniglio,[1] P. Ruggiero[1] and M. Zannetti[2]
[1]Dipartimento di Scienze Fisiche, Università di Napoli, Mostra d'Oltremare, Padiglione 19, 80125 Napoli, Italy
[2]Dipartimento di Fisica, Università di Salerno, 84081 Baronissi (SA), Italy
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The dependence of the scaling properties of the structure factor on space dimensionality, range of interaction, initial and final conditions, and the presence or absence of a conservation law is analyzed in the framework of the large-N model for growth kinetics. The variety of asymptotic behaviors is quite rich, including standard scaling, multiscaling, and a mixture of the two. The different scaling properties obtained as the parameters are varied are controlled by a structure of fixed points with their domains of attraction. Crossovers arising from the competition between distinct fixed points are explicitly obtained. Temperature fluctuations below the critical temperature are not found to be irrelevant when the order parameter is conserved. The model is solved by integration of the equation of motion for the structure factor and by a renormalization group approach.

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I. INTRODUCTION

In growth kinetics one deals with the relaxation to equilibrium of a system quenched from high to low temperature [1]. The processes of interest are those which exhibit scaling [2] in the asymptotic time regime. Denoting with T_I and T_F the initial and final temperatures, these processes can be grouped into three classes characterized by $(T_I > T_c, T_F < T_c)$, $(T_I > T_c, T_F = T_c)$, and $(T_I = T_c, T_F < T_c)$ where T_c is the critical temperature. This subdivision arises from renormalization group arguments [3] whereby the temperature axis (Fig. 1) is controlled by three fixed points at $T = 0, T_c, T = \infty$ and T_c is unstable both with respect to $T = 0$ and $T = \infty$. Such a flow diagram leads naturally to the three universality classes listed above whose basic processes are those originating and terminating in a fixed point. By far, the most studied among these is the phase ordering process from $T_I = \infty$ to $T_F = 0$.

The reason for the continuing interest in this problem is the persistent lack of a full understanding of scaling which is observed both in laboratory [4] and numerical [5] experiments. In terms of the structure factor (Fourier transform of the equal time order parameter correlation function) this asymptotic scaling behavior is of the form

$$C(\vec{k},t) \sim L^{\alpha}(t)F(kL(t)), \tag{1.1}$$

where $L(t)$ is a characteristic length which grows in time with a power law

$$L(t) \sim t^{1/z}. \tag{1.2}$$

A scaling pattern of this type, which we refer to as standard scaling, is completely characterized by the pair of exponents z, α and by the scaling function $F(x)$. These quantities depend to a different extent on the various elements entering in the specification of the process [6] which, in addition to the classes (T_I, T_F) discussed above, include the space dimensionality of the system, the vector dimensionality of the order parameter, the presence or absence of a conservation law, and the short or long range character of interactions.

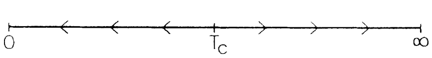

FIG. 1. Renormalization group flow on the temperature axis.

The purpose of this paper is to explore in detail the dependence of the scaling properties on the totality of these elements in the framework of the large-N model [7]. This, at the moment, is the only available nontrivial soluble model with a structure sufficiently rich to be adequate for this kind of investigation. The picture which emerges in the end is quite informative and exposes clearly the profound difference between processes with and without conservation of the order parameter.

We consider a system described by an N-component order parameter $\vec{\phi}(\vec{x}) = (\phi_1(\vec{x}), ..., \phi_N(\vec{x}))$ and by a free energy functional of the Ginzburg-Landau type,

$$\mathcal{H}[\vec{\phi}] = \frac{1}{2}\int d^dx \left[(\nabla\vec{\phi})^2 + r\vec{\phi}^2 + \frac{g}{2N}(\vec{\phi}^2)^2\right] + \mathcal{H}_{\rm LR}[\vec{\phi}], \tag{1.3}$$

where $\mathcal{H}_{\rm LR}[\vec{\phi}]$ contains the long range interaction and will be specified in Sec. II. The Gibbs equilibrium states $P_{\rm eq}[\vec{\phi}] \sim \exp(-\frac{1}{T}\mathcal{H}[\vec{\phi}])$ are parametrized by the temperature T and by the pair of coupling constants $\mu = (r, g)$. In the large-N limit ($N \to \infty$) there is a critical temperature $T_c(\mu) \sim -r/g$ and a phase diagram (Fig. 2) in the three dimensional parameter space (T, μ) with a surface of critical points separating ordered states below it from disordered states above it. The interesting portion of this phase diagram is the $(r \leq 0, g \geq 0)$ sector at or below the critical surface where scaling is to be expected in a quench process. As we shall see in the following, the set of states at $T = 0$ on the g axis plays a special role since it is located at the edge of both the critical surface and the ordering region below it.

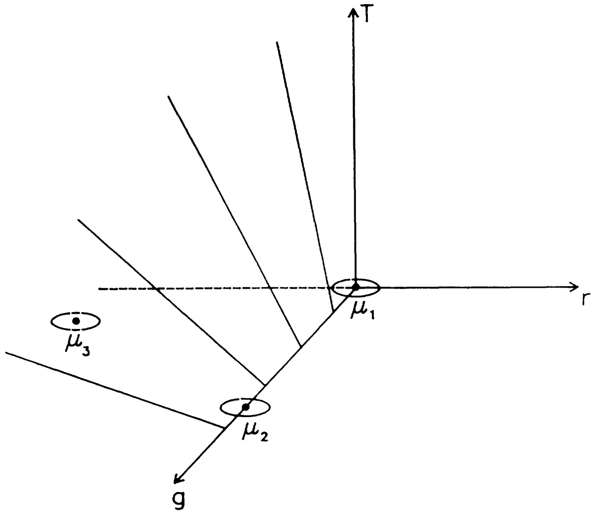

FIG. 2. Manifold of final equilibrium states with the critical surface separating disordered states (above) from ordered states (below).

As anticipated above, the characterization of a process requires, in order, (i) specification of the space dimensionality d; (ii) specification of the initial condition; this we do by taking an initial structure factor of the form

$$C(\vec{k},0) = \frac{\Delta}{k^{\theta}}, \tag{1.4}$$

where Δ is a constant and the value of θ selects the initial state of interest: $\theta = 0$ corresponds to an uncorrelated initial state at infinite temperature ($T_I = \infty$) while $\theta = 2$ corresponds to the critical point ($T_I = T_c$); (iii) choice between a nonconserved order parameter (NCOP) and a conserved order parameter (COP); (iv) specification of the short or long range nature of the interaction; (v) specification of the final state. The interesting subsets in the equilibrium phase diagram are $[T_F = T_c = 0, \mu_1 = (r = 0, g = 0)]$, trivial critical state at zero temperature; $[T_F = T_c > 0, \mu_1 = (r = 0, g = 0)]$, trivial critical states at finite temperature (T axis); $[T_F = T_c = 0, \mu_2 = (r = 0, g > 0)]$, nontrivial critical states at zero temperature (g axis); $[T_F = T_c > 0, \mu_3 = (r < 0, g > 0)]$, nontrivial critical states at finite temperature (critical surface); $[T_F < T_c, \mu_3 = (r < 0, g > 0)]$, phase ordering region. It is convenient to regard the space dimensionality, the initial condition, and the range of the interaction as forming, so to speak, the environment of the process, while the set (T_F, μ) and the specification NCOP or COP as elements of discrimination which we will use to identify processes.

Solving the model analytically and by renormalization group (RG) we arrive at the following picture. The asymptotic scaling properties $[z, \alpha, F(x)]$ depend on (T_F, μ). There is a universality class, under each heading NCOP or COP, for each of the five regions (T_F, μ) listed above. In RG language this means that there are five fixed points (T_F^*, μ^*). The flow in the parameter space and therefore the extension of the universality classes depends on the relative stability of these fixed points. This in turn is regulated by the existence of critical dimensionalities which depend on the environment, i.e., initial condition and range of interaction.

The deep difference between NCOP and COP emerges from how the scaling properties depend on the final state (T_F, μ). The most striking difference is obtained for quenches inside the phase ordering region. It was found previously [8] that when the system is quenched to $(T_F = 0, \mu_3)$ the standard scaling form (1.1) holds only for NCOP, while for COP it is replaced by the more general multiscaling behavior

$$C(\vec{k},t) \sim L^{\alpha(x)}(t) F(x), \tag{1.5}$$

where the exponent α also depends on $x = kL(t)$. We find now that, with some modifications to be discussed below, this basic distinction, NCOP standard scaling and COP multiscaling, holds not just for quenches to $T_F = 0$, but for quenches anywhere in the phase ordering region $(T_F < T_c, \mu_3)$. Furthermore, while temperature perturbations with $0 < T_F < T_c$ are irrelevant for NCOP, it is not so for COP. For quenches elsewhere, i.e., on the critical surface, standard scaling holds both for NCOP and COP. However, while with NCOP (T_F, μ) affects α with no impact on z, the opposite occurs for COP.

The analytical tractability of the large-N model [7–9] allows one to expose nicely the mechanism underlying the picture outlined above and to compute, in addition to the asymptotic properties, the crossovers induced by the competing fixed points. The question of the extension of the properties of the large-N model to finite N needs to be treated with care. We shall comment on this in the concluding section.

The paper is organized as follows: in Sec. II the general features of the large-N model are presented, Sec. III is devoted to the solution of the model by integration of the equation of motion for the structure factor, and in Sec. IV the model is analyzed by RG methods. Concluding remarks are made in Sec. V.

II. THE LARGE-N MODEL

The long range part of the free energy functional (1.3) is of the form

$$\mathcal{H}_{\mathrm{LR}}[\vec{\phi}] = \int d^d x \int d^d x' \vec{\phi}(\vec{x}) \cdot V(\vec{x} - \vec{x}'), \vec{\phi}(\vec{x}'), \tag{2.1}$$

with the large distance behavior $V(\vec{x} - \vec{x}') \sim | \vec{x} - \vec{x}' |^{-(d+\sigma)}$. Considering a time evolution of the order parameter governed by the time dependent Ginzburg-Landau model and neglecting [10, 11] k^2 with respect to k^{σ} for $\sigma < 2$ the equation of motion for the Fourier transform of the order parameter in the large-N limit is given by [7]

$$\frac{\partial \phi_\alpha(\vec{k},t)}{\partial t} = -\Gamma \left[w k^{p+\sigma} + k^p R(t) \right] \phi_\alpha(\vec{k},t) + \eta_\alpha(\vec{k},t), \tag{2.2}$$

where $(\alpha = 1, ..., N)$, w is a coefficient originating in the small momentum expansion of the interaction, Γ is a kinetic coefficient, $p = 0$ for NCOP, $p = 2$ for COP, $\vec{\eta}(\vec{k},t)$ is a Gaussian white noise with expectations

$$\langle \vec{\eta}(\vec{k},t)\rangle = 0, \tag{2.3}$$

$$\langle \eta_\alpha(\vec{k},t)\eta_\beta(\vec{k}',t')\rangle = 2\Gamma T_F k^p \delta_{\alpha\beta}\delta(\vec{k}+\vec{k}')\delta(t-t'), \tag{2.4}$$

and

$$R(t) = r + gS(t), \tag{2.5}$$

with $S(t) = \langle \phi_\alpha^2(\vec{x},t)\rangle$, which is independent of α and must be determined self-consistently [12]. In the following we will let σ and p vary continuously since $\sigma < 2$ describes long range interactions while $0 < p < 2$ describes nonlocal conservation of the order parameter [13].

Integrating Eq. (2.2) with a random initial condition $\vec{\phi}(\vec{k},0)$ and dropping the label α we find

$$\phi(\vec{k},t) = \phi(\vec{k},0)D(\vec{k},t) + \int_0^t dt'\eta(\vec{k},t')\frac{D(\vec{k},t)}{D(\vec{k},t')}, \tag{2.6}$$

where

$$D(\vec{k},t) = \exp\{-\Gamma[wk^{p+\sigma}t + k^p Q(t)]\} \tag{2.7}$$

and

$$Q(t) = \int_0^t dt' R(t'). \tag{2.8}$$

From (2.6) correlation functions of arbitrary order can be obtained forming products of $\phi(\vec{k},t)$ and averaging over both initial condition and thermal noise. For the average order parameter we find

$$\langle \phi(\vec{k},t)\rangle = \langle \phi(\vec{k},0)\rangle D(\vec{k},t), \tag{2.9}$$

which shows that if the initial state is symmetric $\langle \phi(\vec{k},0)\rangle = 0$, as we shall assume in the following, then $\langle \phi(\vec{k},t)\rangle = 0$ for all time, i.e., dynamics does not break the symmetry. The more general case of a time evolution with broken symmetry is outlined in Appendix A.

The structure factor $\langle \phi(\vec{k},t)\phi(\vec{k}',t)\rangle = C(\vec{k},t)\delta(\vec{k}+\vec{k}')$ is given by the sum of two contributions

$$C(\vec{k},t) = C_1(\vec{k},t) + C_2(\vec{k},t), \tag{2.10}$$

where

$$C_1(\vec{k},t) = C(\vec{k},t=0)D^2(\vec{k},t), \tag{2.11}$$

$$C_2(\vec{k},t) = 2\Gamma k^p T_F D^2(\vec{k},t)\int_0^t dt' D^{-2}(\vec{k},t'). \tag{2.12}$$

From (2.10)–(2.12) one can easily verify that the structure factor obeys the equation of motion

$$\frac{\partial C(\vec{k},t)}{\partial t} = -2\Gamma[wk^{p+\sigma} + k^p R(t)]C(\vec{k},t) + 2\Gamma k^p T_F, \tag{2.13}$$

which is closed by the self-consistency condition

$$S(t) = \int \frac{d^dk}{(2\pi)^d} C(\vec{k},t). \tag{2.14}$$

Let us now analyze the final equilibrium states. Since there is no symmetry breaking, it is sufficient to look at the structure factor in the limit $t \to \infty$, obtaining (Appendix B) from Eq. (2.13)

$$C(\vec{k},\infty) = \frac{T_F}{wk^\sigma + \xi^{-\sigma}} + (2\pi)^d M^2 \delta(\vec{k}), \tag{2.15}$$

where $\xi = [R(\infty)]^{-1/\sigma}$ is the equilibrium correlation length. Enforcing the self-consistency condition (2.14) one finds (Appendix B) that there exists a critical temperature

$$T_c = -\frac{r}{gB(0)} \tag{2.16}$$

such that

$$\xi^{-1} > 0 \text{ and } M = 0 \text{ for } T_F > T_c,$$
$$\xi^{-1} = 0 \text{ and } M^2 = M_0^2(T_c - T_F)/T_c \text{ for } T_F \le T_c, \tag{2.17}$$

with $M_0^2 = -r/g$ and

$$B(0) = \int \frac{d^dk}{(2\pi)^d}\frac{1}{wk^\sigma} \propto \frac{1}{d-\sigma}. \tag{2.18}$$

Thus in the (T_F, μ) parameter space (Fig. 2) the critical point as r and g are varied (g must be positive in order to have a well defined theory) spans a surface which lies on the $r \le 0$ sector and separates ordered states underneath it, where the structure factor displays a Bragg peak, from the disordered states above it without Bragg peak. Notice that from (2.16) and (2.18) $\lim_{d\to\sigma} T_c = 0$, implying that σ is the lower critical dimensionality of the model.

As discussed in the Introduction, in the rest of the paper we shall be concerned with the solution of Eq. (2.13) with values of (T_F, μ) lying on or below the critical surface.

III. QUENCH PROCESSES

From (2.10) and (1.4) the formal solution for the structure factor is given by

$$C(\vec{k},t) = \frac{\Delta}{k^\theta} e^{-2\Gamma[wk^{p+\sigma}t + k^pQ(t)]} + 2\Gamma T_F k^p \times \int_0^t dt' e^{-2\Gamma\{wk^{p+\sigma}(t-t') + k^p[Q(t)-Q(t')]\}}. \tag{3.1}$$

We must now extract the scaling properties.

A. NCOP

With $p = 0$ by dimensional analysis we can identify the characteristic length

$$L(t) = (2\Gamma t)^{1/\sigma}, \tag{3.2}$$

which enters (3.1) in the combination $x = kL(t)$. It is worthwhile to rewrite (3.1) with this change of variable,

$$C(\vec{k},t) = \Delta e^{-2\Gamma Q(t)}\frac{L^{\theta}}{x^{\theta}}e^{-wx^{\sigma}} +\sigma T_F L^{\sigma}x^{-\sigma}\int_0^x dx' x'^{\sigma-1}e^{-w(x^{\sigma}-x'^{\sigma})} \times e^{-2\Gamma[Q(t)-Q(t')]}, \quad (3.3)$$

which shows that (3.2) is the only available choice for the characteristic length and therefore necessarily $z=\sigma$ for any process with NCOP. In order to solve for $Q(t)$ we integrate (3.3) over $\vec{k}$ and we use the definitions (2.8) and (2.14) to derive the equation

$$\frac{1}{2\Gamma}\frac{d}{dt}e^{2\Gamma Q(t)} = re^{2\Gamma Q(t)} + g\Delta A(t) +2\Gamma T_F g\int_0^t dt' A_0(t-t')e^{2\Gamma Q(t')}, \quad (3.4)$$

where

$$A(t) = \int \frac{d^dk}{(2\pi)^d}\frac{e^{-2\Gamma wk^{\sigma}t}}{k^{\theta}} = L^{\theta-d}(t)K_d\int_0^{\Lambda L} dx x^{d-\theta-1}e^{-wx^{\sigma}}, \quad (3.5)$$

with $K_d=[2^{d-1}\pi^{d/2}\Gamma(d/2)]^{-1}$, Λ is a momentum cutoff, and $A_0(t)$ is the same as $A(t)$ with $\theta=0$.

1. $T_F=0$

Let us now make a further restriction by considering quenches to zero temperature. With $T_F=0$ (3.4) can be integrated, obtaining

$$e^{2\Gamma Q(t)} = e^{2\Gamma rt}\left[1+2\Gamma g\Delta\left(\int_0^{t_0}dt' e^{-2\Gamma rt'}A(t') + \int_{t_0}^t dt' e^{-2\Gamma rt'}A(t')\right)\right]. \quad (3.6)$$

Next, choosing t_0 such that $\Lambda L(t_0)\sim 1$, from (3.5) we may approximate

$$L^{d-\theta}(t)A(t) \sim \begin{cases} K_d\int_0^{\Lambda L} dx x^{d-\theta-1} = K_d[\Lambda L(t)]^{d-\theta}/(d-\theta) & \text{for } t<t_0 \\ I=\int_0^{\infty} dx x^{d-\theta-1}e^{-wx^{\sigma}} & \text{for } t>t_0 \end{cases} \quad (3.7)$$

and inserting into (3.6) finally we get

$$e^{2\Gamma Q(t)} = \kappa e^{2\Gamma rt} + g\Delta I e^{2\Gamma rt}\int_{t_0}^t dt' e^{-2\Gamma rt'}L^{\theta-d}(t'), \quad (3.8)$$

where $\kappa = 1+g\Delta K_d\Lambda^{d-\theta}(1-e^{-2\Gamma rt_0})/(d-\theta)r$.

The next step is to extract the behavior of $Q(t)$ from (3.8) after specifying the set of coupling constants $\mu=(r,g)$.

$\mu_1=(r=0,g=0)$. In this case $Q(t)\equiv 0$. Thus from (3.3) we get

$$C(\vec{k},t) = \Delta L^{\theta}F(x), \quad (3.9)$$

with

$$F(x) = \frac{e^{-wx^{\sigma}}}{x^{\theta}}. \quad (3.10)$$

Note that $C(\vec{k},t)$ obeys the scaling form (1.1) from beginning to end over the entire time history of the process with $\alpha=\theta$. In this case dynamics propagates the scaling properties of the initial condition.

$\mu_2=(r=0,g>0)$. Setting $r=0$ in Eq. (3.8)

$$e^{2\Gamma Q(t)} = \kappa + g\frac{\Delta I\sigma}{\theta+\sigma-d}\left[L^{\theta+\sigma-d}-L_0^{\theta+\sigma-d}\right], \quad (3.11)$$

which gives

$$C(\vec{k},t) \sim \frac{L^{\theta}}{\kappa+\sigma Ig\Delta[L^{d_c-d}-L_0^{d_c-d}]/(d_c-d)}F(x), \quad (3.12)$$

with $d_c=\theta+\sigma$, $L_0=L(t_0)$, and $F(x)$ given by (3.10). Now scaling is no longer an exact property obeyed over the entire time history, but only asymptotically. Furthermore, Eq. (3.12) shows that d_c is a critical dimensionality, in the sense that different asymptotic behaviors are obtained depending on $d>d_c$ or $d<d_c$. For $d>d_c$ the asymptotic behavior is the same as for the quench to μ_1, with the denominator producing a correction to scaling at early times. Instead for $d<d_c$ there is a crossover time $t^*\sim(\Delta g)^{\frac{\sigma}{d-d_c}}$ such that

$$C(\vec{k},t) \sim \begin{cases} L^{\theta}F(x) & \text{for } t\ll t^* \\ L^{d-\sigma}F(x) & \text{for } t\gg t^*. \end{cases} \quad (3.13)$$

At $d=d_c$ we find a logarithmic correction due to marginality ,

$$C(\vec{k},t) \sim \frac{L^{\theta}}{\ln L/L_0}F(x). \quad (3.14)$$

In order to illustrate the above results we have integrated numerically the equation of motion (2.13) for $p=0$, $\sigma=2$, $d=3$, $r=0$, $T_F=0$, and different values of g. Since the crossover involves only the exponent

α it is convenient to discuss it in terms of $S(t)$. The analytical behavior of $S(t)$ is obtained integrating (3.12) over $\vec{k}$,

$$S(t) \sim \frac{L^{\theta-d}}{\kappa + \sigma I g \Delta [L^{d_c-d} - L_0^{d_c-d}]/(d_c-d)}. \quad (3.15)$$

The case $d > d_c$ is realized taking $\theta = 0$, which yields $d_c = 2$. The double logarithmic plot of the numerical solution for $S(t)$ shows (Fig. 3) that the power law $L^{-d} = t^{-3/2}$ is asymptotically obeyed with deviations from it at early times which become smaller as g decreases, i.e., as μ_1 is approached along the μ_2 axis. Conversely, for $d < d_c$, obtained by setting $\theta = 2$ which implies $d_c = 4$, the numerical solution for $S(t)$ behaves as $L^{-1} \sim t^{-1/2}$ for $t \ll t^*$ and as $L^{-2} \sim t^{-1}$ for $t \gg t^*$ (Fig. 4).

$\mu_3 = (r < 0, g > 0)$. With these values of r and g the quench is made inside the region of the coexisting ordered phases, which is the process normally considered in the kinetics of phase ordering. In a process of this type usually one makes the distinction between the early stage of exponential growth due to the instability generated by $r < 0$ and the late stage characterized by scaling behavior. However, due to the existence of crossovers, we should expect to find a scaling regime also at *early time* with the exponents appropriate to μ_1 or μ_2 when r and g are pushed sufficiently close to μ_1 or μ_2. In fact for short time $(t < 1/2\Gamma|r|)$ Eq. (3.8) yields

$$e^{2\Gamma Q(t)} \sim e^{2\Gamma r t}\left(\kappa + \sigma I \frac{g\Delta}{d_c-d}[L^{d_c-d} - L_0^{d_c-d}]\right), \quad (3.16)$$

which implies for the structure factor

$$C(\vec{k},t) \sim \frac{L^{\theta}}{\kappa + \sigma I g \Delta [L^{d_c-d} - L_0^{d_c-d}]/(d_c-d)} \times F(x) e^{-2\Gamma r t}, \quad (3.17)$$

showing that in front of the exponential actually there is a prefactor identical to (3.12). Hence if $|r|$ is sufficiently small one can detect a scaling regime identical to the one discussed for quenches to μ_2 and *preceding* the usual early time regime of exponential growth.

Instead, when time is large we can set to zero the left hand side of (3.4), obtaining for the structure factor the late stage scaling behavior

$$C(\vec{k},t) \sim M_0^2 L^d F(x) \quad (3.18)$$

for any d. This means that for quenches inside the phase ordering region there is not an upper critical dimensionality such that above it one obtains an asymptotic behavior with $\alpha = \theta$.

The crossover structure is illustrated in Fig. 5 through

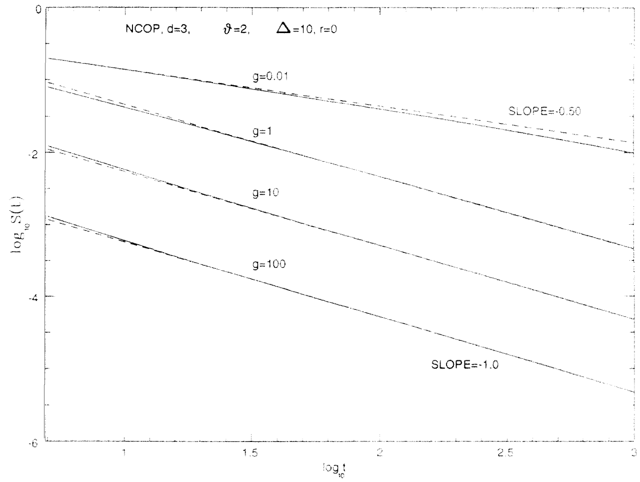

FIG. 4. Behavior of $S(t)$ in a quench to μ_2 for NCOP with $\sigma = 2, \Delta = 10$, and $d < d_c$: $(d = 3, \theta = 2)$. The top dashed line has slope -0.5, the others below have slope -1.0.

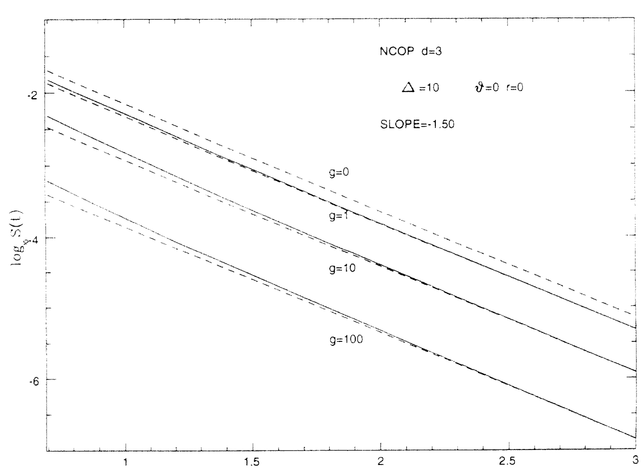

FIG. 3. Behavior of $S(t)$ in a quench to μ_2 for NCOP with $\sigma = 2, \Delta = 10$, and $d > d_c$: $(d = 3, \theta = 0)$. The straight dashed lines have slope $-3/2$.

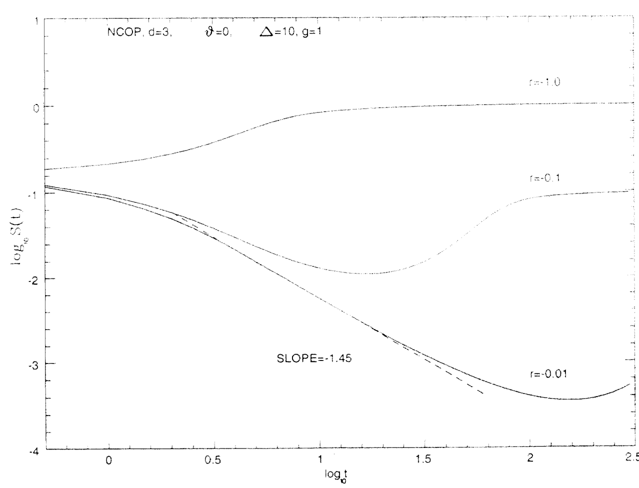

FIG. 5. Behavior of $S(t)$ in a quench to μ_3 for NCOP with $\sigma = 2, \Delta = 10, d = 3, \theta = 0$ at fixed $g = 1$ as r approaches the μ_2 axis. The straight dashed line has slope -1.45.

the behavior of $S(t)$ computed numerically for $d = 3, \theta = 0, \sigma = 2, g = 1$, and decreasing values of r in order to explore the influence on $S(t)$ of the power law associated to the μ_2 axis. While away from μ_2, e.g., at $r = -1.0$, the behavior of $S(t)$ shows only the asymptotic scaling regime $S(t) \sim M_0^2 = 1$, as $|r|$ is decreased and μ_2 is approached for $|r|$ sufficiently small, e.g., $r = -0.01$, $S(t)$ indeed displays at intermediate times the power law $L^{-d} \sim t^{-3/2}$ which we have found previously as an asymptotic behavior in the quenches to μ_2. The case with $\theta = 2$ yields qualitatively similar results.

2. $T_F > 0$

For quenches to a finite final temperature $0 < T_F \leq T_c$ we restrict the choice of the initial condition to high temperature, taking $\theta = 0$ in (1.4).

$\mu_1 = (r = 0, g = 0)$. For quenches to the trivial critical states at finite temperature (T axis in Fig. 2), setting $Q(t) \equiv 0$ in (3.3), we find

$$C(\vec{k}, t) = \Delta e^{-wx^\sigma} + L^\sigma T_F F_0(x), \tag{3.19}$$

where

$$F_0(x) = (1 - e^{-wx^\sigma})/wx^\sigma. \tag{3.20}$$

Thus for these processes the asymptotic scaling behavior is given by $L^\sigma T_F F_0(x)$ while the strength Δ of initial correlations is an irrelevant parameter with correction to scaling behavior $\sim L^{-\sigma}$.

$\mu_3 = (r < 0, g > 0)$ The solution of Eq. (3.4) in the general case has been obtained by Newman and Bray [12]. For large time and for $\sigma < d < 2\sigma$ they find

$$e^{-2\Gamma Q(t)} \sim L^\omega(t), \tag{3.21}$$

with

$$\omega = \begin{cases} \epsilon = 2\sigma - d & \text{for } T_F = T_c \\ d & \text{for } T_F < T_c. \end{cases} \tag{3.22}$$

Inserting into (3.1) we find

$$C(\vec{k}, t) = C(\vec{k}, t_0) \frac{L^\omega(t) e^{-wx^\sigma}}{L^\omega(t_0) e^{-wx_0^\sigma}} + L^\sigma T_F \int_0^{1-(x_0/x)^\sigma} dy (1-y)^{-\omega/\sigma} e^{-wx^\sigma y}, \tag{3.23}$$

where t_0 is some microscopically short time such that scaling holds for $t > t_0$ and where $x_0 = kL(t_0)$. For $T_F < T_c$, recalling that $T_c > 0$ requires $d > \sigma$ and using (3.22) the asymptotic scaling behavior is still given by (3.18) with T_F acting as an irrelevant perturbation whose correction to scaling behavior is given by $\sim L^{\sigma-d}$. For $T_F = T_c$, the role of the two terms in the right hand side of (3.23) is reversed. The thermal contribution provides the dominant scaling term $\sim L^\sigma$ while the first term yields correction to scaling $\sim L^{\sigma-d}$. Taking $\Delta = 0$ and setting the coupling constant g at the fixed point value [14, 15], the first term can be made to vanish ($t_0 \to 0$), obtaining

$$C(\vec{k}, t) \sim L^\sigma T_c F_\epsilon(x), \tag{3.24}$$

with

$$F_\epsilon(x) = \int_0^1 dy (1-y)^{-\epsilon/\sigma} e^{-wx^\sigma y}. \tag{3.25}$$

Notice that $\lim_{\epsilon \to 0} F_\epsilon(x) = F_0(x)$. Thus at the upper critical dimensionality 2σ the nontrivial fixed point merges with the trivial fixed point on the temperature axis.

In summary, with NCOP we have found the following asymptotic scaling properties:

(1) $[T_F = 0, \mu_1]$,

$$C(\vec{k}, t) \sim L^\theta(t) F(x), \tag{3.26}$$

(2) $[T_F = T_c > 0, \mu_1]$,

$$C(\vec{k}, t) \sim L^\sigma(t) T_c F_0(x), \tag{3.27}$$

(3) $[T_F = 0, \mu_2]$,

$$C(\vec{k}, t) \sim \begin{cases} L^\theta(t) F(x) & \text{for } d > d_c \\ \frac{L^\theta(t)}{\ln L(t)} F(x) & \text{for } d = d_c = \theta + \sigma \\ L^{d-\sigma}(t) F(x) & \text{for } d < d_c, \end{cases} \tag{3.28}$$

(4) $[T_F = T_c > 0, \mu_3]$,

$$C(\vec{k}, t) \sim \begin{cases} L^\sigma(t) T_c F_0(x) & \text{for } d \geq 2\sigma \\ L^\sigma(t) T_c F_\epsilon(x) & \text{for } d < 2\sigma, \end{cases} \tag{3.29}$$

(5) $[T_F < T_c, \mu_3]$,

$$C(\vec{k}, t) \sim L^d(t) F(x), \tag{3.30}$$

where $L(t) \sim t^{1/\sigma}$, $x = kL(t)$, and

$$F(x) = e^{-wx^\sigma}/x^\theta, \tag{3.31}$$

$$F_0(x) = (1 - e^{-wx^\sigma})/wx^\sigma, \tag{3.32}$$

$$F_\epsilon(x) = \int_0^1 dy (1-y)^{-\epsilon/\sigma} e^{-wx^\sigma y}, \tag{3.33}$$

with $\epsilon = 2\sigma - d$.

B. COP

Let us now go back to Eq. (3.1) with $p \neq 0$. The important new feature is that now by dimensional analysis we can form the two lengths $L(t) = (2\Gamma t)^{\frac{1}{p+\sigma}}$ and $\lambda(t) = (2\Gamma \mid Q \mid)^{1/p}$. In order to establish the scaling behavior we must know which of the two is the dominant one.

1. $T_F = 0$

Considering first, as before, quenches to zero temperature the structure factor is given by

$$C(\vec{k}, t) = \frac{\Delta}{k^\theta} e^{-[w(kL)^{p+\sigma} + a(k\lambda)^p]}, \tag{3.34}$$

where $a = \text{sgn}(Q)$. To begin with, let us rescale lengths with respect to $L(t)$,

$$C(\vec{k},t) = \Delta e^{-a\beta x^p} L^\theta \hat{F}_>(x), \tag{3.35}$$

where

$$\beta = \left(\frac{\lambda}{L}\right)^p, \tag{3.36}$$

$$\hat{F}_>(x) = \frac{e^{-wx^{p+\sigma}}}{x^\theta}, \tag{3.37}$$

with $x = kL(t)$ and the reason for the notation $\hat{F}_>(x)$ will be clear below. Again, we must solve for $Q(t)$ using the analogue of Eq. (3.4),

$$\frac{dQ}{dt} = r + g\Delta L^{\theta-d} K_d \int_0^{\Lambda L} dx x^{d-\theta-1} e^{-(wx^{p+\sigma}+a\beta x^p)}. \tag{3.38}$$

$\mu_1 = (r = 0, g = 0)$. Since in this case $Q(t) \equiv 0$, implying $\lambda(t) \equiv 0$, clearly $L(t)$ is the dominant length and $z = p + \sigma$. From (3.35) we have, as in the NCOP case, that standard scaling is exactly obeyed over the entire time history with

$$C(\vec{k},t) = \Delta L^\theta \hat{F}_>(x). \tag{3.39}$$

When the nonlinearity is present it is no longer possible, contrary to the NCOP case, to solve directly for $Q(t)$. Hence we now make statements about the solution of (3.38) for large time by consistency checks on the assumption that either one of the two lengths $L(t)$ and $\lambda(t)$ prevails over the other.

$\mu_2 = (r = 0, g > 0)$. From (2.5) and (2.8) in this case a is positive. Let us first suppose that L prevails over λ, i.e., $\beta(t) \to 0$ as $t \to \infty$. For large time Eq. (3.38) can be replaced by

$$\frac{dQ}{dt} = g\Delta L^{\theta-d} K_d \int_0^\infty dx x^{d-\theta-1} e^{-wx^{p+\sigma}} \tag{3.40}$$

and integrating we find

$$Q(t) \sim \frac{L^{d_c+p-d}}{d_c+p-d} + \text{const}, \tag{3.41}$$

where $d_c = \theta + \sigma$. Inserting this result into (3.36) it follows that

$$\beta(t) \sim \begin{cases} L^{-p} & \text{for } d > d_c + p \\ L^{-p}\ln L & \text{for } d = d_c + p \\ L^{d_c-d} & \text{for } d < d_c + p, \end{cases} \tag{3.42}$$

which is consistent with the assumption if $d > d_c$. Thus for $d > d_c$ from (3.35) we find

$$C(\vec{k},t) \sim L^\theta \left[1 - \beta(t) x^p\right] \hat{F}_>(x). \tag{3.43}$$

This result together with (3.42) shows that there exists yet another critical dimensionality $\tilde{d}_c = d_c + p$ affecting the behavior of corrections to scaling.

Assuming next that $\beta(t)$ takes a constant value c, we find that this is consistent only for $d = d_c$ and from (3.35) it follows that

$$C(\vec{k},t) \sim L^\theta \hat{F}(x), \tag{3.44}$$

with

$$\hat{F}(x) = \frac{1}{x^\theta} e^{-[wx^{p+\sigma}+cx^p]}, \tag{3.45}$$

showing that there is scaling with the same exponents as at μ_1 , but with a modified scaling function.

Finally, assuming $\beta(t) \to \infty$, from (3.38) it follows that

$$\beta(t) \sim L^{\frac{p}{(d-\theta+p)}(d_c-d)}, \tag{3.46}$$

which for consistency requires $d < d_c$. Therefore we must now scale with respect to λ, obtaining

$$C(\vec{k},t) \sim \lambda^\theta [1 - w\beta^{-\sigma}(t) x'^{p+\sigma}] \hat{F}_<(x'), \tag{3.47}$$

where

$$\hat{F}_<(x') = \frac{e^{-x'^p}}{x'^\theta} \tag{3.48}$$

and $x' = k\lambda(t)$, $\lambda(t) \sim t^{1/z}$ with

$$z = d + p - \theta. \tag{3.49}$$

This result is interesting because a pattern qualitatively different from the corresponding case with NCOP is obtained. In the latter case when the nonlinearity becomes relevant below d_c the exponent α changes from the value θ to the dimensionality dependent value $(d - \sigma)$, while the exponent z and the scaling function remain the same as for the quench to the trivial fixed point μ_1. With COP instead we find that α always keeps the trivial value θ, while there is a change in the scaling function and the growth exponent picks up the value (3.49) dependent on the space dimensionality of the system.

$\mu_3 = (r < 0, g > 0)$. For the quench in the phase ordering region Eq. (3.38) can be rewritten in the form

$$\frac{dQ}{dt} = r + \frac{g\Delta}{L^{d-\theta}} K_d I(\beta), \tag{3.50}$$

where

$$I(\beta) = \int_0^\infty dx x^{d-\theta-1} e^{-\beta f(x)}, \tag{3.51}$$

$$f(x) = ax^p + w\beta^{-1} x^{p+\sigma}. \tag{3.52}$$

Making the assumptions to be verified *a posteriori* that β is asymptotically divergent and that a is negative we can make a saddle point evaluation of (3.51),

$$I(\beta) \sim 2\left[\frac{\pi}{w\sigma(p+\sigma)}\right]^{\frac{1}{2}} e^{\frac{w\sigma}{p}u} u^\gamma, \tag{3.53}$$

where

$$u = \left[\frac{p\beta}{w(p+\sigma)}\right]^{\frac{p+\sigma}{\sigma}} \tag{3.54}$$

and

$$\gamma = \left[2(d-\theta) - (p+\sigma)\right]/2(p+\sigma). \tag{3.55}$$

For large time we can set to zero the left hand side of (3.50), obtaining $I(\beta) = -rL^{d-\theta}/gK_d\Delta$. Next, using

(3.53) and taking the logarithm we find the asymptotic relation

$$u = \frac{p(d-\theta)}{w\sigma}\ln L - \frac{p\gamma}{w\sigma}\ln u. \quad (3.56)$$

Inserting the leading contribution into (3.54) we obtain

$$\beta(t) \sim (\ln L)^{\frac{\sigma}{(p+\sigma)}} \quad (3.57)$$

consistently with the assumption. Thus in this case $\lambda(t)$ and $L(t)$ diverge in the same way up to a logarithmic factor. As fundamental length we choose [8] the inverse of the wave vector $k_m(t)$ where $C(\vec{k},t)$ reaches its maximum value

$$k_m(t) = L^{-1}u^{\frac{1}{p+\sigma}}. \quad (3.58)$$

Inserting this result in (3.35) and introducing the variable $\bar{x} = k/k_m$ we obtain

$$C(\vec{k},t) \sim \frac{\Delta L^\theta}{u^{\theta/(p+\sigma)}}\frac{e^{u\varphi(\bar{x})}}{\bar{x}^\theta}, \quad (3.59)$$

where

$$\varphi(\bar{x}) = \frac{p+\sigma}{p}\bar{x}^p - \bar{x}^{p+\sigma}. \quad (3.60)$$

Next iterating once (3.56) and inserting the result into (3.59) we obtain the asymptotic expression [8, 10]

$$C(\vec{k},t) \sim \frac{\left[k_m^{\theta-d}(k_m L)^{\frac{p+\sigma}{2}}\right]^{\frac{p}{\sigma}\varphi(\bar{x})}}{(k_m\bar{x})^\theta}, \quad (3.61)$$

which, up to a logarithmic factor in the amplitude, is in the multiscaling form (1.5)

$$C(\vec{k},t) \sim k_m^{-\alpha_0(\bar{x})}\frac{1}{\bar{x}^\theta}, \quad (3.62)$$

with

$$\alpha_0(\bar{x}) = \frac{(d-\theta)p}{\sigma}\varphi(\bar{x}) + \theta. \quad (3.63)$$

This exponent $\alpha_0(\bar{x})$ is plotted in Fig. 6 for $\theta = 0$, $\sigma = 2$ showing that the transition from multiscaling to standard scaling is a smooth one as p is varied continuously from $p = 2$ to $p = 0$.

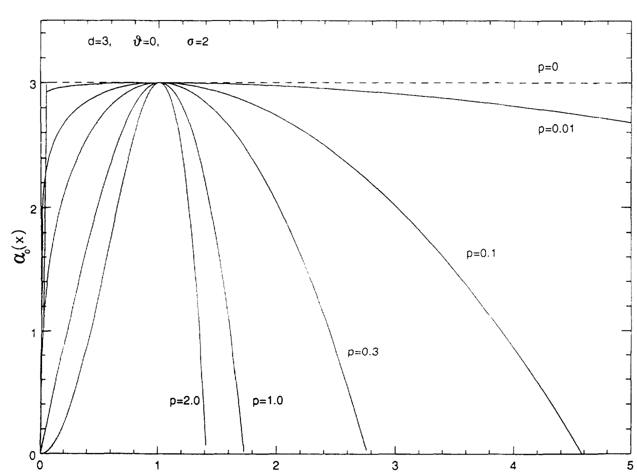

FIG. 6. Plot of $\alpha_0(x)$ for different values of p. Multiscaling goes over smoothly to standard scaling as $p \to 0$.

Notice that from (3.58) and $k_m^{-1} \sim t^{1/z}$, the exponent z develops a time dependence which asymptotically is given by

$$z \sim (p+\sigma)\left[1 + \frac{\ln\ln t}{\ln t}\right]. \quad (3.64)$$

2. $T_F > 0$

$\mu_1 = (r = 0, g = 0)$. Again, with $Q(t) \equiv 0$ and $\theta = 0$ it is straightforward to derive from (3.1) the standard scaling result

$$C(\vec{k},t) = \Delta e^{-wx^{p+\sigma}} + L^\sigma T_F \hat{F}_0(x), \quad (3.65)$$

where

$$\hat{F}_0(x) = (1 - e^{-wx^{p+\sigma}})/wx^\sigma \quad (3.66)$$

and the same considerations made about (3.19) apply here.

$\mu_3 = (r < 0, g > 0)$. For quenches to $\mu_3 = (r < 0, g > 0)$ now we cannot solve explicitly for $Q(t)$. Then we proceed differently by rewriting (3.1) as

$$C(\vec{k},t) = \Delta e^{u(t)\varphi(\bar{x})} + 2\Gamma T_F k^p \int_0^t dt' e^{u(t)\varphi(\bar{x}) - u(t')\varphi(\bar{x}')}, \quad (3.67)$$

where $\varphi(\bar{x})$ has been defined in (3.60) and $u(t)$ is related to $Q(t)$ by (3.54). In the following we will drop the bar over x. If by analogy with what we have found at $T_F = 0$ we make the ansatz

$$e^{u(t)} = cL^\rho(t), \quad (3.68)$$

with $\rho > 0$, the quantities to be determined are the constant c and the exponent ρ. Inserting into (3.67), and assuming $k_m^{-1}(t) \sim L(t)$ up to logarithmic factors we find

$$C(\vec{k},t) = \Delta(cL^\rho)^{\varphi(x)} \times \left[1 + T_F L^\sigma \frac{1}{wx^\sigma}\int_0^x dx' x'^{p+\sigma-1}(x/x')^{\rho\varphi(x')} \times e^{-\rho\varphi(x')\ln L}\right]. \quad (3.69)$$

For large L the integral in the right hand side can be evaluated by steepest descent. Keeping in mind that $\rho > 0$ and that $\varphi(x)$ behaves as in Fig. 6, we must distinguish between $x < x^*$ and $x > x^*$ where $x^* = (\frac{p+\sigma}{p})^{1/\sigma}$ is the nontrivial zero of $\varphi(x)$. For $x < x^*$ the exponential reaches the maximum value at $x' = 0$, while for $x > x^*$ the maximum is at $x' = x$, yielding

$$C(\vec{k},t) \sim \Delta(cL^\rho)^{\varphi(x)} + \frac{T_F L^\sigma}{wx^\sigma}(cL^\rho)^{\varphi(x)} \quad (3.70)$$

for $x < x^*$ and

$$C(\vec{k},t) \sim \Delta(cL^\rho)^{\varphi(x)} + \frac{T_F L^\sigma}{wx^\sigma} \quad (3.71)$$

for $x > x^*$.

We now determine c and ρ by using the self-consistency condition (2.14). From (3.54), where β is defined in (3.36), and (3.68) follows $R(t) = \dot{Q}(t) \sim (\rho \ln L)^{\frac{\sigma}{p+\sigma}} L^{-\sigma}$. On the other hand, $R(t)$ can be computed from (2.5) with $S(t)$ obtained by integration of (3.70) and (3.71) over $\vec{k}$. Thus for $T_F = 0$ dropping logarithmic factors one finds

$$L^{-\sigma} = r + g\Delta K_d L^{-d} \int_0^{\Lambda L} dx x^{d-1} e^{\varphi(x)\ln(cL^\rho)}. \tag{3.72}$$

For large L the integral picks up the dominant contribution at $x = 1$, giving

$$L^{-\sigma} = r + g\Delta K_d c^{\sigma/p} L^{(\rho\sigma - pd)/p}. \tag{3.73}$$

The right hand side can vanish only if $\rho = pd/\sigma$, in agreement with (3.63), and $c^{\sigma/p} \sim M_0^2$.

For $T_F \neq 0$, the first terms in the right hand sides of (3.70) and (3.71) are asymptotically negligible with respect to the second ones, yielding in place of (3.72)

$$L^{-\sigma} = r + gT_F L^{\sigma-d} K_d \left[\int_0^{x^*} dx x^{d-\sigma-1} (cL^\rho)^{\varphi(x)} + \int_{x*}^{\Lambda L} dx x^{d-\sigma-1} \right]. \tag{3.74}$$

Furthermore, taking into account that for large L the dominant contribution to the first integral is obtained at $x = 1$ and that the contribution at x^* is negligible in the second one, finally we obtain

$$L^{-\sigma} = r\left(\frac{T_F - T_c}{T_c}\right) + gT_F K_d c^{\sigma/p} L^{\rho\frac{\sigma}{p}+\sigma-d}. \tag{3.75}$$

For $T_F < T_c$ this implies $\rho = p(d-\sigma)/\sigma$ and $c^{\sigma/p} \sim M_0^2(\frac{T_c - T_F}{T_F T_c})$. Inserting into (3.70) and (3.71) one finds

$$C(\vec{k}, t) \sim T_F \frac{L^{\alpha(x)}}{wx^\sigma}, \tag{3.76}$$

with

$$\alpha(x) = \begin{cases} \frac{p}{\sigma}(d-\sigma)\varphi(x) + \sigma & \text{for } x < x^* \\ \sigma & \text{for } x > x^*, \end{cases} \tag{3.77}$$

showing that the structure factor obeys multiscaling for $x < x^*$ and standard scaling for $x > x^*$. Notice that the two behaviors match at x^* since $\varphi(x^*) = 0$.

If the quench is made on the critical surface ($T_F = T_c$), then Eq. (3.75) gives $\rho = \frac{p}{\sigma}(d - 2\sigma)$, which is negative below the upper critical dimensionality contradicting the initial assumption $\rho > 0$. Hence we take $\rho = 0$ which implies standard scaling with $u(t)$ constant. From (3.68) and (3.36) we have $Q(t) = bL^p(t)$. Inserting in (3.1) and taking $\Delta = 0$ for simplicity, we obtain

$$C(\vec{k}, t) = L^\sigma T_c \hat{F}_\epsilon(x), \tag{3.78}$$

with

$$\hat{F}_\epsilon(x) = \frac{1}{wx^\sigma} \int_0^x dx' x'^{p+\sigma-1} e^{-[w(x^{p+\sigma} - x'^{p+\sigma}) + b(x^p - x'^p)]}. \tag{3.79}$$

From the self-consistency condition (2.14) now we find

$$b = \frac{p+\sigma}{2\Gamma p} g K_d L^{2\sigma-d} T_c \left\{ \int_0^\infty dx x^{d-1} \left(\hat{F}_\epsilon(x) - \frac{1}{wx^\sigma} \right) - \int_{\Lambda L}^\infty \left(\hat{F}_\epsilon(x) - \frac{1}{wx^\sigma} \right) \right\}, \tag{3.80}$$

where we have used the definition (2.16) of T_c. From (3.79) for large x we have $\hat{F}_\epsilon \sim \frac{1}{wx^\sigma}(1 - cbx^{-\sigma})$ where $c = \frac{p}{p+\sigma} \int_0^\infty d\psi \psi e^{-\psi}$. Inserting into (3.80) and defining $f(b) = \int_0^\infty dx x^{d-1}[\hat{F}_\epsilon(x) - \frac{1}{wx^\sigma}]$ we have

$$b\left[1 - \frac{(p+\sigma) g K_d T_c \Lambda^{d-2\sigma}}{2\Gamma p(d - 2\sigma)}\right] = \frac{p+\sigma}{2\Gamma p} g K_d L^{2\sigma-d} f(b) \tag{3.81}$$

from which follows $b = 0$ for $d > 2\sigma$, while for $d < 2\sigma$ the value of b is given by the condition $f(b) = 0$. Solving this equation in the ϵ expansion [15] one finds $b \sim \epsilon$ as for NCOP, which implies the same structure of fixed points. Namely, for $\epsilon \to 0$ the nontrivial fixed point still merges with the trivial one and $\lim_{\epsilon\to 0} \hat{F}_\epsilon(x) = \hat{F}_0(x)$.

The summary of the asymptotic properties with COP is given hereafter:

(1) $[T_F = 0, \mu_1]$,

$$C(\vec{k}, t) \sim L^\theta(t) \hat{F}_>(x), \tag{3.82}$$

(2) $[T_F = T_c, \mu_1]$,

$$C(\vec{k}, t) \sim L^\sigma(t) T_c \hat{F}_0(x), \tag{3.83}$$

(3) $[T_F = 0, \mu_2]$,

$$C(\vec{k}, t) \sim \begin{cases} L^\theta(t)\hat{F}_>(x) & \text{for } d > d_c \\ L^\theta(t)\hat{F}(x) & \text{for } d = d_c = \theta + \sigma \\ \lambda^\theta(t)\hat{F}_<(x') & \text{for } d < d_c, \end{cases} \tag{3.84}$$

where $x = kL(t)$ and $x' = k\lambda(t)$,

(4) $[T_F = T_c > 0, \mu_3]$,

$$C(\vec{k}, t) \sim \begin{cases} L^\sigma(t) T_c \hat{F}_0(x) & \text{for } d \geq 2\sigma \\ L^\sigma(t) T_c \hat{F}_\epsilon(x) & \text{for } d < 2\sigma, \end{cases} \tag{3.85}$$

(5) $[0 < T_F < T_c, \mu_3]$,

$$C(\vec{k}, t) \sim T_F \frac{L^{\alpha(x)}(t)}{wx^\sigma}, \tag{3.86}$$

(6) $[T_F = 0, \mu_3]$,

$$C(\vec{k}, t) \sim L^{\alpha_0(x)}(t) \frac{1}{x^\theta}, \tag{3.87}$$

with

$$L(t) \sim t^{1/(p+\sigma)}, \tag{3.88}$$

$$\lambda(t) \sim t^{1/(d+p-\theta)}, \tag{3.89}$$

$$\hat{F}_>(x) = \frac{e^{-wx^{p+\sigma}}}{x^\theta}, \tag{3.90}$$

$$\hat{F}(x) = \frac{1}{x^\theta} e^{-[wx^{p+\sigma}+cx^p]}, \tag{3.91}$$

$$\hat{F}_<(x') = \frac{e^{-x'^p}}{x'^\theta}, \tag{3.92}$$

$$\hat{F}_0(x) = \frac{1}{wx^\sigma}(1 - e^{-wx^{p+\sigma}}), \tag{3.93}$$

$$\hat{F}_\epsilon(x) = \frac{1}{wx^\sigma}\int_0^x dx' x'^{p+\sigma-1} \times e^{-[w(x^{p+\sigma}-x'^{p+\sigma})+b(x^p-x'^p)]}, \tag{3.94}$$

$$\alpha_0(x) = \frac{(d-\theta)p}{\sigma}\varphi(x) + \theta, \tag{3.95}$$

$$\alpha(x) = \begin{cases} \sigma + \frac{p}{\sigma}(d-\sigma)\varphi(x) & \text{for } x < x^* \\ \sigma & \text{for } x > x^*, \end{cases} \tag{3.96}$$

$$\varphi(x) = \frac{p+\sigma}{p}x^p - x^{p+\sigma}. \tag{3.97}$$

IV. RENORMALIZATION GROUP

We now discuss the scaling properties of the model within the RG approach to the problem [16]. Let us recall that in static critical phenomena the Wilson RG equations are obtained performing the following operations on the equilibrium probability distribution $P_{\rm eq}[\vec{\phi}; T_F, \mu]$: (i) elimination of hard modes $\vec{\phi}(\vec{k})$ with $\Lambda/l < k \leq \Lambda$ where $l > 1$, (ii) rescaling of wave vectors and order parameter

$$\vec{k}' = l\vec{k},$$
$$\vec{\phi}'(\vec{k}') = l^{-y}\vec{\phi}(\vec{k}),$$

(iii) requirement of form invariance of $P_{\rm eq}[\vec{\phi}; T_F, \mu]$. These operations generate recursion relations for the parameters (T_F, μ) which allow one to describe scaling in terms of the geometry of the fixed points and their domains of attraction.

In quench processes one deals with a time dependent probability distribution $P[\vec{\phi}; t, T_F, \mu]$. Therefore RG transformations performed on this object are expected to give recursion relations for (t, T_F, μ) with a fixed point structure which accounts for the variety of scaling behaviors obtained in Sec. III. In order to implement the procedure outlined above, we should construct $P[\vec{\phi}; t, T_F, \mu]$ and then carry out renormalization. However, since the stochastic process is Gaussian and all the equal time information is in $C(\vec{k}, t)$, we can work directly with the equation of motion (2.13). First we separate soft and hard modes

$$C(\vec{k}, t) = C_s(\vec{k}, t) + C_h(\vec{k}, t), \tag{4.1}$$

with

$$C_s(\vec{k}, t) = \begin{cases} C(\vec{k}, t) & \text{for } 0 \leq k \leq \Lambda/l \\ 0 & \text{for } \Lambda/l < k \leq \Lambda, \end{cases} \tag{4.2}$$

$$C_h(\vec{k}, t) = \begin{cases} 0 & \text{for } 0 \leq k \leq \Lambda/l \\ C(\vec{k}, t) & \text{for } \Lambda/l < k \leq \Lambda. \end{cases} \tag{4.3}$$

The equation of motion for either component is given by

$$\frac{\partial C_{s,h}(\vec{k}, t)}{\partial t} = -2\Gamma[wk^{p+\sigma} + k^p R(t)]C_{s,h}(\vec{k}, t) + 2\Gamma k^p T_F, \tag{4.4}$$

with

$$R(t) = r + gS_s(t) + gS_h(t), \tag{4.5}$$

$$S_{s,h}(t) = \int \frac{d^dk}{(2\pi)^d} C_{s,h}(\vec{k}, t). \tag{4.6}$$

Then we proceed to eliminate hard modes. Integrating the equation of motion for $C_h(\vec{k}, t)$ we find

$$C_h(\vec{k}, t) = C_h(\vec{k}, 0)e^{-2\Gamma[wk^{p+\sigma}t + k^p Q(t)]} + 2\Gamma k^p T_F \int_0^t dt' e^{-2\Gamma\{wk^{p+\sigma}(t-t') + k^p[Q(t)-Q(t')]\}}, \tag{4.7}$$

where $Q(t) = Q_s(t) + Q_h(t)$. After integrating over $\vec{k}$ we should solve for $S_h(t)$ and insert the result into the equation for $C_s(\vec{k}, t)$. However, considering that we must eliminate modes with $k > L^{-1}(t)$ and that in the scaling regime these have already equilibrated, with a good approximation we can set

$$C_h(\vec{k}, t) \sim C_h(\vec{k}, \infty) = \frac{T_F}{wk^\sigma + R(\infty)} \tag{4.8}$$

and we are left with Eq. (4.4) for $C_s(\vec{k}, t)$ with

$$R(t) = r + gS_s(t) + gS_h(\infty). \tag{4.9}$$

Next we carry out the rescalings for $k < \Lambda/l$,

$$C'(\vec{k}', 0) = l^{d-2y(0)}C(\vec{k}, 0), \tag{4.10}$$

$$\vec{k}' = l\vec{k}, \tag{4.11}$$

$$t' = l^{-z}t, \tag{4.12}$$

$$C'(\vec{k}', t') = l^{d-2y(x)}C(\vec{k}, t), \tag{4.13}$$

where we have allowed for an order parameter scaling index dependent on the invariant quantity $x = kt^{1/z}$. From

(4.4) and (4.13) we obtain the transformed equation of motion

$$\frac{\partial C'(\vec{k}',t')}{\partial t'} = -2\Gamma\left[l^{z-\sigma-p}wk'^{p+\sigma} + l^{z-p}k'^{p}R(l^z t') + \Gamma^{-1}\frac{dy(x)}{dt'}\ln l\right]C'(\vec{k}',t') + 2\Gamma k'^{p}l^{z-p+d-2y(x)}T_F, \tag{4.14}$$

which can be rewritten in the same form as the original equation of motion (2.13),

$$\frac{\partial C'(\vec{k}',t')}{\partial t'} = -2\Gamma[w'k'^{p+\sigma} + k'^{p}R'(t')]C'(\vec{k}',t') + 2\Gamma k'^{p}T'_F \tag{4.15}$$

defining

$$w' = l^{z-\sigma-p}w, \tag{4.16}$$

$$R'(t') = l^{z-p}R(l^z t') + \frac{x\frac{dy}{dx}\ln l}{\Gamma z t'}, \tag{4.17}$$

$$T'_F = l^{z-p+d-2y(x)}T_F. \tag{4.18}$$

In order to preserve the self-consistent structure we must require

$$R'(t') = r' + g'S'(t'),$$

$$S'(t') = \int \frac{d^d k'}{(2\pi)^d}C'(\vec{k}',t').$$

Since the left hand side of (4.17) depends only on t', the x dependence on the right hand side must disappear, implying

$$y(x) = c\ln x + y(0). \tag{4.19}$$

This form of $y(x)$ diverges at $x=0$ and we must necessarily have $c=0$.

So far we have managed to map the process described by (2.13) in the new process governed by (4.15) with new parameters Δ', w', μ', T'_F. We are going to be interested in those processes (fixed points) whose parameters $\Delta^*, w^*, \mu^*, T^*_F$ do not change under renormalization. For these values of the parameters, since the form of the original equation of motion (4.15) and the self-consistent structure have been preserved, we have $C'(\vec{k}',t') = C(\vec{k},',t')$. The existence of such fixed points is compatible only with the standard scaling choice $y(x)\equiv y$, which seems to exclude the multiscaling solution found in the preceding section. We shall comment on this later on.

We proceed to extract recursion relations. From (1.4) and (4.10) it follows that

$$C'(\vec{k}',0) = l^{d+\theta-2y}\Delta/k'^{\theta}, \tag{4.20}$$

implying

$$\Delta' = l^{d+\theta-2y}\Delta. \tag{4.21}$$

Next, rewrite (4.17) as

$$r' + g'S'(t') = l^{z-p}[r + gS_s(l^z t') + gS_h(\infty)], \tag{4.22}$$

where for quenches to final states with $R(\infty)=0$ one has

$$S_h(\infty) = T_F B(0)(1-l^{\sigma-d}) \tag{4.23}$$

and $B(0)$ is defined in (2.18). Since from (4.13) we have

$$S'(t') = l^{2(d-y)}S_s(t) \tag{4.24}$$

inserting (4.23) and (4.24) in (4.22) we find $r' = l^{z-p}[r + gT_F B(0)(1-l^{\sigma-d})]$ and $g' = l^{z-p+2(y-d)}g$. Introducing the scaling field

$$\tau = r + gT_F B(0) = r(T_c - T_F)/T_c \tag{4.25}$$

the whole set of recursion relations is given by

$$\Delta' = l^{d+\theta-2y}\Delta, \tag{4.26}$$

$$w' = l^{z-\sigma-p}w, \tag{4.27}$$

$$\tau' = l^{z-p}\tau, \tag{4.28}$$

$$g' = l^{z-p+2(y-d)}g, \tag{4.29}$$

$$T'_F = l^{z-p+d-2y}T_F. \tag{4.30}$$

We emphasize that the use of the same scaling index y for $t>0$ as well as for $t=0$ leads to fixed points as processes for which scaling holds over the entire time history. For processes where scaling invariance is only asymptotic it is not necessary that there exist a nontrivial fixed point solution of (4.26)–(4.30).

A. Fixed points

The next step is to look for fixed points $(\Delta^*, w^*, \tau^*, g^*, T^*_F)$ of the recursion relations and to extract from them the exponents z and α. These exponents are determined by imposing that two of the parameters have a finite fixed point value. In the remaining recursion relations we consider the trivial fixed point solution and the corresponding domain of attraction. In order to show how this works in practice let us begin by requiring that Δ^* and w^* be finite. From this, using (4.26) and (4.27) it follows that

$$z = p + \sigma \tag{4.31}$$

and

$$\alpha = 2y - d = \theta. \tag{4.32}$$

Inserting these values into (4.28), (4.29), and (4.30) we obtain

$$\tau' = l^{\sigma}\tau, \tag{4.33}$$

$$g' = l^{d_c-d}g, \tag{4.34}$$

$$T'_F = l^{\sigma-\theta}T_F, \tag{4.35}$$

with $d_c = \theta+\sigma$. The trivial solution is $\tau^* = g^* = T_F^* = 0$, which coincides with $(T_F = 0, \mu_1)$. The corresponding domain of attraction, considering that for quenches from high temperature to the critical surface $\theta \leq \sigma$, is given by the g axis, i.e., $(T_F = 0; \mu_2)$ for $d > d_c$. Otherwise, for $d < d_c$, this fixed point is unstable, and (4.31) and (4.32) do not apply for quenches to $(T_F = 0, \mu_2)$. This coincides with what we have found in Sec. III.

Next, let us require that Δ^* and g^* be finite. Then we find $z = d + p - \theta$, $\alpha = \theta$ and the remaining recursion relations

$$w' = l^{d-d_c}w, \tag{4.36}$$

$$\tau' = l^{d-\theta}\tau, \tag{4.37}$$

$$T'_F = l^{d-2\theta}T_F, \tag{4.38}$$

with the solution $w^* = \tau^* = T_F^* = 0$. w flows to zero for $d < d_c$. This fixed point for COP corresponds to the quench to $(T_F = 0, \mu_2)$. For NCOP this corresponds to the case of independent particles and it is possible to show that in this case the amplitude of the structure factor vanishes. Thus in order to treat quenches to $(T_F = 0, \mu_2)$ for NCOP with $d < d_c$ we must require that w^* and g^* be simultaneously finite. This immediately reproduces the results $z = \sigma$ and $\alpha = d - \sigma$ of (3.13). The ensuing recursion relations for the other parameters,

$$\Delta' = l^{d_c-d}\Delta, \tag{4.39}$$

$$\tau' = l^{\sigma}\tau, \tag{4.40}$$

$$T'_F = l^{2\sigma-d}T_F, \tag{4.41}$$

yield a trivial solution which is unstable under all perturbations. The meaning of Δ flowing to infinity can be understood from the result of Sec. III where the crossover time t^* vanishes for $\Delta \to \infty$. In this limit we have a fixed point in the sense specified above that the same scaling behavior applies over the whole history of the process.

So far we have dealt with fixed points with $T_F^* = 0$ and $\tau^* = 0$, namely, with quenches to zero temperature critical points. In order to analyze quenches on the critical surface at finite temperature, we must require T_F^* and w^* finite as it is usually done in static critical phenomena. From these conditions it follows that $z = p + \sigma$, $\alpha = \sigma$, and

$$\Delta' = l^{\theta-\sigma}\Delta, \tag{4.42}$$

$$\tau' = l^{\sigma}\tau, \tag{4.43}$$

$$g' = l^{2\sigma-d}g. \tag{4.44}$$

Thus for quenches to finite temperature on the critical surface Δ is irrelevant and the attractive fixed point goes from trivial to nontrivial as the dimensionality goes from above to below the upper critical dimensionality 2σ.

Finally let us come to the discussion of quenches inside the phase ordering region. For this we require that w^* and the fixed point ratio

$$\left(\frac{\tau}{g}\right)^* = \left[M_0^2\left(\frac{T_F - T_c}{T_c}\right)\right]^* \tag{4.45}$$

be finite, obtaining $z = p + \sigma$ and $\alpha = d$. The other recursion relations

$$\Delta' = l^{\theta-d}\Delta, \tag{4.46}$$

$$T'_F = l^{\sigma-d}T_F \tag{4.47}$$

show that temperature perturbations in the ordering region are irrelevant and that Δ flows to zero. Now, with $\Delta^* = T_F^* = 0$ the structure factor $C(\vec{k}, t)$ vanishes identically, namely, the scaling form (3.18) is obeyed with $F(x) \equiv 0$. This means that a nontrivial scaling solution for a quench in the phase ordering region with $\alpha = d$ is necessarily asymptotic and cannot be made to hold over the entire history of the process by any choice of the parameters.

This completes the analysis of the fixed point structure of the phase diagram and the derivation of exponents. The RG treatment of the problem presented above reproduces the whole structure found in Sec. III except for multiscaling in the quench below T_c with $p \neq 0$. The point is that the RG procedure we have followed above yields the exponents z and α within a standard scaling framework, but gives no information on the scaling function. If one goes further by performing the scaling ansatz in the equation of motion an equation for $F(x)$ is obtained and it turns out that the scaling function vanishes if $T_F < T_c$ and $p \neq 0$. In order to recover multiscaling through the RG approach the set of transformations must be properly generalized. We do this only for the $(T_F = 0, \mu_3)$ case [17].

As we have seen in Sec. III, when there is multiscaling z is weakly time dependent. Let us then generalize the set of transformations (4.10)–(4.13) by allowing z to depend on t with the constraint

$$\lim_{t\to\infty} z(t) = z_\infty = p + \sigma. \tag{4.48}$$

With these modifications and $T_F = 0$ in place of (4.14) we find

$$\begin{aligned}\frac{\partial C'(\vec{k}', t')}{\partial t'} &= -2\Gamma[wk'^{p+\sigma} + k'^{p}l^{\sigma}R(t)]C'(\vec{k}', t') \\ &\quad -2\Big\{\Gamma(z - z_\infty)[wk'^{p+\sigma} + k'^{p}l^{\sigma}R(t)] \\ &\quad +\frac{dy(x)}{dt'}\Big\}C'(\vec{k}', t')\ln l. \end{aligned}\tag{4.49}$$

Imposing the requirement of form invariance we find that (4.49) is of the form (4.15) if in place of (4.17) the following conditions are satisfied:

$$R'(t') = l^{\sigma}R(t), \tag{4.50}$$

$$\frac{dy}{dt'} = -\Gamma[z(t') - z_\infty][wk'^{p+\sigma} + k'^p R'(t')]. \qquad (4.51)$$

This latter equation holds also for unprimed variables and using

$$\frac{dx}{dt} = \frac{x}{zt}\left[1 - \frac{t}{z}\frac{dz}{dt}\ln t\right] \qquad (4.52)$$

we obtain

$$\frac{dy}{dx} = -\frac{\Gamma z[z - z_\infty]t^{1-(p+\sigma)/z}}{[1 - \frac{t}{z}\frac{dz}{dt}\ln t]} \times [wx^{p+\sigma-1} + x^{p-1}t^{\sigma/z}R(t)]. \qquad (4.53)$$

In order to get rid of the time dependence on the right hand side we must have

$$-\Gamma z(z - z_\infty) = ct^{(z-z_\infty)/z}\left(1 - \frac{t}{z}\frac{dz}{dt}\ln t\right), \qquad (4.54)$$

$$t^{\sigma/z}R(t) = b, \qquad (4.55)$$

where b and c are constants. Integrating (4.53) we find

$$y(x) = c\left[\frac{wx^{p+\sigma}}{p+\sigma} + \frac{b}{p}x^p\right] + y(0). \qquad (4.56)$$

From (4.55) and the definition (2.5) of $R(t)$ it follows that the sign of b is determined by the parameters $\mu = (r, g)$. In particular, b is a negative quantity at μ_3. Imposing $y_{\max} = d$, as appropriate for quenches to μ_3, taking the position of the maximum at $x = 1$, and using $y(0) = (d+\theta)/2$ from (4.56) we find $b = -w$, $c = (\theta - d)p(p+\sigma)/2\sigma$, and

$$2y(x) - d = \alpha_0(x), \qquad (4.57)$$

where $\alpha_0(x)$ coincides with (3.63). Hence, for quenches to μ_3, by allowing for a time dependence in the growth exponent z, we have recovered via RG the multiscaling behavior of the exact analytical solution.

The explicit time dependence of $z(t)$ is obtained by extracting the asymptotic behavior from (4.54),

$$z(t) \simeq z_\infty\left[1 + \frac{\ln\ln t}{\ln t}\right], \qquad (4.58)$$

which is consistent with the assumption (4.48) and reproduces (3.64). Finally, inserting this result into (4.50) we find

$$R(t) \sim \left(\frac{\ln t}{t}\right)^{\frac{\sigma}{p+\sigma}} \qquad (4.59)$$

in agreement with (3.57).

V. CONCLUSIONS

In this paper we have investigated in detail the solution of the large-N model for growth kinetics with the aim of giving a comprehensive view of the influence on the scaling properties of the various elements which enter into the specification of the problem. These are the presence or the absence of a constraint on the order parameter (COP or NCOP), the initial condition, the structure of the phase diagram of final equilibrium states, the range of the interaction and the dimensionality of space.

What the model shows, apart from quenches to the trivial fixed point, is that scaling properties are quite different with and without conservation law. This is due to the existence of only one divergent length $L(t)$ for NCOP and of two divergent lengths $L(t)$ and $\lambda(t)$ for COP. It is the interplay between these two lengths which leads to phenomena not observed with NCOP such as (i) a change in the growth law when crossing the critical dimensionality for the quenches to $(T_F = 0, \mu_2)$ and (ii) multiscaling for quenches inside the phase ordering region. About multiscaling, the availability of the rich variety of cases illustrated in the paper should allow speculation about its origin. Thus we have found that for NCOP in no circumstance is there multiscaling and the same holds true for COP, except for the quenches below T_c. What, then, is the peculiarity of these latter processes? One possible interpretation is that in these processes the system orders and tends to do so by condensing, i.e., by growing a peak which scales as L^d, at $\vec{k} = \vec{0}$. This is fine with NCOP, but with COP there is a conflict with the conservation law which prevents anything from happening at $\vec{k} = \vec{0}$. In this case the peak is formed at some $\vec{k}_m \neq \vec{0}$. The compromise realized in the large-N model is multiscaling whereby the behavior $L^{\alpha_0(x)}$ of the structure factor interpolates smoothly between the behaviors L^0 at $\vec{k} = \vec{0}$ and L^d at $\vec{k}_m$. This picture fits nicely with the absence of multiscaling in any process with NCOP and in all processes with COP on the critical surface. In fact, in the latter case there is nothing to condense at $\vec{k} = \vec{0}$ and in the former there are no constraints at $\vec{k} = \vec{0}$. However, with this mechanism for multiscaling there should be nothing special about $N = \infty$ and multiscaling should be found also for $N < \infty$. Numerical simulations [18] so far have reported no evidence for multiscaling for $N = 1$ and $N = 2$ in two and three dimensions. This could mean that multiscaling disappears for $N \leq d$, when localized topological defects appear in the system. However, an argument against the existence of multiscaling for any finite N is the result of Bray and Humayun [19]. By analyzing an equation of motion for the structure factor which includes the first order correction in $1/N$, they have reached the conclusion that multiscaling does not survive for $N < \infty$ since the correction term sustains a standard scaling solution.

What we can say is that both standard scaling and multiscaling imply scale invariance of the structure factor due to the presence of a divergent length. So far we have not found a criterion to predict *a priori* which one should hold. Only a direct calculation, either analytical or numerical, can discriminate between the two. It must be emphasized that these concepts apply also to other models. For example, in the DLA (diffusion limited aggregation) model in two dimensions it has been found numerically [20] that multiscaling holds. In any case, even if there is not a general criterion, multiscaling should be more likely to occur in situations where the width of the interface becomes very large.

Let us then comment on those features of the $N = \infty$ solution which we believe to be of general validity. The crossover structure which emerges as the parameters of the quench are moved over the manifold of final equilibrium states is a generic feature which is expected to hold beyond the large-N model. The main point of our analysis is that it is quite possible, before the true asymptotic behavior is reached, to detect a preasymptotic scaling behavior due to a less stable fixed point lying in the neighborhood of the final equilibrium state. It is natural to pose the question of the observability of these effects. Here we suggest (Appendix A) that the duration of this preasymptotic behavior can be magnified and observed in off-critical quenches [21].

Furthermore, the crossover picture we have illustrated suggests the possibility of observing a crossover in the growth law (1.2) in the symmetrical quench of a system with scalar ($N = 1$) COP. In that case asymptotically $L(t)$ grows according to (1.2) with $z = 3$. On the other hand, in the trivial theory ($r = 0, g = 0$) one has $z = 4$ for COP, irrespective of the order parameter being a scalar or a vector. Thus, for a quench to ($T_F = 0, \mu_3$) sufficiently close to ($T_F = 0, \mu_1$) with $N = 1$, it should be possible to observe the influence of the trivial fixed point at early time, producing a crossover in (1.2) from $z = 4$ to $z = 3$.

APPENDIX A

Let us consider a process where symmetry breaking along one direction is allowed, e.g.,

$$\langle \phi_\alpha(\vec{x}, t) \rangle = N^{1/2} M(t) \delta_{\alpha,1}. \tag{A1}$$

This may be due to nonsymmetrical initial conditions, or to the presence of an external field, or to both of these circumstances [22]. Introducing the external field along the 1 direction from (1.3) and dropping the long range term we obtain the equation of motion for the order parameter

$$\frac{\partial \phi_\alpha(\vec{x}, t)}{\partial t} = -\Gamma(i\nabla)^p \left[\left(-\nabla^2 + r + \frac{g}{N} \sum_{\beta=1}^{N} \phi_\beta^2(\vec{x}, t) \right) \times \phi_\alpha(\vec{x}, t) - h_\alpha \right] + \eta_\alpha(\vec{x}, t), \tag{A2}$$

where $h_\alpha = N^{1/2} h \delta_{\alpha,1}$.

Defining the fluctuation field $\psi(\vec{x}, t)$ by

$$\phi_1(\vec{x}, t) = N^{1/2} M(t) + \psi(\vec{x}, t) \tag{A3}$$

and inserting into (6.2) we obtain the pair of equations

$$\frac{\partial (N^{1/2} M + \psi)}{\partial t} = -\Gamma(i\nabla)^p \left[-\nabla^2 \psi + r(N^{1/2} M + \psi) + \frac{g}{N} (N^{1/2} M + \psi) \sum_{\beta \neq 1} \phi_\beta^2 + g \left(N^{1/2} M^3 + 3 M^2 \psi + \frac{3}{N^{1/2}} M \psi^2 \frac{1}{N} \psi^3 \right) - N^{1/2} h \right] + \eta_1(\vec{x}, t), \tag{A4}$$

$$\frac{\partial \phi_\beta}{\partial t} = -\Gamma(i\nabla)^p \left[-\nabla^2 \phi_\beta + r \phi_\beta + \frac{g}{N} \sum_{\gamma \neq 1} \phi_\gamma^2 \phi_\beta + g \left(M^2 + \frac{2}{N^{1/2}} M \psi + \frac{1}{N} \psi^2 \right) \phi_\beta \right] + \eta_\beta, \tag{A5}$$

with $\beta \neq 1$.

Taking the large-N limit we replace $\frac{1}{N} \sum_{\beta \neq 1} \phi_\beta^2(\vec{x}, t)$ by $\langle \phi_\beta^2(\vec{x}, t) \rangle = S_\perp(t)$ and collecting terms of the same order of magnitude, from (6.4) and (6.5) we obtain the set of equations

$$\frac{\partial M(t)}{\partial t} = -\Gamma(i\nabla)^p \Big\{ [r + g M^2(t)] M(t) + g S_\perp(t) M(t) - h \Big\}, \tag{A6}$$

$$\frac{\partial \psi(\vec{x}, t)}{\partial t} = -\Gamma(i\nabla)^p \left[-\nabla^2 + r + 3g M^2(t) + g S_\perp(t) \right] \times \psi(\vec{x}, t) + \eta_1(\vec{x}, t), \tag{A7}$$

$$\frac{\partial \phi_\beta(\vec{x}, t)}{\partial t} = -\, \Gamma(i\nabla)^p \left[-\nabla^2 + r + g M^2(t) + g S_\perp(t) \right] \times \phi_\beta(\vec{x}, t) + \eta_\beta(\vec{x}, t). \tag{A8}$$

Notice that the behavior of $\psi(\vec{x}, t)$ is immediately obtained once the pair of coupled equations (A6) and (A8) have been solved. Furthermore, for COP Eq. (A6) is trivial since $M(t)$ does not change in time and keeps the initial value $M(0)$. In this case Eq. (A8) for the transverse components, after Fourier transforming, can be rewritten as

$$\frac{\partial \phi_\beta(\vec{k}, t)}{\partial t} = -\Gamma k^p \left[-\nabla^2 + \tilde{r} + g S_\perp(t) \right] \phi_\beta(\vec{x}, t) + \eta_\beta(\vec{x}, t), \tag{A9}$$

where

$$\tilde{r} = r + g [M(0)]^2. \tag{A10}$$

Thus $\tilde{r}$ can be modulated by varying $M(0)$. In particular, a quench to μ_2 can be realized as an off-critical quench to $T_F = 0$ and at the edge of the coexistence region $M^2(0) = -r/g$.

APPENDIX B

In the limit $t \to \infty$ the left hand side of Eq. (2.13) vanishes and we have

$$0 = -2\Gamma[wk^{p+\sigma} + k^p R]C(\vec{k},\infty) + 2\Gamma k^p T_F, \quad \text{(B1)}$$

where R stands for $R(\infty)$.

Let us first consider $p = 0$ and a system in a finite volume V. Assuming $R > 0$ from (B1) it follows that

$$C(\vec{k},\infty) = \frac{T_F}{wk^\sigma + R}, \quad \text{(B2)}$$

where R must satisfy the self-consistency condition

$$R = r + g\frac{1}{V}\sum_{\vec{k}} \frac{T_F}{wk^\sigma + R}, \quad \text{(B3)}$$

which always admits a solution with $R > 0$.

In the infinite volume limit Eq. (B3) can be rewritten as

$$R = r + gT_F B(R) + g\frac{T_F}{VR}, \quad \text{(B4)}$$

where, allowing for the possibility that the solution $R \to 0$, the zero wave vector term in the summation has been separated out and

$$B(R) = \int \frac{d^d k}{(2\pi)^d}\frac{1}{wk^\sigma + R} \quad \text{(B5)}$$

is a non-negative monotonously decreasing function of R with a maximum at $B(0) = K_d\Lambda^{d-\sigma}/[w(d-\sigma)]$. Λ is a momentum cutoff and $K_d = [2^{d-1}\pi^{d/2}\Gamma(d/2)]^{-1}$. From Eq. (B4) it follows that when $r < 0$ there is a critical temperature

$$T_c = -\frac{r}{gB(0)} \quad \text{(B6)}$$

such that for $T_F > T_c$ there exists a solution with $R > 0$ and therefore the last term in the right hand side can be neglected, but for $T_F < T_c$ Eq. (B4) can only have the solution $R = 0$ provided that R vanishes in the infinite volume limit as $R \sim 1/V$. Hence, defining the constant

$$M^2 = \frac{T_F}{VR} \quad \text{(B7)}$$

and inserting into (B4) we find

$$M^2 = M_0^2(T_c - T_F)/T_c, \quad \text{(B8)}$$

with $M_0^2 = -r/g$. Notice that for $T_F = T_c$ Eq. (B4) admits the solution $R = 0$ provided $R \sim V^{-x}$ with $0 < x < 1$. In conclusion, the structure factor is given by

$$C(\vec{k},\infty) = \begin{cases} T_F/(wk^\sigma + R) & \text{with } R > 0 \text{ for } T_F > T_c \\ T_F/wk^\sigma + (2\pi)^d M^2\delta(\vec{k}) & \text{for } T_F \le T_c. \end{cases} \quad \text{(B9)}$$

For $p \neq 0$ Eq. (B2) applies only for $\vec{k} \neq 0$. Due to the conservation law $C(\vec{k} = \vec{0},\infty)$ is determined by the initial condition. If we consider an initial state without symmetry breaking we have $\phi(\vec{k} = \vec{0}) = 0$ and $C(\vec{k} = \vec{0}, t = 0) = 0$. Then Eq. (B3) must be replaced by

$$R = r + \frac{g}{V}\sum_{\vec{k}\neq 0}\frac{T_F}{wk^\sigma + R}. \quad \text{(B10)}$$

Solving the above equation for R we find that there exists a temperature

$$\tilde{T}(V) = -\frac{r}{g}\frac{1}{V\sum_{\vec{k}\neq\vec{0}} wk^\sigma} \quad \text{(B11)}$$

such that $R \ge 0$ for $T_F \ge \tilde{T}(V)$, while $R < 0$ for $T_F < \tilde{T}(V)$ with $|R| < k_{\min}$ where $k_{\min} \sim V^{-1/d}$ is the minimum value of the wave vector in the summation. When the infinite volume limit is taken from (B11) it follows that $\tilde{T}(V) \to T_c$ and the anlogue of (B4) is

$$R = r + gT_F B(R) + \frac{g}{V}\frac{T_F}{wk_{\min}^\sigma + R}. \quad \text{(B12)}$$

For $T_F > T_c$ there is a solution $R > 0$ and the last term vanishes, while for $T_F < T_c$ Eq. (B12) admits the solution $R = 0$ provided $(k_{\min}^\sigma + R) \sim 1/V$. Writing

$$M^2 = \frac{T_F}{V(wk_{\min}^\sigma + R)} \quad \text{(B13)}$$

in place of (B7), in the end we recover the results (B8) and (B9).

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