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Vortical scales for two- and three-dimensional turbulence

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Correlation analysis of two-dimensional (2D) turbulence is performed on the basis of the Navier-Stokes equations. It is found that within the inertial range of scales ($L\,\mathrm{Re}^{-1/2} \ll r \ll L$; L is the external scale, Re is the Reynolds number) there is a physically distinguished scale $l_c = L\,\mathrm{Re}^{-1/4}$—a natural scale for coherent vortex patches. A hierarchy of vortical scales for 2D and 3D turbulence is found from a covariance analysis of more detailed spatial structure of the vorticity field.

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Spatial differentiation of nonlinear equations can help to reveal physical effects, which are hidden in the original form of equations. In particular, the passage from the Navier-Stokes equations for the velocity to the equations for the vorticity illuminates the effect of self-amplification of vorticity (stretching of vortex filaments) in three-dimensional (3D) turbulence [1,2]. Similarly, additional differentiation reveals the effect of self-amplification of vorticity gradient (VG) in 2D turbulence [3–6]:

$$\frac{\partial s_i}{\partial t} + v_k \frac{\partial s_i}{\partial x_k} = -\frac{\partial v_k}{\partial x_i} s_k + \nu \Delta s_i + \psi_i \ , \quad \frac{\partial v_i}{\partial x_i} = 0 \ , \tag{1}$$

$$s_i = \frac{\partial \omega}{\partial x_i} \ , \quad \omega = \epsilon_{ik} \frac{\partial v_k}{\partial x_i} \ , \quad \psi_i = \frac{\partial \phi}{\partial x_i} \ ,$$
$$\phi = \epsilon_{ik} \frac{\partial f_k}{\partial x_i} \ , \quad \frac{\partial f_i}{\partial x_i} = 0 \ . \tag{2}$$

Here v_i, ω, s_i, and f_i are correspondingly velocity, vorticity, VG, and external random force in 2D incompressible fluid, ν is kinematic viscosity and ϵ_{ik} is the unit antisymmetric tensor. We use the concept of self-amplification, because the tensor of deformation rates, responsible for amplification, is expressed in terms of local characteristics (vorticity in 3D and VG in 2D) [2–6].

The statistical balances of vorticity and VG in homogeneous 2D turbulence have the forms [3–6]

$$\frac{1}{2} \frac{\partial}{\partial t} \langle \omega^2 \rangle = -\gamma + \langle \phi \omega \rangle \ , \quad \gamma = \nu \left\langle \left[\frac{\partial \omega}{\partial x_i} \right]^2 \right\rangle \ , \tag{3}$$

$$\frac{1}{2} \frac{\partial}{\partial t} \langle s_i^2 \rangle = -\left\langle \frac{\partial v_k}{\partial x_i} s_i s_k \right\rangle - \nu \left\langle \left[\frac{\partial s_i}{\partial x_k} \right]^2 \right\rangle + \langle \psi_i s_i \rangle \ . \tag{4}$$

Here $\langle \ \rangle$ indicates statistical averaging, all fields are taken at the same space-time location, and γ is the mean rate of enstrophy dissipation. The first terms in the right-hand side (rhs) of (1) and (4) represent the effect of self-amplification, connected with the compression of fluid elements in the direction of VG. More detailed statistical analysis of this effect is done by the conditional averaging of (1) with fixed VG at the same point [3–6].

Spatial differentiation of nonlinear equations, followed by a covariance analysis, can also help to find new characteristic scales. In particular, such an approach revealed a physically distinguished scale (21) for the "vortex strings" in 3D turbulence [7]. The covariance analysis of VG performed below leads to a new characteristic scale, which we associate with coherent vortex patches in 2D turbulence. Corresponding analysis for various spatial derivatives of vorticity reveals a hierarchy of vortical scales for 2D and 3D turbulence.

From (1), by a standard procedure [7–9], we get an equation for the correlation tensor of VG in homogeneous isotropic 2D turbulence:

$$\frac{\partial S_{ij}}{\partial t} - 2 \frac{\partial}{\partial r_i} \langle v_k s_k s_j' \rangle = 2\nu \Delta S_{ij} + \Psi_{ij} \ , \tag{5}$$

$$S_{ij}(\mathbf{r}) = \langle s_i s_j' \rangle \ ,$$
$$\Psi_{ij}(\mathbf{r}) = 2 \langle \psi_i s_j' \rangle = -\frac{\partial^2 \Phi(r)}{\partial r_i \partial r_j} \ , \tag{6}$$
$$\Phi = 2 \langle \phi \omega' \rangle \ .$$

Here prime indicates that fields are taken at the point $\mathbf{x}' = \mathbf{x} + \mathbf{r}$. For the Gaussian, δ-correlated in time and statistically stationary random forces we have [6–11]

$$\Phi(r) = \int_{-\infty}^{\infty} d\tau \langle \phi(t,\mathbf{x}) \phi(t+\tau, \mathbf{x}+\mathbf{r}) \rangle \ , \quad \Phi(0) = 2\gamma \ . \tag{7}$$

The second equality in (7) corresponds to enstrophy balance (3) for statistically stationary turbulence. The external scale and Reynolds number for such turbulence is naturally defined by (compare with Refs. [6,10])

$$L^{-2} = -\frac{1}{2\gamma} \left. \frac{d^2 \Phi}{dr^2} \right|_{r=0} \ , \quad \mathrm{Re} = \gamma^{1/3} L^2 \nu^{-1} \ . \tag{8}$$

From (5) and (6) for statistically stationary turbulence we have

$$-2 \frac{\partial}{\partial r_i} \langle v_k s_k s_i' \rangle = 2\Delta \frac{\partial}{\partial r_k} \langle v_k \omega \omega' \rangle = 2\nu \Delta S - \Delta \Phi \ ,$$
$$S \equiv S_{ii} = -\Delta \langle \omega \omega' \rangle \ . \tag{9}$$

This is a balance between self-induced generation of VG correlations, viscous smoothing, and the influence of large-scale motion. Let us note that Eq. (9) is simply the result of application of the Laplace operator to the balance equation for the vorticity correlation. Such a procedure will lead us to a new vortical scale, because the external force by definition is characterized by only one scale L (8), while the balances of vorticity and VG correlations depend also on viscosity.

Let us consider the inertial range, corresponding to the spectral enstrophy flux in 2D turbulence [12–14]:

$$l_0 = L\,\mathrm{Re}^{-1/2} \ll r \ll L\ . \tag{10}$$

The main parameter in this range is γ. Neglecting possible logarithmic correction, we have from dimensional arguments:

$$S(r) \sim \gamma^{2/3} r^{-2}\ . \tag{11}$$

From (8) for $r \ll L$ we get

$$\Delta\Phi(r) \approx -4\gamma L^{-2}\ , \tag{12}$$

where we take into account that $\Phi(r)$ is even (isotropy). Substitution of (11) and (12) into the balance (9) shows, that when we approach the internal scale l_0, the contribution of external force for large Re became negligible: $\sim \mathrm{Re}^{-1}$. Thus, self-production of VG is balanced by viscous smoothing [3–6]. However, within the inertial range (10), the influence of large-scale forcing became comparable with viscous smoothing at the scale

$$l_c = L\,\mathrm{Re}^{-1/4}\ . \tag{13}$$

It seems that l_c is a natural scale for coherent vortex patches in 2D turbulence. We plan to check this by a specially designed numerical experiment with sufficiently high Re.

We now extend the above presented covariance analysis to the m-order spatial derivatives of the vorticity field. In the inertial range (10), the corresponding terms in the correlation balance are

$$\nu\Delta^m S(r) \sim \nu\gamma^{2/3} r^{-2(1+m)]}\ ,\quad \Delta^m\Phi(r) \sim (-1)^m \gamma L^{-2m} \quad (m=1,2,\ldots)\ . \tag{14}$$

We see that the influence of large-scale forcing becomes important starting at the scale

$$l_m = L\,\mathrm{Re}^{-1/[2(1+m)]}\ ,\quad l_1 \equiv l_c \quad (l_m \to L \ \text{ when } m \to \infty)\ . \tag{15}$$

Thus, for $m > 1$ the influence of external forcing is shifting to larger scales. This hierarchy of scales can be useful in a statistical description of a net of thin and long vortex layers—reminiscent of the processes of coalescence and destruction of more compact vortices. At the same time, the hierarchy represents an unusual "cascade" with scale factor, depending on Re and m:

$$\frac{l_{m+1}}{l_m} = \mathrm{Re}^{1/[2(1+m)(2+m)]}\ . \tag{16}$$

This hierarchy deserves more detailed study in the future.

Let us now turn to a 3D turbulent vorticity field. The basic equation is the balance of vorticity correlations [7], which for statistically stationary turbulence can be written in the form

$$2\Delta\frac{\partial}{\partial r_k}\langle v_k v_i v_i'\rangle = 2\nu\Delta\Omega - \Delta F\ ,\qquad \Omega(r) = \langle \omega_i \omega_i'\rangle\ ,\quad F(0) = 2\epsilon\ . \tag{17}$$

Here the left-hand side represents self-induced generation of vorticity correlations and $F(r)$ corresponds to large-scale random forcing, supplying energy at the rate ϵ. The external scale, Reynolds number, and inertial range are defined by

$$L^{-2} = -\frac{1}{2\epsilon}\frac{d^2F}{dr^2}\bigg|_{r=0}\ ,\quad \mathrm{Re} = \epsilon^{1/3} L^{4/3} \nu^{-1}\ , \tag{18}$$

$$l_\nu \equiv L\,\mathrm{Re}^{-3/4} \ll r \ll L\ , \tag{19}$$

where l_ν is the Kolmogorov internal scale. In the inertial range:

$$\Omega(r) \sim \epsilon^{2/3} r^{-4/3}\ ,\quad \Delta F \approx -6\epsilon L^{-2}\ . \tag{20}$$

Comparison of two terms in the rhs of (17) gives the first scale, which has been associated with "vortex strings" [7]:

$$l_1^{(3)} \equiv l_s = L\,\mathrm{Re}^{-3/10}\ . \tag{21}$$

[Superscript (3) means 3D]. In the correlation balance for m-order derivatives of 3D vorticity we have terms:

$$\nu\Delta^{1+m}\Omega \sim \nu\epsilon^{2/3} r^{-(10/3)-2m}\ ,\quad \Delta^{1+m}F \sim \epsilon L^{-2(1+m)}\ . \tag{22}$$

Comparison of these terms gives a hierarchy of increasing scales:

$$l_{1+m}^{(3)} = L\,\mathrm{Re}^{-3/(10+6m)} \quad (m=0,1,2,\ldots)\ . \tag{23}$$

We can suggest again that this hierarchy corresponds to the statistical structure of the net of vortex tubes and sheets—reminiscent of the processes of formation and destruction of vortex strings.

We hope that the presented results and interpretations will stimulate more detailed studies of vortex structures in 2D and 3D turbulence.

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