RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 49, NUMBER 2 FEBRUARY 1994

Controlling unstable periodic orbits by a delayed continuous feedback

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(Received 22 October 1993)

A method is presented for stabilizing unstable periodic orbits of a dynamical system by applying continuous feedback on a control parameter. The feedback signal is proportional to the difference between two values of a dynamical variable, separated by a time equal to the unstable orbit periodicity. This method has been checked numerically and experimentally on the control of the unstable orbits of a CO_2 laser with modulated losses.

PACS number(s): 05.45.+b, 42.60.Mi

The interest in the control of dynamical systems has recently increased after the presentation by Ott, Grebogi, and Yorke (OGY) [1] of a general method allowing conversion of a chaotic motion into a periodic one. The main idea relies on the fact that a chaotic attractor has typically embedded in it a dense set of periodic orbits, which can be stabilized by a feedback technique. The interest of the OGY method is twofold. It opens the way to an efficient mastery of chaotic phenomena: in addition to simply suppressing chaos by stabilizing unstable periodic orbits [2], or steady states [3,4], methods derived from the OGY one make use of chaos to direct trajectories to targets [5], generate aperiodic orbits [6], and communicate [7]. Another interest has also grown in using stabilization methods as investigation tools of dynamical systems. This has been made possible because the application of the OGY method leads to stable states which are identical to unstable states (periodic or steady) of the original system. Modifications of the OGY method have allowed us to stabilize and characterize unstable periodic orbits [8,9] which are embedded or not embedded in a chaotic attractor, as well as unstable steady states [4]. A drawback, shared by the feedback techniques based on the OGY idea, is that these techniques cannot be applied to very fast systems. The limiting factor is the use of a Poincaré section, which implies the processing of discontinuous signals. In this paper, we propose to overcome this limit with a stabilization technique involving continuous feedback [10], and which verifies the two following properties: (1) the stable periodic orbit of the controlled system coincides with the unstable one of the original (uncontrolled) system; (2) the feedback procedure is applicable without knowing *a priori* the location of the periodic orbit. These two properties allow us to use this feedback technique as a means to study an unstable periodic orbit, and to easily track it when a control parameter is varied as in Ref. [8]. This technique is checked numerically and experimentally in the case of a CO_2 laser with modulated losses.

Let us consider a dynamical system with an accessible control parameter m and a measurable dynamical variable $x(t)$. We suppose that an unstable orbit of period T exists for a given value m_0 of m, our aim being to stabilize it by applying a correction $p(t)$ on the control parameter $[m(t) = m_0 + p(t)]$. The proposed continuous-feedback method presents common points with previous stabilization techniques involving discontinuous corrections, and it is thus important to recall some of their properties. To ensure that the system with feedback contains an orbit rigorously identical to the considered unstable periodic orbit, we require that the correction $p(t)$ vanishes when the system is stabilized. In the original method proposed by Ott *et al.*, the correction is proportional to the difference between two unstable components of vectors taken from a Poincaré section: $(X_n^u - X_F^u)$ where $\mathbf{X}_F$ denotes the position of the unstable orbit and $\mathbf{X}_n$ the current one. It is also worth noticing that the measure of unstable components is not always needed, and in many cases the use of an arbitrary dynamical variable instead of X_n^u is sufficient. Applied to the original OGY method, this leads to the simple proportional feedback technique, and has succeeded in experimental systems [11]. Another modification of the OGY method has been performed in order to allow an easy tracking of periodic orbits, without knowing *a priori* the periodic orbit location. The main idea consists in using a correction proportional to $(X_n^u - X_{n-1}^u)$ [8]. Here, as for the OGY method, one can simply use in many cases a dynamical variable instead of X_n^u as has been shown in the case of a laser system [8]. The choice of a continuous-feedback procedure satisfying the conditions mentioned above (the vanishing of the correction after stabilization) is not unique. We use here the simplest one, which takes advantage of the simple proportional feedback idea, and which does not involve the fixed point location. The applied correction is proportional to the difference between two successive values of an arbitrary dynamical variable $x(t)$:

$$p(t) = \alpha[x(t) - x(t-T)] \tag{1}$$

with α the gain parameter of the delayed continuous feedback (DCF). The T-periodic orbit remains a solution of the system with feedback, but the associated set of Floquet multipliers is changed. The efficiency of the method depends on the possibility of choosing values of α so that all these Floquet multipliers are in the stable domain. As in the case of the simple proportional feedback [11] and the case of Ref. [8], we can expect that this method will be often efficient for systems near a bifurcation point and/or very dissipative systems. However, the linear sta-

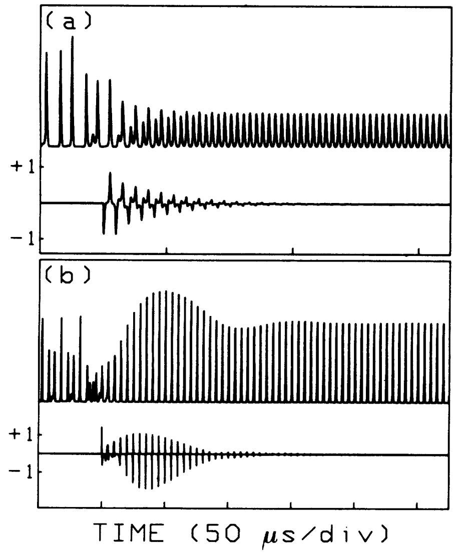

FIG. 1. Typical transient calculated after the activation of the feedback : (a) stabilization of the unstable T orbit, (b) stabilization of the unstable $2T$ orbit. The upper trace represents the laser intensity in arbitrary units, and the lower trace the correction applied to the modulator in units of the modulation index $[p(t)/m]$.

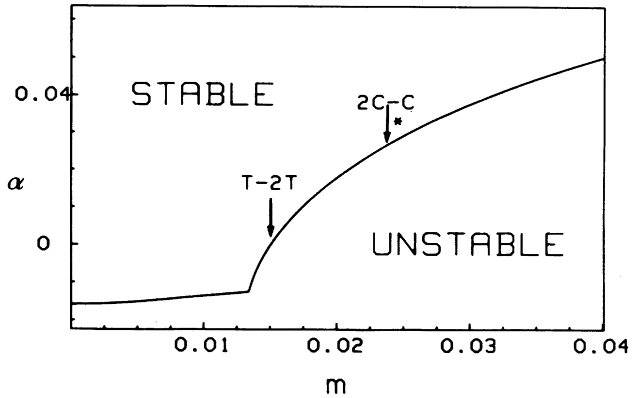

FIG. 2. Stability domain of the T-periodic orbit vs the modulation index m and the gain parameter α. In the absence of feedback, the T-stable orbit is destabilized at $m = 0.015$, and the unstable orbit is embedded in the chaotic attractor for $m > 0.0237$. The asterisk indicates the case presented in the Fig. 1(a).

bility analysis of the orbit is made difficult by the presence of a delayed term, and we will concentrate here on checking its efficiency numerically and experimentally.

The DCF method has been implemented on a CO_2 laser with a modulated parameter (in our case, a loss modulation using an intracavity electro-optic modulator). When the modulation index is increased, the laser undergoes a classical period-doubling cascade leading to chaos and periodic windows are commonly observed [12]. The numerical simulations have been performed on the basis of the well-known two-level model without detuning [13]. Thus the behavior of the laser can be described by the following set of two coupled nonlinear equations:

$$\dot{I} = 2I[AD - 1 - k(t)],$$
$$\dot{D} = \gamma[1 - D(1 + I)], \tag{2}$$

where the time t is in units of the cavity lifetime. In these equations, A is the pump parameter and I and D are the laser intensity and the population inversion, respectively. γ is the ratio of the population relaxation rate to the cavity damping rate. $k(t)$ is the additional loss introduced by the modulator and is equal to $k_0(t) = m \cos 2\pi f t$, where m is the modulation index, and f the modulation frequency. For the stabilization of an unstable nT-periodic orbit ($T = 1/f$), a DCF term is added to the loss modulation :

$$k(t) = k_0(t) + \alpha[I(t) - I(t - nT)], \tag{3}$$

where α is the gain of the feedback loop. The numerical parameters used in the simulations correspond to the values already used by Dangoisse *et al.* [13] to describe the experimental situation of our laser. The frequency is fixed at 400 kHz, the cavity damping rate is estimated to be 6×10^7 s^{-1}, and the population relaxation rate is calculated to be 2.5×10^5 s^{-1}. In the case of the numerical simulations presented here, the pump parameter A is fixed at 1.1 and the modulation index m at 0.0246, in a parameter domain where the laser dynamics is chaotic. We have performed the numerical simulations with a Runge-Kutta algorithm. The possibilities to stabilize and the transient signal duration depend significantly on initial conditions. Figure 1 shows the calculated signals when the feedback loop is activated: Fig. 1(a), in the case of the stabilization of the unstable T-periodic orbit ($\alpha = 0.033$), and Fig. 1(b), in the case of the unstable $2T$-periodic orbit ($\alpha = 0.1$). In both cases, the upper trace represents the CO_2 laser intensity and the lower trace, the correction applied to the system. We verify that, when the unstable orbit is stabilized, the magnitude of the correction signal vanishes below the numerical accuracy. The numerical study of the T-periodic orbit stability versus α and m reveals a great robustness of the DCF method. For any value of m, in the investigated domain (0 to 0.3 corresponding to an unstable Floquet multiplier going up to -7.33), there exists a value of α beyond which the T-periodic orbit is stabilized. In addition, by increasing α up to 10, we have not detected an upper limit value leading to destabilization. Figure 2 represents the stability domain for the values of m usually reached experimentally. At the boundary, two types of bifurcations are observed : a Hopf bifurcation for the small values of m (before the angular point at $m = 0.0134$) and a subharmonic one for $m > 0.0134$. One can remark that a single value of the feedback gain α leads to the stabilization for a large range of the modulation index m. This fact allows us to keep α constant and to stabilize a particular unstable orbit, even when a control parameter (here m) is swept in a wide range. Therefore it is possible to track an unstable orbit not only when it is embbeded in the chaotic attractor but also when it is not.

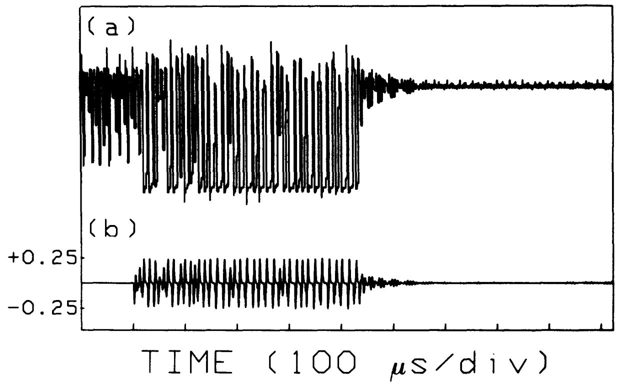

FIG. 3. Stabilization of the unstable T-periodic orbit: (a) periodic sampling of the laser power in arbitrary units. (b) Correction applied to the modulator in units of the modulation index $[p(t)/m]$.

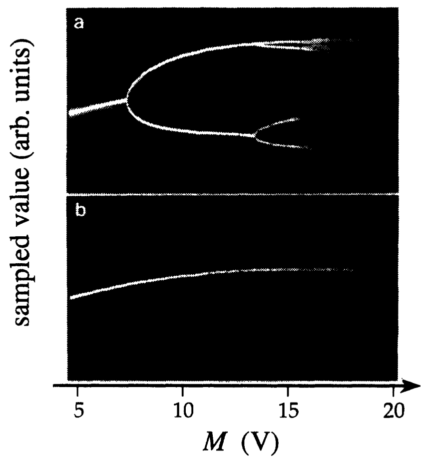

FIG. 4. Bifurcation diagram without (a) and with (b) stabilization of the unstable T-periodic orbit. The stable T-periodic orbit is destabilized for a modulation index of $M = 7.2$ V and the unstable orbit is embedded in the chaotic attractor for $M > 16.8$ V.

After the numerical simulations, the DCF method has been checked experimentally on a CO_2 laser. The experimental setup [13] consists essentially of a sealed-off waveguide containing the active medium, placed inside a cavity in which an electro-optic modulator is inserted. The control parameter is the modulation index applied to the modulator near its mechanical resonance frequency ($f = 365$ kHz). The laser power is monitored by an Hg-Cd-Te photovoltaic detector and its output is used to modulate a laser diode emitting at 845 nm. Thus the time delay (2.74 μs) is obtained by propagating the laser diode light in a 600-m-long fiber. The difference $x(t) - x(t-T)$ is directly obtained by two silicium photodiodes connected top to bottom. The difference, amplified with an adjustable gain, is added to the modulation signal [14].

Stabilization on the unstable T-periodic orbit has been achieved with the DCF method. A typical transient observed when the feedback control is switched on is represented in Fig. 3 [compare with Fig. 1(a)]. When the trajectory lies on the T-unstable cycle, the correction applied to the system decreases and amounts to a very small value (less than 2% of the modulation index). The remaining fluctuations are the superposition of two components with the same order of magnitude. The first is a random signal which corrects only the noise in the system. The second is a residual T-periodic modulation which arises from an imperfection of the delay line and could be suppressed by using a monomode fiber. It is essential to note that the obtained magnitude of this spurious signal does not represent a fundamental limit, and may be decreased by improving the feedback device. More precisely, because the remaining signal is close to a sinusoidal one, its effect is to slightly shift the modulation index (less than 2% of m). One can therefore conclude that the stabilized orbit is identical to the unstable T-periodic orbit existing in the original system, for a modulation index which is negligibly different from m. In the experimental situation, the technical limitation of the correction magnitude leads to transients longer than the experimental ones. However, this does not prevent stabilization for any initial conditions. As in the numerical simulations, one can follow the unstable orbit when a control parameter is slowly varied without modifying any parameter of the feedback system (Fig. 4). Moreover, we have checked that, all along the sweep, the correction applied to the system remains small, showing that the stabilized orbit is identical to the unstable one existing without stabilization. So, it is also possible to characterize the unstable T-periodic orbit in a large domain (whether or not it is embedded in a chaotic attractor) and to follow it down to the T-$2T$ bifurcation where this orbit becomes stable. In this way, we show experimentally that the unstable T-periodic orbit embedded in the chaotic attractor after the $C2$-C transition of the inverse cascade [15] comes from the destabilization of the T-stable orbit through the T-$2T$ bifurcation.

To conclude, we have proposed here an alternative method to suppress chaos in a dynamical system by locking it to unstable orbits, using continuous feedback. This method, which appears to be efficient in the case of a CO_2 laser with modulated losses, has allowed us to track and characterize an unstable orbit in a large domain of control parameters. This method is of practical interest for the stabilization of numerous fast systems for which the usual methods of control are hardly applicable. The DCF method requires, *a priori*, the knowledge of the unstable orbit periodicity and may be easily applied to the nonautonomous system in which this period is fixed by the forcing term. However, it is also possible to extend its efficiency to autonomous systems, even with swept parameters, if in addition, we make use of a predictor-corrector procedure for the adjustment of the delay time [9].

P. Niay and M. Douay are gratefully acknowledged for their collaboration in the experiments. We thank D. J. Gauthier for calling our attention to Pyragas's works. The Laboratoire de Spectroscopie Hertzienne is "associé au CNRS No. 249."

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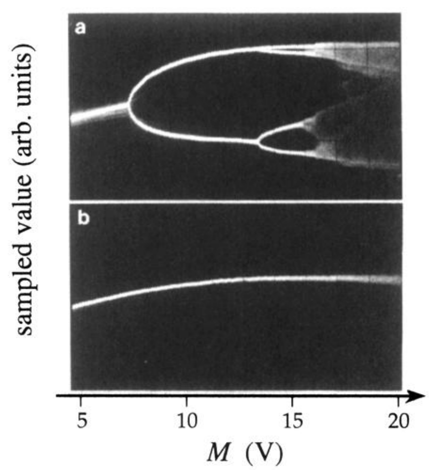

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