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Sign-singular measure and its association with turbulent scalings

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Turbulent quantities such as vorticity, which oscillate in sign on very fine scales, have recently been characterized by sign-singular measures [E. Ott, Y. Du, K. R. Sreenivasan, A. Juneja, and A. K. Suri, Phys. Rev. Lett. **69**, 2654 (1992)] and quantified by the so-called cancellation exponent. Here, the connection between the cancellation exponent and other known exponents for velocity structure functions and multifractal spectrum of the energy dissipation field is discussed. Comparison with high-Reynolds-number experimental data in one dimension and direct measurements of vorticity in a plane in moderate-Reynolds-number flows reveals excellent internal consistency. Estimates for second-order cancellation exponent are presented.

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Recently, a quantity called the cancellation exponent has been introduced to describe the tendency for vorticity field in high-Reynolds-number fluid turbulence (or magnetic field at high magnetic Reynolds numbers) to oscillate in sign on very fine scales [1]. This leads to the concept of *sign-singular measures*, which are introduced in analogy to multifractal probability measures. Consider a measure μ_s of a finite interval X. Let $A \subset X$ such that $\mu_s(A) \neq 0$. The measure μ_s is said to be sign singular if, for any such interval A, there is an interval $B, B \subset A$, such that $\mu_s(A)\mu_s(B) < 0$.

To characterize sign-singular measures quantitatively, the cancellation exponent has been introduced [1,2]. In particular, for high-Reynolds-number fluid turbulence,

$$\chi_s(r) = \sum_i \frac{|\int_{C_i} \boldsymbol{\omega}\, d\mathbf{x}|}{|\int_V \boldsymbol{\omega}\, d\mathbf{x}|}, \tag{1}$$

where the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$, $\mathbf{v}$ being the velocity field. The domain V is divided into a grid of cubes C_i of edge length r. The cancellation index is defined as

$$\kappa = \lim_{r\to 0} \frac{\ln \chi_s(r)}{\ln 1/r}. \tag{2}$$

Consider local mean value of the vorticity

$$\omega_i = \frac{1}{r^D}\left|\int_{C_i} \boldsymbol{\omega}\, d\mathbf{x}\right|, \tag{3}$$

in D-dimensional space, and its global average

$$\langle\omega(r)\rangle = \frac{\sum_i \omega_i}{N} = \chi_s(r)\left|\int_V \boldsymbol{\omega}\, d\mathbf{r}\right|, \tag{4}$$

where the number of the r-sized cubes $N = 1/r^D$ (assuming $V = 1$), in complete analogy with the generalized dimension formalism [3].

For simplicity, we start with the one-dimensional model, $D = 1$, for which the vorticity $\omega = \partial_x v$, v being the "velocity." Then, $|\omega_i|$ is simply $|v(x+r) - v(x)|/r$, and therefore, according to Eq. (4),

$$\langle\omega(r)\rangle = \frac{\langle|v(x+r) - v(x)|\rangle}{r}. \tag{5}$$

The numerator of (5) represents a *structure function* of a random process (cf. comment [16] in [1]).

For fully developed turbulence, we expect $\langle|v(x+r) - v(x)|\rangle \sim r^{\alpha}$; for Kolmogorov turbulence, $\alpha = 1/3$. Therefore

$$\langle\omega(r)\rangle \sim \frac{1}{r^{1-\alpha}} = \frac{1}{r^{\beta_1}}, \tag{6}$$

which corresponds to the scaling of the vorticity field. For Kolmogorov turbulence, $\beta_1 = 2/3$. From Eqs. (2), (4), and (6), one has

$$\beta_1 = \kappa. \tag{7}$$

This result can be easily understood. For Kolmogorov turbulence, the vorticity scales as $\omega(r) = \omega(l)(l/r)^{2/3}$, where l is the size of the energy containing eddies. Therefore the contribution to the integral (3) by large eddies, with scale $r' > r$, is $\sim (l/r')^{2/3} r$, while that of small eddies, $r' < r$, is $\sim (l/r')^{2/3} r'$. Thus the main contribution comes from "resonant" vortices $r' \sim r$, so that the integral $\sim (l/r)^{2/3} r$, and $\omega_i \sim 1/r^{2/3}$ (recalling that $D = 1$). Now, Eq. (4) simply averages this expression, resulting in Eq. (6).

These expressions can be generalized for the three-dimensional case. Indeed, now the contribution of large eddies is $\sim (l/r')^{2/3} r^3$, while that from small eddies is $\sim (l/r')^{2/3} r'^3$. Then, the contribution comes again from

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resonant vortices, resulting in Eq. (6). Note that the contribution of small eddies can be larger in pathological cases, such as when vorticity lines are parallel to the box sides. Suppose, for the "worst" case, that the vorticity lines are lying in the y-z plane. Then, the integration over y and z in Eq. (3) is trivial. It is easy to see that the contribution of small eddies is $\sim (l/r')^{2/3}r'r^2$, with the same conclusion.

Definition (2) applies in the singular limit as the Reynolds number Re$\to\infty$. For any large but finite Re, the small scale variation is limited at r_* (Kolmogorov scale). The definitions (1) and (2) have a meaning in the scaling range $l \gg r \gtrsim r_*$, which corresponds to the inertial range. This gives us another means for verifying Eq. (7) and estimating κ [4].

When $r \lesssim r_*$, we have $|\int_{C_i}\boldsymbol{\omega}\,d\mathbf{x}| \approx \int_{C_i}|\boldsymbol{\omega}|d\mathbf{r}$ which, in combination with Eq. (6), gives $\chi_s \approx \langle|\boldsymbol{\omega}|\rangle/\omega(l) = (l/r_*)^{\beta_1}$. Thus, for Kolmogorov turbulence, $\langle|\boldsymbol{\omega}|\rangle/\omega(l) = \mathrm{Re}^{1/2}$ [5]; further, $r_* \sim \mathrm{Re}^{-3/4}$, so that $\chi_s \sim r_*^{-2/3}$, yielding $\kappa = 2/3$ as before.

The three-dimensional analog of the structure function form (5) can also be constructed. Indeed,

$$\frac{1}{r^3}\int_{C_i}\boldsymbol{\omega}\,d\mathbf{x} = \frac{1}{r^2}\oint\{\bar{\mathbf{v}}\cdot d\mathbf{s}\}_{\mathbf{x}}, \tag{8}$$

where the contour integral is taken on the surface of the r cube, and $\{\}_{\mathbf{x}}$ means three components and the contour lies on a plane normal to the coordinate $\mathbf{x}$ (e.g., the x component corresponds to the $x=$const plane, etc.). The overbar corresponds to the averaging along the coordinate, say, for the x component, $\bar{\mathbf{v}} = (1/r)\int_x^{x+r}\mathbf{v}\,dx$.

To get a better understanding of Eq. (8), let us write down its x component, and average

$$\left\langle\frac{1}{r^3}\left|\int_{C_i}\omega_x d\mathbf{x}\right|\right\rangle = \left\langle\frac{1}{r^3}\left|\int_{C_i}(\partial_y v_z - \partial_z v_y)d\mathbf{x}\right|\right\rangle = \frac{1}{r}\langle|\overline{v_z(y+r)-v_z(y)}+\overline{v_y(z)-v_y(z+r)}|\rangle$$
$$= \frac{2}{r}\langle|\overline{v_i(\mathbf{x}+\mathbf{r}_\perp)}-\overline{v_i(\mathbf{x})}|\rangle, \tag{9}$$

where, in addition to the x averaging, the overbar corresponds to the average parallel to the velocity component (i.e., z average for v_z and y average for v_y). In the last inequality yet another mean value has been introduced, averaging over all directions lying in the $x=$const plane. The vector $\mathbf{r}_\perp$ points in one such direction. The final expression is analogous to the structure function because of the presumed isotropy of the process.

We now specify random fields that can be treated with a cancellation index. A process with $\Delta v \sim r^\alpha$ has the energy spectrum $E(k) \sim k^{-2\alpha-1}(\sim k^{-5/3}$ for Kolmogorov turbulence). We may indicate by Kolmogorov-type turbulence any isotropic random process with converging energy ($\int E(k)dk < \infty$) and diverging vorticity [$\int E(k)k^2dk \to \infty$ as Re$\to\infty$]. It is then clear that α should satisfy

$$0 < \alpha < 1, \tag{10a}$$

and, for the vorticity field itself, that

$$0 < \beta_1 < 1. \tag{10b}$$

It is the property (10b) of β_1 that makes the cancellation index of the vorticity field an interesting quantity to measure. Indeed, if $\alpha > 1$, i.e., $\beta_1 = 1-\alpha < 0$, then the main contribution to the integral in Eq. (3) would come from large eddies, and $\kappa = 0$, independent of β_1 [cf. Eq. (7)]. If, on the other hand, $\alpha < 0$, then small eddies contribute, acting like a noise, at least for the one-dimensional section of the random process, typical for the laboratory signal [6]. In such a case, $\kappa = 1$, again independent of β_1.

Finally, if the vector-potential structure functions satisfy condition (10a), i.e., magnetic field $\mathbf{B}$ behaves like $|\mathbf{B}| \sim 1/r^{\beta_1}$, where β_1 satisfies Eq. (10b), the cancellation index κ is directly related to the spectrum exponent, according to Eq. (7).

We saw that the index κ corresponds to the first-order structure function [see Eq. (5) or (9)]. The second-order structure function is similarly related to

$$\langle\omega(r)^2\rangle = \frac{\sum_i\omega_i^2}{N} \sim \frac{1}{r^{\beta_2}}; \tag{11}$$

cf. (6). Clearly, in analogy with Eqs. (5) and (9),

$$\langle\omega(r)^2\rangle \sim \frac{\langle|v(x+r)-v(x)|^2\rangle}{r^2}$$

or

$$\langle\omega(r)^2\rangle \sim \left(\frac{2}{r}\right)^2\langle|\overline{v_i(\mathbf{x}+\mathbf{r}_\perp)}-\overline{v_i(\mathbf{x})}|^2\rangle. \tag{12}$$

The question now is the relation between the exponent β_2 and other known quantities. Obviously, if there is no intermittency, then $\langle\omega(r)^2\rangle \sim \langle\omega(r)\rangle^2$, and $\beta_2 = 2\kappa$. In general, this will not be true.

The probability measure has been defined [7–9]

$$\mu_\omega(C_i) = \frac{\int_{C_i}|\boldsymbol{\omega}|d\mathbf{x}}{\int_V|\boldsymbol{\omega}|d\mathbf{x}}, \quad \mu(C_i) = \frac{\int_{C_i}\epsilon\,d\mathbf{x}}{\int_V\epsilon\,d\mathbf{x}}. \tag{13}$$

Here ϵ means the energy dissipation rate, as usual. We also use $\chi(r) = \sum_i\mu_\omega(C_i)^2$ and a local mean value

$$|\boldsymbol{\omega}|_i = \frac{1}{r^D}\int_{C_i}|\boldsymbol{\omega}|d\mathbf{x}. \tag{14}$$

Now,

$$\langle|\boldsymbol{\omega}(r)|^2\rangle = \frac{\sum_i |\boldsymbol{\omega}|_i^2}{N} = \frac{\chi(r)}{r^D}\left(\int_V |\boldsymbol{\omega}|d\mathbf{x}\right)^2 \sim \left(\frac{r}{l}\right)^{D_2^{(\omega)}-D}\left(\int_V |\boldsymbol{\omega}|d\mathbf{x}\right)^2, \tag{15a}$$

and

$$\langle\epsilon(r)^q\rangle \sim r^{-(D-D_q)(q-1)}. \tag{15b}$$

The dimensions $D_p^{(\omega)}$ and D_q are based on different measures (13).

According to [10] the structure functions are related to μ measure:

$$\langle|v(x+r)-v(x)|^p\rangle \sim r^{(p/3-1)D_{p/3}+1}. \tag{16}$$

The measure here corresponds to a linear section of the process, and so, in order to match with this formula we put $D=1$ hereafter. We can then compare Eq. (16) with Eqs. (5) and (9). Taking into account Eqs. (6) and (7) we get

$$\kappa = \frac{2}{3}D_{1/3} \quad \left(= \frac{2}{3}(D_{1/3}^{(3)}-2)\right). \tag{17}$$

Here $D_q^{(3)}$ corresponds to three-dimensional measurements ($D_q^{(3)} = D_q + 2$, see [10,11]).

The scaling exponents in Eq. (16) have been measured by several independent investigators [12], all of which are in reasonable agreement with each other. In particular [10], $D_{1/3} = 0.96$, so that, according to Eq. (17), one has $\kappa = 0.64$. If we invoke Taylor's hypothesis and relate the cancellation exponent of $\Delta v/\Delta t$ in high-Reynolds-number turbulence to that of the vorticity [see Eq. (5)], this estimate is identical to the measurement in [1].

For $p=2$, we compare with the second-order structure function, Eq. (12). It follows from Eq. (11) that

$$\beta_2 = 1 + \frac{1}{3}D_{2/3}. \tag{18}$$

From the knowledge that $D_{2/3} = 0.92$ [11], $\beta_2 = 1.31$.

Consider $\langle\omega(r)^2\rangle$ at the Kolmogorov scale $r = r_*$. It follows from Eq. (15a) that

$$\langle\omega(r_*)^2\rangle \sim \langle|\boldsymbol{\omega}|^2\rangle \sim \omega(l)^2\left(\frac{r_*}{l}\right)^{-1-D_{2/3}1/3}. \tag{19a}$$

We also have, according to Eqs. (16) and (17),

$$\langle\omega(r_*)\rangle \sim \langle|\boldsymbol{\omega}|\rangle \sim \omega(l)\left(\frac{r_*}{l}\right)^{-D_{1/3}2/3}. \tag{19b}$$

Noting that in addition to Eq. (19a), $\langle\omega(r_*)^2\rangle \sim \langle|\boldsymbol{\omega}(r_*)|^2\rangle$, and $\int_V|\boldsymbol{\omega}|d\mathbf{x} = \langle|\boldsymbol{\omega}|\rangle$, it is possible to compare Eqs. (19) and (15) to give

$$1 - D_2^{(\omega)} = 1 + \frac{1}{3}D_{2/3} - \frac{4}{3}D_{1/3}$$

or (20)

$$D_2^{(\omega)} = \frac{4}{3}D_{1/3} - \frac{1}{3}D_{2/3}.$$

Finally, eliminating $D_{2/3}$ with the help of Eq. (20) and $D_{1/3}$ with Eq. (17), we may write Eq. (18) as

$$\beta_2 = 1 - D_2^{(\omega)} + 2\kappa, \tag{21a}$$

or, for D-dimensional fields, as

$$\beta_2 = D - D_2^{(D,\omega)} + 2\kappa. \tag{21b}$$

For nonintermittent turbulence, $D_2^{(\omega)} = 1$, and β_2 indeed equals 2κ, as already mentioned. Since the formula (21) contains quantities involving only the $\boldsymbol{\omega}$ field, it is conceivable that β_2 can be obtained without involving the dimensions D_q for the energy dissipation. Indeed, suppose that there is *only one* scaling regime in the inertial range $l < r < r_*$. This implies that

$$\langle\omega(r)^2\rangle = \omega(l)^2\left(\frac{l}{r}\right)^{\beta_2}. \tag{22}$$

Since $\langle\omega(r_*)^2\rangle \approx \langle|\boldsymbol{\omega}|^2\rangle$, using the previous results that $|\int_V \boldsymbol{\omega}\, d\mathbf{x}| \approx \omega(l)$ and $\langle|\boldsymbol{\omega}|\rangle = \int_V |\boldsymbol{\omega}|d\mathbf{x} \approx \chi_s(r_*)\omega(l)$, we get from Eq. (15) (at $r = r_*$)

$$\langle|\boldsymbol{\omega}|^2\rangle = \omega(l)^2\left(\frac{r_*}{l}\right)^{D_2-D}\left(\frac{r_*}{l}\right)^{-2\kappa}. \tag{23}$$

This formula (in terms of magnetic fields) has been obtained in Ref. [13]. Comparing Eq. (23) with Eq. (22), we recover Eq. (21) [and now, backwards, from Eqs. (21), (17) and (18) would follow Eq. (20)]. On the other hand, the formula (21) can be related to the *spectrum of cancellation exponents* introduced in [14], where κ_2 is defined from Eq. (2) with $\chi_s^{(2)} \sim \sum_i |\int_{C_i} \boldsymbol{\omega}\, d\mathbf{x}|^2$. Combining this with the definition (11) it easy to see that $\beta_2 = D + \kappa_2$. Indeed, substitution of this expression into the formula (55) of Ref. [14] with $q=2$ gives Eq. (21).

In order to obtain an estimation of β_2, according to Eq. (21) and independent of Eq. (18), we use the relationship between $D_q^{(\omega)}$ and $D_p^{(\omega^2)}$, namely,

$$(q-1)D_q^{(\omega^2)} = (2q-1)D_{2q}^{(\omega)} - qD_2^{(\omega)}, \tag{24}$$

see [15]. This formula is obtained by expressing all the quantities involved at Kolmogorov cutoff scale $r = r_*$. Putting $q = 1/2$, we have

$$D_2^{(\omega)} = D_{1/2}^{(\omega^2)}. \tag{25}$$

Now, from experimental data, $D_{1/2}^{(\omega^2)} = 0.94$ [9], and substitution of $\kappa = 0.64$ into (21) results in $\beta_2 = 1.34$, in good agreement with the estimate obtained from Eq. (18). Therefore, from Eq. (21a), the second-order cancellation exponent $\kappa_2 = 0.34$.

Expression (21) also holds for magnetic fields, but formulas (17), (18), and (20) *do not*. The point is that the vector-potential structure functions are not related to the energy dissipation fluctuation, unlike (16).

While there is thus good internal consistency in high-Reynolds-number measurements, it should be emphasized that the measurements were made at a single point in space, which were interpreted as one-dimensional cuts by involving Taylor's frozen flow hypothesis. It is conceivable that one-dimensional cuts can miss rare events and effectively one-dimensional objects such as vortex filaments. It would therefore be far better to measure vorticity directly, at least one component of it. Vorticity measurements in a plane have been made in the wake of a circular cylinder using particle image velocimetry. The cancellation exponent for one-dimensional cuts of these vorticity measurements have already been obtained in Ref. [1]; the scaling was quite unambiguous, and the first-order cancellation exponent had a value of 0.45. We now obtain the cancellation exponent for a component of vorticity in a *plane*, make consistency checks and obtain an estimate for the second-order cancellation exponent. We summarize the results here while relegating experimental details to a later publication.

As already remarked, the flow was the turbulent wake behind a circular cylinder. The Reynolds numbers based on the cylinder diameter and the oncoming uniform velocity were 1100 and 4500. Measurements were made in a water tunnel at a distance of 50 diameters downstream of the cylinder. The vorticity component ω_y in the x-z plane, where x is the direction of the mean flow and z along the length of the cylinder, was estimated from the velocity field obtained from particle image velocimetry. From scaling experiments, it was determined that the cancellation exponent was 0.84, and that the exponent β_2 was 1.74. The scaling was unambiguous in both cases. Substitution of these values into Eq. (21b) results, for the two-dimensional case, in $D - D_2^{(\omega)} = 0.06$, which shows small effects of intermittency. Equation (25) then yields a $D_{1/2}^{(\omega^2)}$ of 0.94, in excellent agreement with the measurements of Ref. [9]. Further, from the relation $\beta_2 = D + \kappa_2$, we obtain $\kappa_2 = 0.26$. Recall that κ_2 for the different conditions of high Reynolds numbers was 0.34.

In conclusion, we have shown that the sign-singular measure is relevant to turbulent vorticity and magnetic fields. It is also directly associated with the generalized dimension of dissipation and (vorticity)^2. This observation makes it possible, among other things, to compare theoretical expressions with experimental data, and make predictions about high-order cancellation exponents. The agreement is very good. We have explicitly considered the second-order cancellation exponent, and provided estimates from measurements along a line and in a plane.

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