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Self-organized criticality in the "game of Life"

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The question of saturation of self-organized criticality in the "game of Life" is analyzed through finite-size scaling. We have carried out connection-machine simulations for lattices up to 1024×1024. In that range we find no evidence of saturation. We do, however, find finite-size exponents that seem inconsistent with a single length-scale picture at criticality.

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The evidence of self-organized criticality in the "game of Life" [1,2] presented by Bak, Chen, and Creutz [3] has been questioned by Bennett and Bourzutschky [4], who claim that the criticality is an artifact resulting from small lattices. They find that for sizes larger than about $L \times L = 100 \times 100$, the average "equilibration time" saturates at $\langle t \rangle = 200 \pm 10$. We have carried out extensive connection-machine studies for lattices up to 1024×1024. Our data are consistent with those found by Bak, Chen, and Creutz, although we estimate the scaling exponents to be about 10% smaller than theirs. In contrast to Bennett and Bourzutschky, we do not find saturation for L up to 1024. However, we do observe a crossover in the finite-size scaling, and our finite-size analysis gives exponents that seem inconsistent with simple theoretical arguments.

The game of Life is defined on a square lattice, where every site is a live site or a dead site. The evolutionary rules for the sites are simple and based on the eighth nearest neighbors: (1) Each live site will remain alive the next time step if it has two or three live neighbors, otherwise it will die. (2) At a dead site new life will be born only if there are exactly three live neighbors. Initiating the game of Life from some random configuration generally leads to a "rest" state with about 3% live sites, partly consisting of simple cyclic life configurations. Our simulations were carried out using open boundary conditions, and the state was identified as being at rest when it was identical to the state observed two or six time steps earlier (longer periods never occurred in our simulations). To test for self-organized criticality, the rest state is perturbed by flipping a single, randomly selected dead site to life. Since the density of life is low, the state often returns to the same rest state in one time step. To reduce the number of these "flip fails" (and thereby the computer time), we disregard the dead sites for which the 20 nearest neighbor sites are also dead, then flipping a dead site randomly selected among the rest. In this way we reduce the number of flip fails to 22%.

The evolutionary change caused by a single-site perturbation is called an avalanche. The liftetime of an avalanche is the number of time steps before the state again returns to a rest state. The size of an avalanche is the space- and time-sum of the number of live sites that

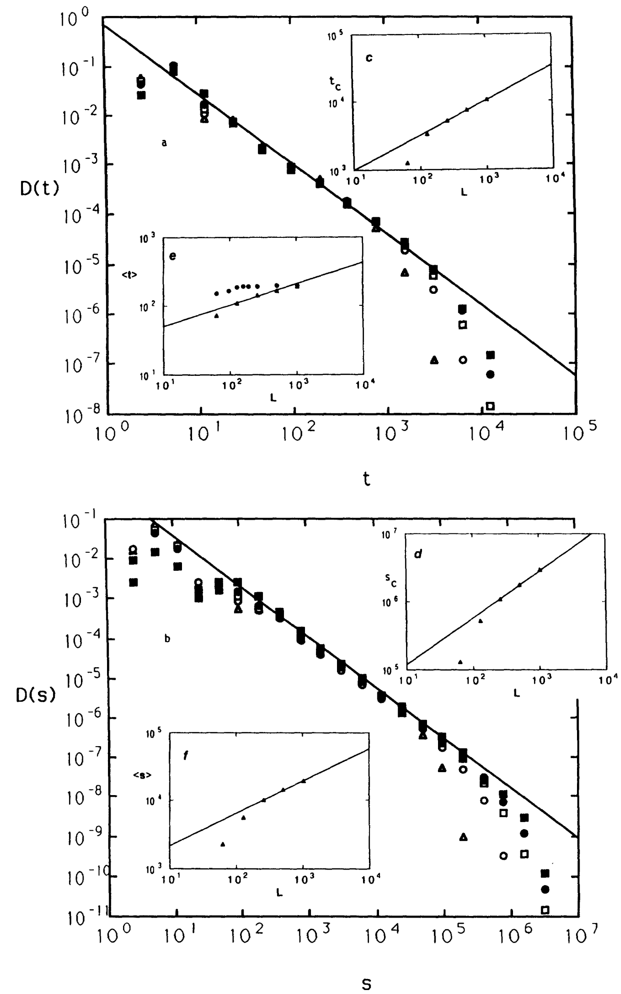

FIG. 1. (a) and (b) Log-log plots of the normalized distributions of lifetimes t and sizes s for avalanches caused by single-site perturbations in the game of Life, simulated on $L \times L$ lattices with $L = 64$ (△), $L = 128$ (○), $L = 256$ (□), $L = 512$ (●), and $L = 1024$ (■). The fits have slopes 1.41 and 1.27. (c) and (d) Log-log plot of the critical lifetime t_c and size s_c versus L. The fits have slopes 0.52 and 0.69. (e) and (f) Log-log plot of the average lifetime $\langle t \rangle$ and size $\langle s \rangle$ versus L. The fits have slopes 0.31 and 0.48. The dots are from Ref. [4].

were not alive two time steps earlier. For the state to be self-organized critical the distribution of lifetimes and sizes must be power law distributions with finite-size scaling properties. It is important that the distributions are found from repeated random single-site perturbations; in contrast, e.g., the distribution of decay times to a rest state from random initial conditions may not follow a power law.

The normalized distributions $D(t)$ and $D(s)$ of avalanche lifetimes and sizes (> 1), found for different lattice sizes, are shown in Figs. 1(a) and 1(b). The total number of time steps for each lattice size was between 1 000 000 and 5 000 000. In agreement with the original work of Bak, Chen, and Creutz we find that the distributions follow power laws (over three orders of magnitude). $D(t) \propto t^{-b}$ and $D(s) \propto s^{-\tau}$, with $b = 1.4$, and $\tau = 1.3$. However, corrections to scaling as well as an exponential finite-size cutoff (variations from the power law) are observed. We point out that both are real reproducible features. For size 64×64 the rest state with all sites dead is obtained after $\sim$1 000 000 timesteps. We do not find any evidence of transient states, e.g., for $L = 512$ the distributions obtained between 100 000 and 2 000 000 time steps are identical to those obtained between 2 000 000 and 5 000 000 time steps. Anyhow, we always disregarded the first 100 000 time steps.

Based on our data, finite-size analysis was performed, assuming that the distributions follow the scaling forms $D(t) = t^{-b}f(t/t_c)$ and $D(s) = s^{-\tau}g(s/s_c)$, where f and g are scaling functions, and $t_c \propto L^z$ and $s_c \propto L^\delta$. The resulting values of t_c and s_c are shown in Figs. 1(c) and 1(d). There seems to be a crossover at $L \simeq 100$ to finite-size scaling with $z = 0.5$ and $\delta = 0.7$. The exponents z and δ are, however, determined within one decade only, which calls for caution. Theoretically, the fact that $z < 1$ and $\delta < 1$ implies that for sufficient large L, $t_c \ll L$ and $s_c \ll L$, i.e., an avalanche with the critical lifetime t_c and critical size s_c will necessarily extend over only a small part of the lattice, hence t_c and s_c are not related with an avalanche of the size of the lattice. This seems to be inconsistent with the usual single length-scale picture at criticality, and we urge further studies in this direction.

We have in Figs. 1(e) and 1(f) shown the computed average values $\langle t\rangle$ and $\langle s\rangle$. In agreement with the scaling properties above, we find $\langle t\rangle \propto L^{(2-b)z}$ and $\langle s\rangle \propto L^{(2-\tau)\delta}$, where $(2-b)z \simeq 0.3$ and $(2-\tau)\delta \simeq 0.5$. The small value for the $\langle t\rangle$ exponent implies that effects from small avalanches and corrections to scaling may be severe, and we conclude that $\langle t\rangle$ is not a good parameter to test criticality; one should rather consider higher moments. Contrary to the results for $\langle t\rangle$ by Bennett and Bourzutschky, we do not find saturation. Their averages disregard lifetimes less than 7. Taking this into account our values of $\langle t\rangle$ only change by a number of order unity. A change from open to periodic boundary conditions may, however, change $\langle t\rangle$ by more. Further studies on this point seem needed.

In summary, finite-size scaling does not show saturation of self-organized criticality in the game of Life on lattices up to 1024×1024. The low values of the finite-size exponents are, on the other hand, not consistent with the usual single length-scale picture at criticality.

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