Static parametric fluctuations give nonstatistical behavior in uncoupled chaotic systems

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We show that an ensemble of logistic maps, with parameters distributed in some range inside a chaotic region, cannot be statistical. This is so because any parameter range for this map includes periodic windows. The effects of this periodicity in the averages of the system can be estimated, and are also apparent in the power spectra of average values. As a counterexample, we show that the behavior for tent maps, where chaotic regions do not include periodic windows, is statistical.

PACS number(s): 05.45.+b, 05.90.+m

I. INTRODUCTION

It was shown in a recent work by Sinha [1] that independent chaotic maps under the influence of global noise behave in a nonstatistical manner. In particular, it was found that when a collection of logistic maps in their chaotic regime has time-dependent but spatially homogeneous fluctuations in its nonlinear parameter, the average value of x shows persistent fluctuations even in the large-N limit. This behavior is similar to that of an ensemble of chaotic maps under weak global coupling, a problem that has been studied in several recent works [2–5].

It was also claimed in that work that when the applied noise is static but space dependent, the behavior of this uncoupled chaotic system is statistical. This means that, for instance, if we have a large collection of logistic maps with their parameters distributed in some narrow range in the chaotic region, the average value of x should converge to some fixed value with fluctuations that die out as $1/\sqrt{N}$. At first sight, this seems quite plausible, since this just reflects the statistical behavior one expects from a collection of independent chaotic oscillators, where each and every one of them is characterized by an invariant probability distribution with finite support. But upon more careful examination one has to realize that there is a failure in this reasoning. This failure lies in the fact that the statistical superposition mentioned before works only if all the mappings included in the parameter region of interest are purely chaotic. By "purely chaotic" we mean here the absence of any periodic behavior, i.e., we exclude cases of the type known as "periodic chaos" [6], where the motion covers in a periodic way a finite collection of distinct chaotic windows.

But it is well known [7] that this is impossible to do with any smooth distribution of parameters in the logistic map, since there is at least a periodic window between any two different points in parameter space where the map is chaotic, and this periodic window (or windows) represents a nonzero fraction of the parameter range.

Therefore, as we increase the size of the lattice on which we are working, we will be at the same time maintaining some fraction of the elements moving in a periodic way. And this part of the system will spoil the convergence of the average, by keeping persistent fluctuations whose origin is simply the periodicity of the map in these windows.

In this paper we show that the intrusion of periodic windows does alter the statistical properties of a collection of chaotic maps whose parameters are distributed. We also show that in typical cases this effect is extremely small, thus explaining the results reported in Ref. [1]. We show how this effect depends on the initial configuration of the lattice, and how the effect of periodic windows on the fluctuations of the average can be estimated.

II. UNCOUPLED LOGISTIC MAPS WITH PARAMETRIC NOISE

The model used for this work is

$$x_{n+1}^i = 1 - a^i (x_n^i)^2 , \qquad (2.1)$$

where i is the space index and n is the time index. The values of the parameter a are given by

$$a^i = a + \sigma \epsilon^i , \qquad (2.2)$$

where ϵ^i is a random number uniformly distributed between -1 and 1, σ is the (small) amplitude of the parameter fluctuations, and a is just the center of the distribution. This is denoted as "case (iii)" in Ref. [1], where a slightly different prescription $a^i = a(1 + \sigma' \epsilon^i)$ was used. It is clear that these two prescriptions are identical if one makes $\sigma = a\sigma'$.

In order to test whether or not this system is statistical we calculate the instantaneous mean value h_n of the variable x_n^i, over large size lattices, and study the time evolution of this average. In particular, we check its mean-square deviation (MSD), which is defined by $\langle (\langle h \rangle - h)^2 \rangle$, where the angular brackets are time averages. We also check its power spectrum, which for a su-

perposition of purely chaotic systems should be broad.

A comment should be made here about the meaning of these averages. What we want to know here is whether a single lattice, made out of many elements, can be statistical, in the sense that averages over those elements obey the central limit theorem and the law of large numbers. We are not considering the different problem of an ensemble of lattices [8], which depending on the conditions of the problem may or may not be statistical.

Before going over the numerical results, let us try to give an estimate of the size we can expect these effects to have. For this, we can do the following approximation: we can separate the average h_n into two parts. One comes from the points of the lattice where a falls in the purely chaotic region, and another comes from those points with a in a periodic window,

$$h_n = h_n^c + h_n^p \equiv \frac{1}{N} \sum_{a^i \text{ chaotic}} x_n^i + \frac{1}{N} \sum_{a^i \text{ periodic}} x_n^i . \qquad (2.3)$$

Now, the part that comes from chaotic a will converge towards some fixed value h^* in the infinite lattice limit, with uncorrelated fluctuations ξ_n. These fluctuations will have zero mean and a mean-square deviation that decays as $1/N$ (multiplied by some coefficient of order 1).

For the periodic part we will take into account only the largest periodic window, of periodicity K. At some arbitrarily chosen time $n=0$, after the transients have died, a fraction w_k will have been attracted to the kth point in the cycle, denoted x_k. It is clear that the value x_k changes along the window, and is not even well defined at its end, where the motion is over narrow chaotic strips. However, since for narrow periodic windows these changes are small, we will just approximate the whole interval, including the chaotic strips, by a single representative value of x_k.

With these approximations the value for the periodic part of the average is now

$$h_n^p = \Delta \sum_{k=1}^{K} w_k^n x_k , \qquad (2.4)$$

where $w_k^n = w_{(k-n) \bmod K}$ is the fraction of points in the lattice with value x_k at time n, and Δ is the relative width of the periodic window, assumed to be small. The time average of h is

$$\langle h \rangle = \langle h^c \rangle + \langle h^p \rangle = (1-\Delta)h^* + \frac{\Delta}{K} \sum_{k=1}^{K} x_k , \qquad (2.5)$$

where the $1/K$ comes from the time average of w_k^n. The mean-square deviation of h becomes

$$\langle (h - \langle h \rangle)^2 \rangle \approx \frac{1}{N} + \Delta^2 \left\langle \left[\sum_{k=1}^{K} \left[w_k - \frac{1}{K} \right] x_k \right]^2 \right\rangle \qquad (2.6)$$

$$\approx \frac{1}{N} + \Delta^2 \sum_{i,j=1}^{K} \left[\langle w_i w_j \rangle - \frac{1}{K^2} \right] x_i x_j , \qquad (2.7)$$

where we are taking $(1-\Delta)^2 \approx 1$. Notice that if one could choose the initial conditions for the lattice so as to cover equally the K basins of attraction of the map f^K, then all the w_k would be equal to $1/K$ and the second term would

be zero. Therefore we are considering here an effect that is strongly dependent on the distribution of the initial conditions. In general, if this distribution is homogeneous between some two values—not too close to each other—and covers a good part of the $(-1,1)$ range, there will be small but nonzero deviations from the $1/K$ mean value. This effect is the one that induces persistent fluctuations on the mean values for a lattice of logistic maps.

III. NUMERICAL RESULTS

A. Estimate of nonstatistical effects

To verify what have been said above, we have done a numerical estimate of the size of the effects one may expect in a simulation of a logistic map lattice, in order to see under which conditions we may expect to find them. The first limiting factor here is the relative width of the periodic window, which for typically small cases (say of order 10^{-3}) already makes the possible effects visible only for lattices of 10^6 points or more. Besides this, we also have to check what are the typical values for the fractions w_k for a uniformly distributed set of initial conditions. Our numerical results show that these fractions tend to deviate from the even value $1/K$ by a small amount—of the order of a few percent—for initial conditions with some bias (for instance, x_0^i chosen between 0 and 1), and even for initial conditions distributed homogeneously in the whole $(-1,1)$ range. This adds another factor of 10^{-3}–10^{-4} or smaller to our estimate for the saturation point of the MSD, and means that in typical cases one should not see any nonstatistical effects for lattices of less of 10^9–10^{10} points. This explains the null results found in Ref. [1], where lattices up to 10^4 points were used, and means that in order to see the nonstatistical behavior of these systems in smaller lattices one has to look for some specific conditions, in particular, a parameter range that includes small but still appreciable periodic windows.

In our simulations we have used the parameters $a=1.96$ and $\sigma=0.02$, which gives us a^i in the range 1.94–1.98. This range in parameter space includes a narrow four-window around $a=1.941$, which takes close to $\frac{1}{18}$ of the covered range. We have tested the saturation value given by Eq. (2.7) using an initial distribution with x_0^i between -0.5 and 0.5, which introduces some bias. The results obtained for three different points inside the window—one of them in its chaotic part—were consistent with each other, and the final estimate for the saturation point of the MSD is around 1.5×10^6. The actual values of w_k and x_k for the three tested points are given in Table I.

B. Actual simulation of the lattice

We have simulated the dynamics of this systems on lattices of sizes up to 633 960 points, with the same ranges of a and of initial conditions given above. The first 5000 iterations were discarded as a transient, and the statistics were collected over 50 runs of 1024 iterations each. Since

TABLE I. Values of x for the four cycle, and fractions of initial conditions attracted to them after 5000 iterations. The statistics were compiled over 50 runs, on lattices of 40 000 points. Here we show the results for three different values of a inside the four-window. Errors in the fractions w are all of order 2×10^{-4}.

a	x_1	x_2	x_3	x_4	w_1	w_2	w_3	w_4	Estimated MSD
1.9410	-0.9410	-0.7186	-0.0022	0.9999	0.2506	0.2376	0.2698	0.2420	1.1×10^{-6}
1.9415	-0.9412	-0.7199	-0.0062	0.9999	0.2500	0.2355	0.2740	0.2405	1.6×10^{-6}
1.9425	-0.9425	-0.7255	-0.0225	0.9990	0.2502	0.2315	0.2809	0.2374	2.7×10^{-6}

these effects are sensitive to fluctuations in the distribution of initial conditions, we have repeated the simulation four times, each with a different set of initial conditions. The results of this calculation are given in Fig. 1, which shows the beginning of the saturation of the MSD of h as N grows, and in Fig. 2, which shows the power spectrum of h.

In Fig. 1 we can see that for large n the MSD has deviated strongly for the $1/N$ behavior, and is clearly starting to saturate, with values that approach our previous estimate of 1.5×10^{-6} from above. The bars give the total spread obtained for the four repetitions, i.e., they go from the minimum to the maximum value obtained for the MSD. As a comparison (and control, in order to test that this effect is not just some roundoff effect from the computer), we are including the results from the same calculations performed on lattices of tent maps,

$$x_{n+1}^i = 1 - a^i |x_n^i| , \qquad (3.1)$$

using exactly the same parameters and run times as in the logistic case. This system is expected to show perfectly good statistical behavior in this case, because it does not have any periodic windows in the range of a considered; the tent map is purely chaotic for all values of a in the range 1.94–1.98. In the figure we see that the behavior of the MSD for tent map lattices is perfectly statistical, following the $1/N$ law. We have not included spread bars for these points since here the spreads are negligible.

In Fig. 2 we have plotted the power spectrum of h for both the logistic and the tent lattices, for $N = 633\,960$. The spikes corresponding to the frequencies $\frac{1}{2}$ and $\frac{1}{4}$ are quite evident in the spectrum for the logistic map. They arise from the periodic part of the mean field, h^p, and include contributions from the purely periodic part (fundamental and subharmonics), and the periodically chaotic [6] final part of the window. The noisy background is formed by the purely chaotic part of the mean field, h^c, which comes from most of the included range in a, with some contribution from the periodically chaotic segment. Notice that periodic chaos contributes to both types of spectrum; in fact, the typical power spectrum of this kind of motion is quite similar to that shown in Fig. 2. The origins of these two are different, however. In "periodic chaos" one single degree of freedom hops periodically from one chaotic window to another. In the present situation, we are thinking about the addition of many degrees of freedom. The average we perform in getting the mean field h reduces the power of the spectrum of h^c for nonzero frequencies, while maintaining the (relatively weak) component h^p.

We should mention that periodic spikes are visible even for much smaller lattices, and appear in power spectra well before saturation of the MSD. For comparison, we also plot the power spectrum for the mean value for a lattice of tent maps. It is evident that there is no periodicity at all in this case.

IV. CONCLUSIONS

We have shown that nonstatistical behavior appears for lattices of uncoupled chaotic maps when these maps are subject to static parametric fluctuations. This effect is due to the intrusion of periodic windows in the chaotic parameter sector, which makes it impossible to say that a given parameter range is purely chaotic. The magnitude
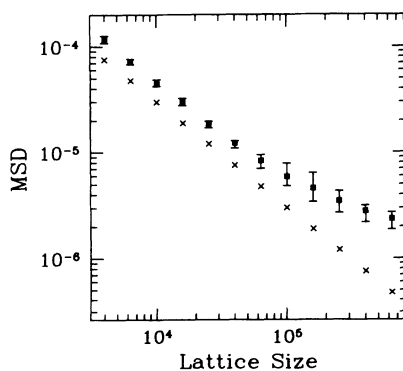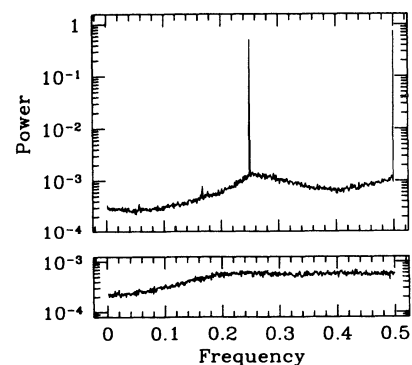

FIG. 1. Mean-square deviation for the lattice average h vs lattice size. The squares correspond to lattices of logistic maps, and the bars show the spread of the MSD. The expected saturation value for the MSD is around 1.5×10^{-6}. The crosses correspond to lattices of tent maps, whose MSD does not saturate.

FIG. 2. Power spectrum of the lattice average h for lattices of logistic maps (top), and tent maps (bottom). The vertical scale is the same for both figures.

of this effect can be calculated, and the results from actual simulations agree with other estimates. These effects disappear in cases where it is possible to set a parameter range where the evolution of the maps is purely chaotic, and in the case of the tent map. (Notice that tent maps were used in Ref. [8].)

This nonstatistical behavior is manifested in the saturation of the MSD of the average h as the lattice size N grows, and in the appearance of sharp spikes in the power spectrum of h. However, for most typical cases, the saturation values for the MSD of the average h are so small that they affect only extremely large lattices. Also, the effect is quite sensitive to the distribution one chooses for the initial conditions in the lattice. In principle, it may even be possible to produce a distribution of initial conditions that cancels the effects of at least the largest periodic window, and makes the system behave statistically for even larger sizes of the lattice.

The situation with the power spectrum is different. The spikes that signal periodicity in the lattice appear even for small lattice sizes, even though their behavior becomes consistent only as N grows. For large lattices, one can observe very clearly the effect of the periodic windows on the evolution of the average. The signals coming from the whole window contribute to this periodic effect, including the subharmonics and the periodic chaos. The noisy background comes mainly from the mappings outside the periodic window, with some contributions from the periodic chaos section inside.

This is a very simple model, whose nonstatistical behavior is easy to understand, so much so that it can be estimated beforehand. We believe, however, that there has to be a connection with the more complex but similar phenomena one finds in the case of globally coupled chaotic mappings. (A review is given in Ref. [9].) Both of them show saturation of the MSD, peaks in the power spectrum (broad in the coupled case), and in both cases the nonstatistical effects disappear for the continuously chaotic examples of the tent map. In the uncoupled case the explanation of this fact is simple; with no periodic windows one gets invariant distributions for any values of a, which gives finally simple statistical behavior. For the coupled case the connection between continuous (in parameter space) chaos and statistical behavior has been only postulated and discussed within a static approximation [10], but still the similarity between the two modes seems to imply a deeper connection.

[1] S. Sinha, Phys. Rev. Lett. **69**, 3306 (1992).

[2] K. Kaneko, Phys. Rev. Lett. **65**, 1391 (1990); Physica D **55**, 368 (1992).

[3] G. Perez, S. Sinha, and H. Cerdeira, Physica D **63**, 341 (1993); G. Perez, C. Pando-Lambruschini, S. Sinha, and H. A. Cerdeira, Phys. Rev. A **45**, 5469 (1992).

[4] S. Sinha, D. Biswas, M. Azam, and S. V. Lawande, Phys. Rev. A **46**, 6243 (1992); **46**, 3193 (1992).

[5] H. Chaté and P. Manneville, Europhys. Lett. **14**, 409 (1991); **17**, 291 (1992); J. Hemmingsson, Jülich Report No. HLRZ 85/91, 1991 (unpublished); J. A. C. Gallas, P. Grassberger, H. J. Herrmann, and P. Ueberholz, Physica A **180**, 19 (1992); P.-M. Binder, B. Buck, and V. A. Macaulay, J. Stat. Phys. **68**, (1992); P.-M. Binder and V. Priv-man, Oxford Report No. OUTP 92-08S, 1992 (unpublished).

[6] S. Thomae and S. Grossmann, J. Stat. Phys. **26**, 485 (1981); S. Grossmann and S. Thomae, Z. Naturforsch. **32a**, 1353 (1977).

[7] P. Collet and J. P. Eckmann, *Iterated Maps on the Interval as Dynamical Systems* (Birkhaüser, Boston, 1980).

[8] A. S. Pikovsky, Phys. Rev. Lett. **71**, 653 (1993).

[9] G. Perez, in *Proceedings of the Adriatico Research Conference "Mesoscopic Systems and Chaos, a Novel Approach"* (World Scientific, Singapore, in press).

[10] G. Perez and H. A. Cerdeira, Phys. Rev. A **46**, 7492 (1992).