RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 49, NUMBER 1 JANUARY 1994

Signatures of chaos in quantum billiards: Microwave experiments

A. Kudrolli and S. Sridhar
Physics Department, Northeastern University, 360 Huntington Avenue, Boston, Massachusetts 02115

Akhilesh Pandey and Ramakrishna Ramaswamy
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110067, India
(Received 7 June 1993)

The signatures of classical chaos and the role of periodic orbits in the wave-mechanical eigenvalue spectra of two-dimensional billiards are studied experimentally in microwave cavities. The survival probability for all the chaotic cavity data shows a "correlation hole," in agreement with theory, that is absent for the integrable cavity. The spectral rigidity $\Delta_3(L)$, which is a measure of long-range correlation, is shown to be particularly sensitive to the presence of marginally stable periodic orbits. Agreement with random-matrix theory is achieved only after excluding such orbits, which we do by constructing a special geometry, the Sinai stadium. Pseudointegrable geometries are also studied, and are found to display intermediate behavior.

PACS number(s): 05.45.+b, 03.65.Sq, 84.40.Cb

The quantum mechanics, and more generally the wave mechanics, of systems which are classically chaotic, is currently of great interest [1]. It has been realized that classical chaos manifests itself in unique, and often universal ways, in the eigenvalue spectra and the eigenfunctions of the corresponding quantum systems [2–4]. It has also been recognized that the periodic orbits (PO's) of the classical system, although of measure zero, play a crucial role in organizing the quantum behavior [1,5]. Although several major theoretical developments have occurred, new results concerning the quantum signatures of classical chaos are continually emerging. Furthermore, the experimental manifestations of quantum chaos are only recently being studied, and real experimental systems displaying quantum chaos are still relatively few [6–10].

Recently two-dimensional (2D) electromagnetic systems, particularly microwave cavities, have emerged as a very useful laboratory tool to study the issues of wave chaos [7–9]. These experiments exploit the exact equivalence of Maxwell's equations with Schrödinger's equation for the 2D stationary TM modes of thin microwave cavities, viz., $(\nabla^2+k^2)\psi=0$. One powerful capability of these experiments is the ability to observe directly eigenfunctions, and this has led to a direct observation of scars in wave functions [7,11].

The theme of the present work is to test experimentally, via highly precise measurements using microwave cavities, the signatures of classical chaos in wave-mechanical spectra. A very important advantage of the microwave experiments is the ability to address essentially arbitrary 2D geometries, which are not accessible via numerical simulation. This latter advantage is exploited in the present work, in which we construct several billiards, whose classical dynamics is integrable, chaotic, or even pseudointegrable. By manipulating the cavity shape, we are able to *experimentally* demonstrate and quantitatively study the influence of PO's and their stability on the statistical features of the eigenvalue spectra. Furthermore, pseudointegrable systems [12], which are not usually found in atomic or nuclear systems, but are nevertheless of importance from both theoretical and practical viewpoints, are studied here experimentally.

We analyze the eigenvalue spectra using the nearest-neighbor spacing $P(s)$ for short-range correlations, and the spectral rigidity $\Delta_3(L)$ for longer-range correlations [13]. To measure the consequences in the time domain, we also study the Fourier transform of the spectral autocorrelation function $\langle\langle (P(t)\rangle\rangle$ which can be interpreted as the survival probability [14,15]. Particularly striking is the signature observed in the survival probability $\langle\langle P(t)\rangle\rangle$, which shows, for the chaotic cavities, the so-called "correlation hole" [16]. We also show, via direct experiment, the influence of the nature of the stability of PO's: Only for a new geometry which has only isolated and unstable PO's do we find quantitative agreement with random-matrix theory.

The experiments were carried out in thin (height 6 mm) copper cavities shaped in the form of a rectangle (21.8 cm $\times$35.3 cm with golden-mean aspect ratio), a chaotic Sinai billiard (21.8 cm$\times$44.0 cm with a 4.95 cm quarter disk), a Sinai stadium (a Bunimovich stadium with an off-center disk in the middle to eliminate all nonisolated PO's), and a pseudointegrable geometry (Fig. 2, top). Details of the experimental setup to measure the eigenvalue spectrum of the cavities are described in Ref. [17]. The cavities were studied in the transmission mode using an HP8510 Network Analyzer. Figure 1(a) displays a typical experimental transmission spectrum for one of the cavities. The resonance frequencies f_n are easily identified from the transmission spectrum, which is converted to an energy spectrum $E_n=f_n^2$, and subsequently unfolded [2].

An important aspect of our work is the separation of measurement artifacts from intrinsic system properties.

 ©1994 The American Physical Society

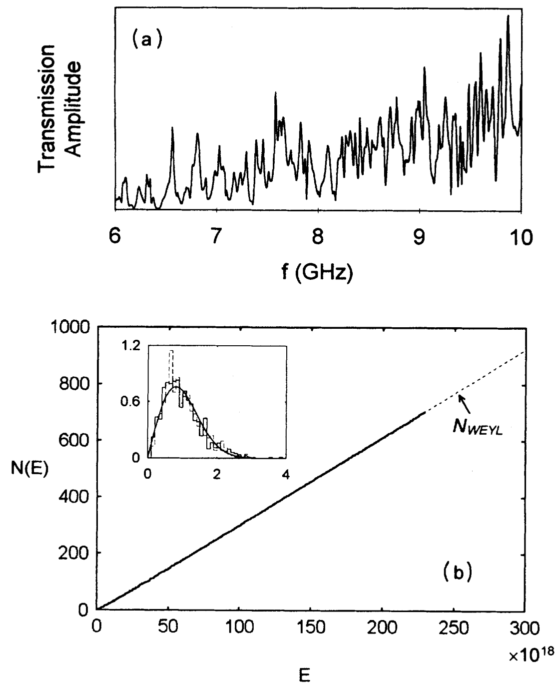

FIG. 1. (a) Typical transmission spectrum for the cavities studied. The peaks correspond to resonances. (b) The cumulative density of states, $N(E)$ vs E for the Sinai-stadium cavity. Note the excellent agreement with the Weyl law (dotted). Inset: the spacing statistics, $P(s)$ vs s for the Sinai (solid) and Sinai stadium (dashed) along with the Wigner surmise.

Preliminary experiments indicated that in order to ensure that the measurement procedure does not influence the properties of the system, and to minimize missing levels, several precautions were needed. The transmission method we employ is substantially superior to reflection methods, since differences from zero need to be observed, leading to more precise identification of levels, rather than differences from unity in the latter. Another important feature is our employment of variable coupling, which ensures that the coupling mechanism does not perturb the eigenstates, in our case to less than 10^{-4}. This is confirmed by the agreement of individual resonance frequencies and eigenfunctions, with calculations for both a rectangle and a Sinai billiard [11]. Multiple coupling-probe locations were necessary and were used in order to ensure that accidental occurrence of nodes at a particular coupling site did not lead to a missed level. In practice we have found that four locations leading to four traces, as in Fig. 1(a), are adequate.

"Good" data are obtained, using the Weyl formula for $N(E)$ as a yardstick, only when the above precautions are taken. This is shown in Fig. 1(b), where we display the experimental staircase for the first 704 levels for the Sinai-stadium cavity. The agreement with the Weyl formula is very good and shows that no levels were missed. This was also true for the other geometries, except for the rectangle, where the inherent tendency for level clustering leads to several near degeneracies, no matter what the aspect ratio of the cavity. The nearest-neighbor spacing distribution for the eigenmodes of the cavities shows level repulsion clearly for the chaotic cavities [inset of Fig. 1(b)], which also show good agreement with the Wigner surmise [13].

The results for $\Delta_3(L)$, shown in Fig. 2, are found to depend crucially on the nature of the periodic orbits present in the geometry. It has been conjectured that the spectra of time invariant quantal systems in which the classical motion is strongly chaotic have Gaussian-orthogonal-ensemble (GOE) fluctuations [18]. For the Sinai billiard, which is strongly chaotic, we see that $\Delta_3(L)$ follows the GOE curve up to $L \sim 10$ and then a linear rise is seen. In a Sinai billiard, both isolated and nonisolated orbits are present [19]. The presence of these nonisolated orbits, which are marginally stable, leads to stronger fluctuations in the energy spectrum, causing a linear rise in Δ_3. This form of the rise was seen to be the same when the experiment was repeated with a larger disk (radius $=10$ cm). It is noted that the presence of random spurious levels in a GOE sample would also lead to a linear rise, which is clearly not the case here, and further demonstrates the high quality of the data. The theoretical form for the spectral rigidity using semiclassical theory [5] is $\Delta_3(L)=1/\pi^2\ln(L)-0.007$ for $1 \ll L \ll L_{\max}$ (which is the same as in a GOE of random-matrix theory), where $L_{\max}(=hd_{\mathrm{av}}/T_{\min})$ is the outer scale of the spectrum. Here d_{av} is the mean level spacing and $T_{\min}$ is the period of the shortest orbit. The rigidity $\Delta_3(L)$ is then expected to saturate for $L \gg L_{\max}$. $L_{\max}$ can be further evaluated in terms of the area A and the number of levels N; using the Weyl area approximation, we get $L_{\max} \cong 2\sqrt{\pi A N}/l_{\min}$, where $l_{\min}$ is the length of the shortest PO. The experimental $\Delta_3(L)$ saturates at 0.466 for $L \approx 110$, which compares well with the theoretical value of 0.45.

The rigidity in the spectrum is also seen to be very sensitive to the stability of the periodic orbits. Clear differences are seen between the Sinai, Sinai-stadium, and the pseudointegrable billiard. Only for the Sinai stadium which has only marginally stable PO's do the data show

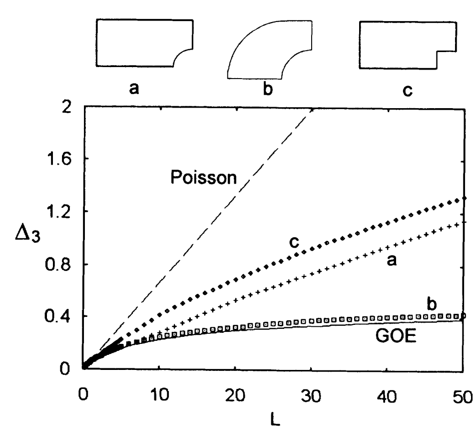

FIG. 2. The spectral rigidity, $\Delta_3(L)$ vs the energy length L, for (a) the Sinai billiard, (b) the Sinai stadium, and (c) the pseudointegrable cavities and the theoretical forms expected when the corresponding classical motion is integrable (Poisson) or strongly chaotic (GOE).

clear agreement with the GOE. The relevance of PO's to the statistical analysis has been known, and in some experiments has been accounted for numerically [9]. The present results represent a direct experimental demonstration of the PO contribution.

Until recently, theoretical work has focused mainly on classifying systems on the basis of statistical properties of the energy spectrum, as discussed above. Of equal fundamental importance is the difference between these systems in the time domain. Several theoretical studies [20] have focused on examining the survival probability, $P(t)$, which is defined as $P(t)=|\langle\phi(t)|\phi(0)\rangle|^2$, and is the recurrence probability of a given initial wave function $\phi(0)$, the wave function itself being defined as

$$|\phi(t)\rangle=\sum_{i=1}^{N} a_i e^{-iE_i t/\hbar}|E_i\rangle \ .$$

This quantity is found to depend on the initial states, a_i. Following earlier work by Pechukas [21], Wilkie and Brumer [14] have shown that $\langle\langle P(t)\rangle\rangle$, the survival probability averaged over initial states and Hamiltonians, shows a distinct difference between the regular and irregular spectrum, but these differences are confined to short times $t\approx 2\pi\hbar/\langle\Delta E\rangle$, the long-time limit being the same in both cases.

In Fig. 3, $\langle\langle P(t)\rangle\rangle$ vs time is shown for the Sinai, the Sinai-stadium, and the rectangle cavities described above. This quantity was computed from the experimental "stick" spectra in the following way. Following Ref. [22], $P(t)$ averaged over initial states was calculated from

$$\langle P(t)\rangle=\frac{3}{N+2}\left\{1+\frac{2}{3N}\sum_{\substack{n,m\\ n>m}}\cos\left[2\pi(f_n^2-f_m^2)\frac{t}{\Delta f^2}\right]\right\} . \tag{1}$$

The average over Hamiltonians to obtain $\langle\langle P(t)\rangle\rangle$ was done by averaging over $n=21$ to 420, each segment containing $N=10$ levels. The time t was appropriately time scaled with respect to the average energy spacing to get quantitative comparison with theory.

We see a rapid dephasing of $\langle\langle P(t)\rangle\rangle$ in time $t=1/N$ (here $N=10$) for all three cases. This corresponds to short time scale dynamics [23] which is not universal. For the chaotic case, $\langle\langle P(t)\rangle\rangle$ goes below the asymptotic value before recovering to it, describing a correlation hole. This is universal and is unambiguously present in the chaotic Sinai billiard spectrum and the Sinai stadium, and is clearly absent in the integrable rectangle spectrum.

A quantitative comparison of our data with random-matrix theory (RMT) is carried out following Ref. [15], who have evaluated the further averaging over the Hamiltonians giving

$$\begin{aligned}\langle\langle P(t)\rangle\rangle&=\overline{P_{\mathrm{av}}(t)}\\&=\frac{3}{N+2}\{1+\Delta_N\circ[\tfrac{1}{3}\delta(t)-\tfrac{1}{3}b_{2\beta}(t)]\} \ .\end{aligned} \tag{2}$$

Here $b_{2\beta}(t)$ is the two-level form factor [13], and $\beta=1$ for the GOE. The $\Delta_N\circ$ denotes the convolution

$$\Delta_N\circ f(t)=f dt'\Delta_N(t')F(t-t')$$

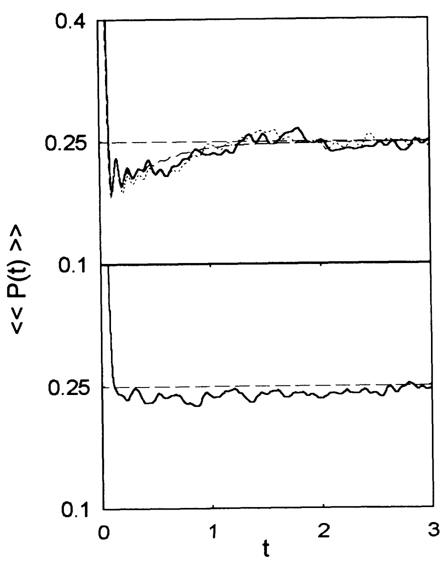

FIG. 3. The survival probability averaged over initial states and Hamiltonians, $\langle\langle P(t)\rangle\rangle$ vs t, for the cavities. The asymptotic value is 0.25. Top: the Sinai (dotted) and the Sinai stadium (solid), along with the theory (dashed) follow the theoretical form obtained from random-matrix theory. Bottom: the rectangle data show no such correlations.

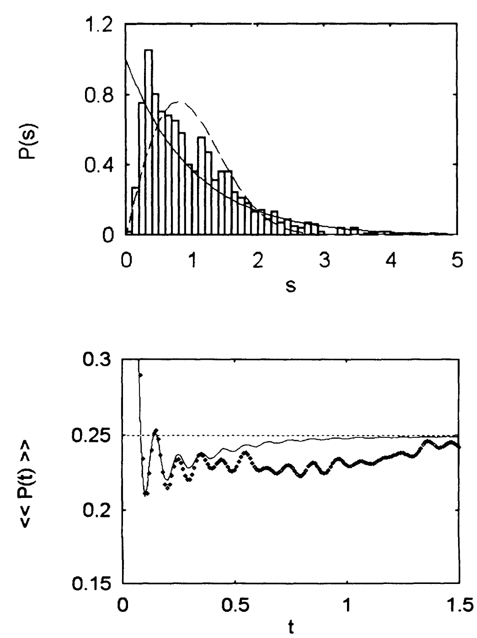

FIG. 4. The spacing statistics, $P(s)$ vs s, along with the integrable (dashed) and chaotic (solid) limits, and the survival probability for the pseudointegrable cavity. Both statistical measures show intermediate behavior.

and

$$\Delta_N(t)=N^{-1}[\sin(\pi N t/\pi t)]^2$$

due to the finite subspace, N, used.

We have compared the above theoretical form with the experimental data for the chaotic case in Fig. 3 and the data are in accord with theoretical expectations. The similarity in the form of the Sinai and the Sinai stadium $\langle\langle P(t)\rangle\rangle$ indicates that the form is not sensitive to the presence of the marginally stable PO's, and only depends on the classical phase space being chaotic.

Another feature of PO's, besides the influence of their stability, is the nature of their proliferation, i.e., power law or exponentially as the length of the PO's increases. We have studied this experimentally by constructing a pseudointegrable cavity. For the pseudointegrable case studied, the PO's occur in one parameter families [12] and the asymptotic form for the proliferation of PO's is the same as for the integrable case, i.e., $\propto l^2$ [24]. But there are differences in the rate of proliferations of PO's [25] which contribute to the spectral rigidity. This causes the rigidity to rise faster than logarithmically, giving rise to the observed deviations from a GOE form, even for small energy lengths, L. For the survival probability, an intermediate situation is found, in that the correlation hole is present but is weaker. For intermediate situations, Ref. [15] has suggested on the basis of arguments based on a fraction of the phase space being chaotic [26] that $b_{2\beta}(t)$ can be written as $\beta b_2(t/\beta)+[(1-\beta)/N]\delta(t)$, where b_2 is the exact expression for the GOE ($\beta=1$). Using this ansatz, we have the best agreement with the data for the pseudointegrable case for $\beta=0.68$ (Fig. 4). Even though this argument is not perfectly applicable to pseudointegrable systems since the phase flow is confined to a surface with genus 2, the same argument gives $\beta=0.65$ for $\Delta_3(L)$, in reasonable agreement with that obtained from the survival probability. The theory of pseudointegrable geometries is not on firm ground yet, and further developments [27] are required—however, our experimental data can be used as quantitative tests of future theories.

The good agreement found between experiment and theory leads to two important conclusions. First, proper experimental procedures can lead to extremely precise data. Also, the results of this paper show that important statistical analysis can be reliably carried out using room-temperature cavities, and higher Q (e.g., superconducting cavities at low temperatures <4.2 K) is in general not needed. Room-temperature studies have important implications in electromagnetics, in view of potential applications in microwave systems.

The approach taken here has been to study the stick spectrum obtained from the measured eigenvalues of the different cavities. In the process, information regarding resonance amplitudes and absorption widths has been discarded, and in effect the real system has been replaced by a corresponding idealized system. Future work will explore the inclusion of amplitudes and widths, and will address the very interesting issue of inverse deconvolution of classical dynamics from the measured wave spectrum.

The authors wish to thank R. Badrinarayanan, D. Hogenboom, J. V. Jose, and N. Whelan for useful discussions. Work at Northeastern was supported by ONR under Grant No. N00014-92-J-1666 and NSF under Grant No. ECS-911485.

[1] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics* (Springer-Verlag, New York, 1990).
[2] O. Bohigas and M.-J. Giannoni, *Mathematical and Computational Methods in Nuclear Physics,* edited by J. S. Dehesa, J. M. G. Gomez, and A. Polls, Lecture Notes in Physics Vol. 209 (Springer, New York, 1984), p. 209.
[3] E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984).
[4] M. L. Du and J. B. Delos, Phys. Rev. Lett. **58**, 1731 (1987).
[5] M. V. Berry, Proc. R. Soc. London Ser. A **400**, 229 (1985).
[6] J. E. Bayfield and P. Koch, Phys. Rev. Lett. **33**, 258 (1974).
[7] S. Sridhar, Phys. Rev. Lett. **67**, 785 (1991).
[8] H.-J. Stockmann and J. Stein, Phys. Rev. Lett. **64**, 2215 (1990).
[9] H.-D. Graf *et al.*, Phys. Rev. Lett. **69**, 1296 (1992).
[10] C. Marcus *et al.*, Phys. Rev. Lett. **69**, 506 (1992).
[11] S. Sridhar and E. J. Heller, Phys. Rev. A **46**, R1728 (1992).
[12] P. J. Richens and M. V. Berry, Physica (Amsterdam) **2D**, 495 (1981).
[13] M. L. Mehta, *Random Matrices,* 2nd ed. (Academic, New York, 1990).
[14] J. Wilkie and P. Brumer, Phys. Rev. Lett. **67**, 1185 (1991).
[15] Y. Alhassid and N. Whelan, Phys. Rev. Lett. **70**, 572 (1993).
[16] L. Leviander, M. Lombardi, R. Jost, and J. P. Pique, Phys. Rev. Lett. **56**, 2449 (1986); J. P. Pique, Y. Chen, R. W. Field, and J. L. Kinsey, *ibid.* **58**, 475 (1987).
[17] S. Sridhar, D. Hogenboom, and B. Willemsen, J. Stat. Phys. **68**, 239 (1992).
[18] O. Bohigas, M. J. Giannoni, and C. Schmit, Phys. Rev. Lett. **52**, 1 (1984).
[19] M. V. Berry, Ann. Phys. (N.Y.) **131**, 163 (1981).
[20] P. Brumer and M. Shapiro, Chem. Phys. Lett. **72**, 528 (1980).
[21] P. Pechukas, Chem. Phys. Lett. **86**, 553 (1982).
[22] Y. Alhassid and R. D. Levine, Phys. Rev. A **46**, 4650 (1992).
[23] E. J. Heller, Phys. Rev. A **35**, 1360 (1987).
[24] E. Gutkin, Physica (Amsterdam) **19D**, 311 (1986).
[25] D. Biswas and S. Sinha, Phys. Rev. Lett. **70**, 916 (1993).
[26] M. V. Berry and M. Robnik, J. Phys. A **17**, 2413 (1984).
[27] F. Haake *et al.*, Phys Rev. A **44**, R6161 (1991); A. Shubo and Y. Shimizu, Phys. Rev. E **47**, 54 (1993).