Roughening interfaces in the dynamics of perturbations of spatiotemporal chaos

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It is shown that the dynamics of linear perturbations of the turbulent regimes in coupled-map lattices is governed by a discrete version of the Kardar-Parisi-Zhang equation [Phys. Rev. Lett. **56**, 889 (1986)]. The asymptotic scaling behavior of the perturbation field is investigated in the case of large lattices. A possible application to spatiotemporal intermittency is discussed.

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Chaos is often defined as a dynamical regime with "sensitive dependence on initial conditions" [1,2]. Quantitatively, this sensitivity is measured by the Lyapunov exponent, which is an averaged exponential growth rate of linear perturbations of the motion under investigation. The Lyapunov exponent is easily computed numerically (although in experiment it is not easy to obtain) and thus serves as a standard tool in studying chaos.

The concept of the Lyapunov exponent may be straightforwardly applied to distributed systems as well. Here some generalizations are also possible. For example, in some problems (e.g., for flow systems) it is useful to define local (or convective, or velocity-dependent) Lyapunov exponents, which measure the growth rate of local-in-space perturbations [3]. In distributed systems one can also calculate the whole spectrum of Lyapunov exponents and study its behavior as the length of the system increases [4].

The aim of the present paper is to study more thoroughly the behavior of perturbations in distributed chaotic systems. While the Lyapunov exponent measures only the averaged growth rate, we investigate some other statistical characteristics of the perturbations in large systems. As a model we choose the simplest coupled-map lattice system [5]. We show that the dynamics of perturbations may be described by the Kardar-Parisi-Zhang (KPZ) equation, derived previously for growing interfaces in a random medium [6].

In a coupled-map lattice (CML) model a field $u(x,t)$ that depends on discrete space $x=1,2,\ldots,L$ and time $t=0,1,2,\ldots$ obeys an evolution equation

$$u(x,t+1)=f(\hat{D}(\epsilon)u(x,t)) \ . \tag{1}$$

Here $f(\ )$ is a nonlinear transformation, and $\hat{D}$ is a linear operator depending on the coupling parameter ϵ. A widely used choice for $\hat{D}$ corresponds to the nearest-neighbor interaction of diffusive type:

$$\hat{D}(\epsilon)v(x)=\epsilon v(x-1)+(1-2\epsilon)v(x)+\epsilon v(x+1) \ . \tag{2}$$

Throughout this paper we assume periodic boundary conditions. If the mapping $u\to f(u)$ is chaotic, spatiotemporal chaos is typically observed in the distributed system (1) [7]. In order to study perturbations of a turbulent state $u^0(x,t)$, we linearize (1) and get for the evolution of the perturbation $w(x,t)$,

$$\begin{aligned} w(x,t+1)&=a(x,t)\hat{D}(\epsilon)w(x,t) \ ,\\ a(x,t)&=f'(\hat{D}(\epsilon)u^0(x,t)) \ . \end{aligned} \tag{3}$$

Our goal is to study the statistical properties of the perturbation field w for large system size L and time t. Before proceeding we would like to discuss the relation of the model (1)–(3) to some other discrete linear models.

The directed polymer in random media is a model involving a directed walk on a square lattice, with bonds in the direction of the walk having random energies $\mu(x,t)$ [8,9]. In the transfer-matrix approach the overall Boltzmann weight $Z(x,t)$ obeys a recursive equation

$$\begin{aligned} Z(x,t+1)=e^{-\mu(x,t)}[&\gamma Z(x-1,t)+Z(x,t)\\ &+\gamma Z(x+1,t)] \ , \end{aligned} \tag{4}$$

where γ is a bare line tension for bonds in the transverse direction. Comparing this equation with (3) we see that γ corresponds to the diffusion constant ϵ, and the random weights $e^{-\mu(x,t)}$ correspond to the factors $a(x,t)$. The two models are thus identical if the statistical properties of $\mu(x,t)$ are chosen properly.

Recently, a simple discrete model of dynamo effect (magnetic field generation by the turbulent fluid motion) has been proposed [10]:

$$H(x,t)=\hat{S}qe^{\xi(x,t)}H(x,t) \ . \tag{5}$$

Here $\hat{S}$ is some diffusion operator, $q=\pm1$ with probability 1/2 and $\xi(x,t)$ are independent Gaussian random variables. The quantity $H(x,t)$ is interpreted as a magnetic field, which is locally amplified by the stochastic velocity field qe^{ξ} of the turbulent flow. We may assume that the local amplification rate depends on the dynamically evolving field $u(x,t)$. We can also use for $\hat{S}$ the simplest form of discrete diffusion operator, namely $\hat{D}$ from Eq. (2). Then, we get a dynamical discrete dynamo model

$$H(x,t+1)=\phi(u(x,t))\hat{D}(\epsilon_m)H(x,t) \ . \tag{6}$$

This equation should be considered simultaneously with (1). For the particular choice $\phi(\ )=f'(\ )$ and $\epsilon_m=\epsilon$, Eq.

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(6) coincides with Eq. (3).

The main point of this paper is the similarity of Eq. (3) to the KPZ equation. Indeed, Eq. (3) may be considered as a discrete analog of the diffusion equation with multiplicative noise

$$\frac{\partial W}{\partial t}=\xi(x,t)W+R\frac{\partial^2 W}{\partial x^2} . \tag{7}$$

This equation with the ansatz $W=\exp(H)$ is transformed into the KPZ equation [6]

$$\frac{\partial H}{\partial t}=\frac{\lambda}{2}\left[\frac{\partial H}{\partial x}\right]^2+\nu\frac{\partial^2 H}{\partial x^2}+\xi(x,t) . \tag{8}$$

This equation describes kinetic roughening of random driven interfaces and has been thoroughly investigated in recent years [11]. If the KPZ equation is derived from Eq. (7), one has $\lambda=2R$, $\nu=R$. In the standard KPZ equation it is assumed that the noise $\xi(x,t)$ is Gaussian and δ correlated,

$$\langle \xi(x,t)\xi(x',t')\rangle=D\delta(x-x')\delta(t-t') ,$$

although finite correlations and deviations from Gaussian distribution do not violate the asymptotic behavior [12] (except for distributions with power-law tails [13]).

We now explore the analogy between the discrete equation (3) and the multiplicative noise equation (7) and apply the ansatz

$$w(x,t)=e^{h(x,t)} . \tag{9}$$

Then we get from Eq. (3) a discrete analog of the KPZ equation:

$$h(x,t+1)-h(x,t)=\ln a(x,t)+\ln[1-2\epsilon+\epsilon\exp(h(x-1,t)-h(x,t))+\epsilon\exp(h(x+1,t)-h(x,t))] . \tag{10}$$

It is worth noting that for the discrete case there is an important restriction in performing the ansatz (9), namely, $w(x,t)$ should be positive for all x,t. In the continuous case this can be ensured by a proper choice of the initial field, while in the discrete case also the condition $a(x,t)>0$ must be fulfilled for all x,t.

It follows from (9) that the exponential growth of the field $w(x,t)$ in time corresponds to the linear motion of the interface position $h(x,t)$; the mean velocity is exactly the Lyapunov exponent [see Eq. (15) below]. Except for this mean motion, the interface $h(x,t)$ also fluctuates [due to fluctuations of $a(x,t)$] and we now can investigate these fluctuations using the correspondence to the KPZ equation.

Because ϵ is an effective diffusion constant corresponding to R in Eq. (7), Eq. (10) corresponds to the KPZ equation (8) with

$$\lambda=2\epsilon,\ \nu=\epsilon .$$

Note that the parameter ϵ is the diffusion constant both in the KPZ equation and in the discrete equation (3). The parameter λ in the KPZ equation describes the change in the growth rate of the tilted interface. For the discrete equation (3) this corresponds, because of the ansatz (9), to the change of the Lyapunov exponent when exponentially growing-in-space perturbations are considered; such generalized Lyapunov exponents have been introduced recently by Politi and Torcini [14]. The problem remains in finding a value for the noise strength D. The values of $a(x,t)$ are produced by chaotic motions in the CML (1) and of course are neither Gaussian nor δ correlated. These differences are, however, not important if the asymptotic behavior coincides with that predicted by the KPZ equation. While a large number of models belong to the universality class of KPZ equation, we have to check this for the perturbation field in CML once more.

We used in the numerical calculations the following "skewed" doubling transformation:

$$f(u)=\begin{cases} bu & \text{for } 0\le u<b^{-1} \\ \dfrac{b}{b-1}u & \text{for } b^{-1}\le u\le 1 . \end{cases} \tag{11}$$

In this transformation the local instantaneous expansion rate $a(x,t)$ takes the values b and $b(b-1)^{-1}$, so varying the parameter b we can consider both cases of weak ($b\approx 2$) and strong ($b\gg 1$) noise. Numerical simulation shows that the CML model (1)–(3) and (11) indeed demonstrates properties of the KPZ equation. If a system of finite length L is considered, then for sufficiently large t a statistically stationary roughened interface appears (Fig. 1) (we consider here only statistical properties of the interface's fluctuations, thus its mean position is always subtracted). The probability distribution density of h obeys Gaussian law (Fig. 2), and the spatial spectrum scales as k^{-2}, as is expected for the KPZ equation [9] (Fig. 3).

From the asymptotic behavior of system (1)–(3) we can estimate the effective noise strength (this procedure has been recently applied to the Kuramoto-Sivashinsky equation [15]). It is known that for a system of length $L\gg 1$, governed by the KPZ equation (8), the averaged saturated width of interface is [6,9,15]

$$\lim_{t\to\infty}\langle (H(x,t)-\langle H(x,t)\rangle)^2\rangle=\frac{DL}{24\nu} . \tag{12}$$

Thus, calculating this quantity for our discrete model (10) we can estimate the value of D. It depends on the nonlinear transformation $f(u)$ and on the coupling constant ϵ (because the statistical properties of the CML depend on ϵ). Results of the calculations of the effective noise strength are presented in Fig. 4.

It is worth noting that the observed field $w(x,t)$ demonstrates highly intermittent properties, as one can see from Fig. 1. In fact, what is observed in the w vs x graph is a narrow region near the maximum of the field $h(x,t)$, due to the exponent in (9).

For the KPZ equation a scaling growth of the width of the interface (starting from the flat one) is predicted for

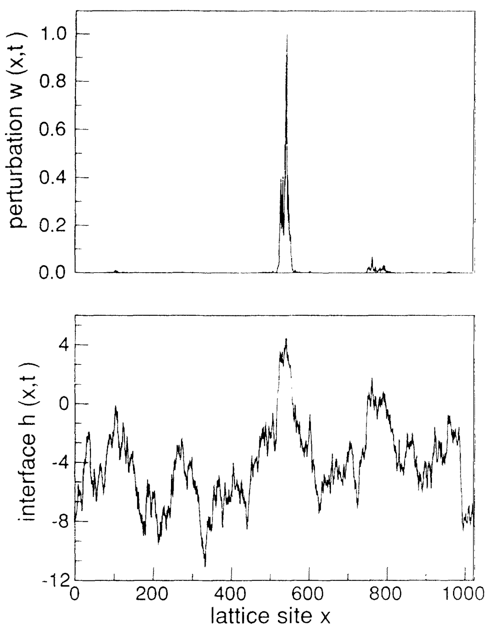

FIG. 1. Snapshot of the fields $w(x,t)$ and $h(x,t)$ for the CML equations (1)–(3) and (11) with $L=1024$, $\epsilon=0.1$, $a=4$.

infinite systems [6,11],

$$\xi^2=\langle (H(x,t)-\langle H(x,t)\rangle)^2\rangle = Ct^{2/3} , \quad (13)$$

where $C=0.16D^{8/3}\lambda^{2/3}\nu^{-4/3}$ [15]. However, as was mentioned in [15], this scaling is observed only for large times and for long systems,

$$t \gg t_c \approx 252\nu^5\lambda^{-4}D^{-2},\ L \gg L_c \approx 152\nu^3\lambda^{-2}D^{-1} , \quad (14)$$

because only for large t and L does the nonlinear term in the KPZ equation dominate. Applying these formulas to the CML model (1)–(3) and (11), we conclude that the scaling (13) may be observed only for systems with sufficiently large b and small ϵ. In Fig. 5 the results of simulations with $b=5$, $\epsilon=0.1$ are presented. The observed exponent is clearly larger than the value 0.5 predicted by linear theory [15], but still slightly less than the asymptotic KPZ value $\frac{2}{3}$, probably due to the still insufficient length of the system.

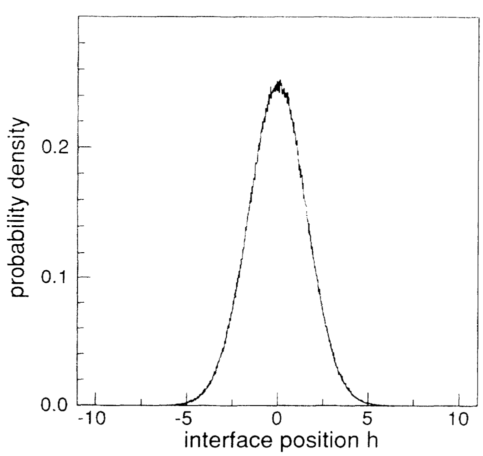

FIG. 2. Probability distribution density of the field $w(x,t)$ for the CML with $\epsilon=0.3$, $b=4$, $L=256$. The curve nearly coincides with the Gaussian one.

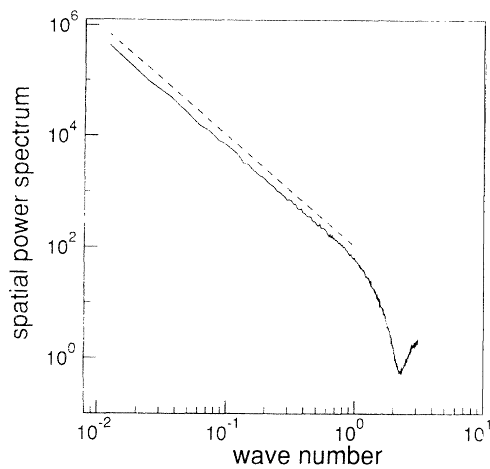

FIG. 3. Spatial spectrum of the field $w(x,t)$. The broken line has slope -2.

From Eqs. (12) and (13) we can estimate the transient time for the interface width growth,

$$t_{\rm tr} \approx 5.5D^{-5/2}\lambda^{-1}\nu^2 .$$

Only for $t > t_{\rm tr}$, when the interface width growth saturates, may the perturbation field be considered as statistically stationary. However, some characteristics of the linear system may be well defined already in the transient regime. Consider, e.g., the Lyapunov exponent, which may be represented as a mean growth rate of the field $h(x,t)$:

$$\Lambda = \lim_{t\to\infty} \frac{\langle h(x,t)\rangle}{t} . \quad (15)$$

As follows from Eq. (8), the mean growth rate is proportional to $\langle (\partial H/\partial x)^2\rangle$ and is thus determined mainly by Fourier harmonics of the perturbation field with high wave numbers. So for calculating the Lyapunov ex-

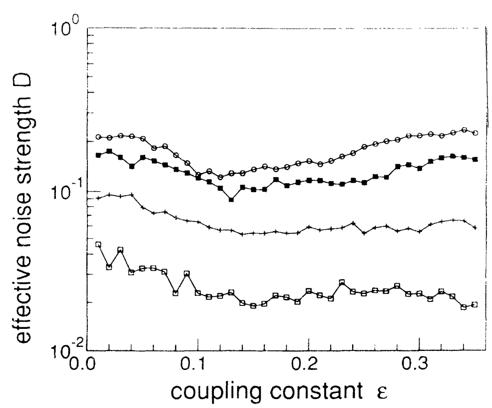

FIG. 4. Effective noise strength D vs coupling ϵ for different values of parameter b in (11): $b=2.5$, 3, 4, 5 (from bottom to top).

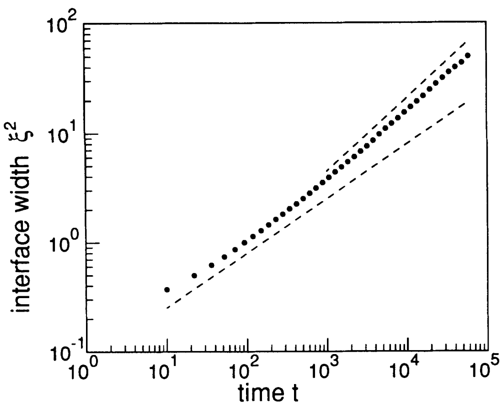

FIG. 5. Growth of the "interface width" ξ^2 in the CML with $\epsilon=0.1$, $b=5$, $L=8000$. The broken lines have slopes 0.5 and $\frac{2}{3}$. One can see a crossover to a nonlinear regime at $t\approx 10^3$.

ponent it is not necessary to wait until the modes with low wave numbers become statistically stationary, and a correct value of Λ may be obtained already for $t<t_{\rm tr}$.

So far we have considered the linear perturbation of the turbulent field. We would like to discuss briefly a situation where such a linearized field appears naturally. Let us consider two coupled CML's of the type (1):

$$\begin{aligned} u(x,t+1)&=(1-\gamma)f(\hat{D}(\epsilon)u(x,t))+\gamma f(\hat{D}(\epsilon)v(x,t)) ,\\ v(x,t+1)&=\gamma f(\hat{D}(\epsilon)u(x,t))+(1-\gamma)f(\hat{D}(\epsilon)v(x,t)) , \end{aligned} \tag{16}$$

where γ is a coupling constant. This system generalizes the two coupled one-dimensional chaotic attractors studied in Refs. [16–18]. If $\gamma=\frac{1}{2}$, the CML's are always synchronized: $u(x,t)\equiv v(x,t)$, while for $\gamma=0$ they are uncoupled and thus uncorrelated. It is clear that there exists a critical value of γ for which an asynchronous regime appears. Near this critical value we can consider the difference between CML's $z(x,t)=u(x,t)-v(x,t)$ as a small perturbation of the synchronous turbulent state $u^0(x,t)$. Thus, we get for $z(x,t)$ the linear equation

$$z(x,t+1)=(1-2\gamma)f'(\hat{D}(\epsilon)u^0(x,t))\hat{D}(\epsilon)z(x,t) . \tag{17}$$

This equation differs from Eq. (3) only in the factor $(1-2\gamma)$, so the field $z(x,t)$ can be considered in the framework of the analogy with growing interfaces developed above. The detailed statistical analysis of the intermittency in the system (16) will be presented elsewhere [19].

In conclusion, we have established the analogy between the perturbations of the turbulent regimes in coupled-map lattices and roughening interfaces. The dynamics of the perturbation field is shown to be governed by the discrete analog of the Kardar-Parisi-Zhang equation. From this analogy the asymptotic properties of the perturbations can be obtained.

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[1] A. J. Lichtenberg and M. A. Lieberman, *Regular and Chaotic Dynamics* (Springer, New York, 1992).
[2] H. G. Schuster, *Deterministic Chaos, An Introduction* (VCH-Verlag, Berlin, 1988).
[3] A. S. Pikovsky, Chaos **3**, 225 (1993).
[4] R. Livi, A. Politi, and S. Ruffo, J. Phys. A **19**, 2033 (1986).
[5] Chaos **2** (3) (1992) (special issue on coupled-map lattices).
[6] M. Kardar, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[7] J. P. Crutchfield and K. Kaneko, in *Directions in Chaos,* edited by B.-L. Hao (World Scientific, Singapore, 1987), Vol. 1.
[8] M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).
[9] J. Krug, P. Meakin, and T. Halpin-Healy, Phys. Rev. A **45**, 638 (1992).
[10] A. D. Poezd, R. Galeeva, D. D. Sokolov, and A. A. Ruzmaikin, Chaos **2**, 253 (1992).
[11] *Dynamics of Fractal Surfaces,* edited by F. Family and T. Vicsek (World Scientific, Singapore, 1991).
[12] E. Medina, T. Hwa, M. Kardar, and Y. C. Zhang, Phys. Rev. A **39**, 3053 (1989).
[13] Y.-C. Zhang, Physica A **170**, 1 (1990).
[14] A. Politi and A. Torcini, Chaos **2**, 293 (1992).
[15] K. Sneppen, J. Krug, M. H. Jensen, C. Jayaprakash, and T. Bohr, Phys. Rev. A **46**, 7351 (1992).
[16] A. S. Pikovsky, Z. Phys. B **55**, 149 (1984).
[17] H. Fujisaka, Prog. Theor. Phys. **71**, 513 (1984).
[18] A. S. Pikovsky and P. Grassberger, J. Phys. A **24**, 4587 (1991).
[19] A. Pikovsky and J. Kurths (unpublished).